A classification of symmetric graphs of order 30*

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Abstract

Let Γ be a simple undirected graph and G a subgroup of $\mathrm{Aut}\,\Gamma$. Γ is said to be *G-symmetric*, if G acts transitively on the set of ordered adjacent pairs of vertices of Γ; Γ is said to be *symmetric* if it is $\mathrm{Aut}\Gamma$-symmetric. In this paper we give a complete classification for symmetric graphs of order 30. (See Theorem 10.)

1 Introduction

Let Γ be a simple undirected graph and G a subgroup of $\mathrm{Aut}\,\Gamma$. Γ is said to be *G-symmetric*, if G acts transitively on the set of ordered adjacent pairs of vertices of Γ; Γ is said to be *symmetric* if it is $\mathrm{Aut}\,\Gamma$-symmetric. The classifications of symmetric graphs of order p, $2p$, $3p$ and kp, where k and p are distinct primes and $5 \leq k < p$, were done in [1], [2], [3] and [4], respectively. After finishing the classification of symmetric pq-graphs, some people tried to do the same thing for symmetric graphs of order $4p$, $6p$. Our main results are in the same spirit. Although the class of graphs of order 30 is only a special class of symmetric graphs of order $6p$, it is a necessary step in classifying that of $6p$ since it is the only situtation where $p < 6$. Thus the problem of classfying symmetric graphs of order $6p$ splits into two cases, the case $p = 5$ and the case $p \neq 5$. The aim of this paper is to classify the symmetric graphs of order 30.

The group- and graph-theoretic notation and terminology used in this paper are standard in general; the reader can refer to [3] and [4] when necessary. Other notation and terminology such as block graphs, orbital graphs of a permutation group used here are the same as in [3] and [5].

Throughout this paper we use $V(\Gamma)$ and $E(\Gamma)$ to denote the vertex and edge sets of Γ, respectively. Two vertices u and v being adjacent is denoted by $u \sim v$ or

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$uv \in E(\Gamma)$. For $v \in V(\Gamma)$, $\Gamma_1(v)$ denotes the neighborhood of v in Γ, that is the set of vertices adjacent to v in Γ. Then Γ is G-symmetric if and only if G acts transitively on $V(\Gamma)$ and for any vertex $v \in V(\Gamma)$, the stabilizer G_v of v in G is transitive on the neighborhood $\Gamma_1(v)$ of v.

Now we consider how the group $PSL(2,29)$ acts on the cosets of $Z_{29} \cdot Z_{14}$. Since this action is doubly transitive, this orbital graph $L_2(29)^{29}_{30} \cong K_{30}$. This is the unique primitive symmetric graph of order 30. So to give the classifaction of symmetric graphs of order 30, it suffices to consider the imprimitive case.

Assume that G acts on Γ imprimitively and that B is a nontrivial block of G. Let $\Sigma = \{B_0, B_1, \cdots, B_{n-1}\}$ be a complete block system of A. We define the *block graph* of Γ corresponding to Σ, which is still denoted by Σ, by

$$\begin{aligned} V(\Sigma) &= \Sigma, \\ E(\Sigma) &= \{B_iB_j \mid \text{there exist } v_i \in B_i, v_j \in B_j \text{ such that } v_iv_j \in E(\Gamma)\}. \end{aligned}$$

G induces an action on Σ. Assume that the kernel of this action is K. Set $\overline{G} = G/K$. Then $\overline{G}$ acts on Σ faithfully. We quote a lemma about block graphs from [5].

Lemma 1.1 *Assume that Γ is G-symmetric. Then*
(1) Σ is $\overline{G}$-symmetric;
(2) If Γ is connected, then so is Σ;
(3) If the induced graph by B_i has an edge, then B_i is a union of several connected components of Γ.

By the above lemma, if Γ is connected, then there is no edge in each induced graph B_i, and if Γ is disconnected then the connected components of Γ must have size a divisor of 30. Using Lemma 1.1,we have

Theorem 1 *Assume that Γ is a disconnected symmetric graph of order 30. Then Γ is isomorphic to one of the following:*

No	*Graph*	*No*	*Graph*	*No*	*Graph*	*No*	*Graph*	*No*	*Graph*
1	$5K_6$	*2*	$5K_{3,3}$	*3*	$5C_6$	*4*	$5C_3[2K_1]$	*5*	$3C_5[2K_1]$
6	$3K_{5,5}$	*7*	$3K_{10}$	*8*	$3C_{10}$	*9*	$3G(2\cdot 5,4)$	*10*	$3K_5[2K_1]$
11	$3O_3$	*12*	$3O_3^c$	*13*	$2K_{15}$	*14*	$2K_{5,5,5}$	*15*	$2G(3\cdot 5,4)$
16	$2F(1)'$	*17*	$2T_6$	*18*	$2T_6^c$	*19*	$2G(3\cdot 5,2)$	*20*	$2K_5[3K_1]$
21	$15K_2$	*22*	$2C_{15}$	*23*	$6C_5$	*24*	$2PG(3,2)$	*25*	$2C_5[3K_1]$
26	$10C_3$	*27*	$30K_1$	*28*	$6K_5$				

Note that the graph $2PG(3,2)$ will be described in section 7. By virtue of this lemma, in the rest of this paper we always assume that Γ is connected. We determine all such graphs in the next six sections. We deal with the cases in which the lengths of the imprimitive blocks are 5, 3, 2, 6, 15, 10 in the sections 3, 4, 5, 6, 7, 8 respectively.

2 Preliminary Results

We use $Z_p = \{0, 1, \cdots, p-1\}$ to denote the cyclic group of order p, written additively. The automorphism group $\mathrm{Aut}Z_p$ of Z_p is isomorphic to Z_{p-1}. For any positive divisor r of $p-1$ we use H_r to denote the unique subgroup of $\mathrm{Aut}Z_p$ of order r, which is isomorphic to Z_r.

The following obvious lemma gives a method of constructing larger symmetric graphs from smaller ones.

Lemma 2.1 *If Σ is a symmetric graph of order n, then the lexicographic product $\Sigma[mK_1]$ of Σ by mK_1 is a symmetric graph of order nm.*

By [1], [2], [3], we have a lemma about block graphs.

Lemma 2.2 *(a) If $|B_i| = 5$ then Σ could be one of the following:*
(1) K_6; (2) $K_{3,3}$; (3) $G(2\cdot 3, 2) = C_6$; (4) $C_3[2K_1]$.
(b) If $|B_i| = 3$, Σ could be one of the following:
(1) K_{10}; (2) $K_{5,5}$; (3) $G(2\cdot 5, 4)$; (4) $K_5[2K_1]$; (5) $G(5,2)[2K_1]$; (6) $G(2\cdot 5, 2)$; (7) O_3 or O_3^c.
(c) If $|B_i| = 2$ Σ could be one of the following:
(1) K_{15}; (2) $K_{5,5,5}$; (3) $C_5[3K_1]$; (4) $G(3\cdot 5, 4)$; (5) $G(3\cdot 5, 2)$; (6) $K_3[5K_1]$; (7) T_6; (8) T_6^c; (9) $F'(1)$; (10) C_{15}.
(d) If $|B_i| = 6$ then Σ is isomorphic to K_5, C_5.
(e) If $|B_i| = 15$ then $\Sigma = K_2$ and if $|B_i| = 5$ then $\Sigma = C_3$

Note that

$$K_2[15K_1] = K_{3,3}[5K_1] = K_{5,5};$$
$$K_3[10K_1] = (K_3[2K_1])[5K_1] = K_{5,5,5}[2K_1];$$
$$K_5[6K_1] = (K_5[2K_1])[3K_1] = (K_5[3K_1])[2K_1];$$
$$G(5,2)[6K_1] = (G(5,2)[3K_1])[2K_1] = (G(5,2)[2K_1])[3K_1].$$

Using Lemma 2.2 and the method of [3], it is not difficult to prove the following:

Theorem 2 *If K acts on B_i unfaithfully, then Γ is isomorphic to one the following:*

No	*Graph*	*No*	*Graph*	*No*	*Graph*	*No*	*Graph*
1	$G(2\cdot 5,2)[3K_1]$	*2*	$K_3[10K_1]$	*3*	$K_5[6K_1]$	*4*	$C_5[6K_1]$
5	$G(2\cdot 5,4)[3K_1]$	*6*	$K_{10}[3K_1]$	*7*	$K_2[15K_1]$	*8*	$K_{15}[2K_1]$
9	$G(3\cdot 5,4)[2K_1]$	*10*	$G(3\cdot 5,2)[2K_1]$	*11*	$T_6[2K_1]$	*12*	$F'(1)[2K_1]$
13	$G(3\cdot 5,1)[2K_1]$	*14*	$T_6^c[2K_1]$	*15*	$O_3[3K_1]$	*16*	$O_3^c[3K_1]$

By virtue of this lemma, we may assume that Γ is not isomorphic to any of the above sixteen graphs and that K acts on B_i faithfully. Now we consider the case that K acts on B_i doubly transitively. Also by the method of [3], we have

Theorem 3 *If K acts on B_i doubly transitively and faithfully, then Γ is isomorphic to $\Sigma[pK_1] - p\Sigma$ (i.e $\Sigma[pK_1]$ minus a 1-factor), where $\Sigma[pK_1]$ is one of the sixteen graphs in Theorem 2 and so we have another sixteen graphs.*

By virtue of this theorem, throughout the rest of this paper we may assume that:
Γ is not isomorphic to $\Sigma[pK_1]$ or $\Sigma[pK_1] - p\Sigma$ and K acts on B_i faithfully and simply-primitively or $K = 1$.

3 A has a block of length r=5

In this section we assume that $\Sigma = \{B_i | i \in Z_6\}$ is a complete block system of A, and that K is the kernel of the action of A on Σ. Set $\overline{A} = A/K$. We also use Σ to denote the corresponding block graph. Then Σ is $\overline{A}$-symmetric.

Now K acts on B_i simply-primitively and faithfully. In this case $K^{B_i} < AGL(1,5)$ is solvable. So K^{B_i} has only one nonequivalent transitive representation of degree 5; and for any $v \in B_i$, $K_v < Z_4$ is semiregular on $B_i - \{v\}$.

Now we define six graphs. Let H_2 be the unique subgroup of $\mathrm{Aut}Z_5 \cong Z_4$ of order 2. Then we define the graph $C_6(5,2)$ by

$$\begin{aligned} V(C_6(5,2)) &= \{(i,x) \mid i \in Z_6, x \in Z_5\}, \\ E(C_6(5,2)) &= \{((i,x),(i+1,y)) \mid i \in Z_6, y - x \in H_2\}; \end{aligned}$$

Assume that $H_4 = \langle t \rangle$, then we define the graph $C_6(5,2)'$ by

$$\begin{aligned} V(C_6(5,2)') &= \{(i,x) \mid i \in Z_6, x \in Z_5\}, \\ E(C_6(5,2)') &= \{((i,x),(i+1,y)) \mid i \in Z_6, y - x \in t^i H_2\}; \end{aligned}$$

we define the graph $K_6(5,2)$ by

$$\begin{aligned} V(K_6(5,2)) &= \{(i,x) \mid i \in Z_6, x \in Z_5\}, \\ E(K_6(5,2)) &= \{((i,x),(j,y)) \mid i,j \in Z_6, y - x \in H_r\}; \end{aligned}$$

we define the graph $K_{3,3}(5,2)$ by

$$\begin{aligned} V(K_{3,3}(5,2)) = & \ \{(i,k,x) \mid i \in Z_2, k \in Z_3, x \in Z_5\}, \\ E(K_{3,3}(5,2)) = & \ \{((i,k,x),(j,l,y)) \mid j - i = 1, k,l \in Z_3, y - x \in H_2\}; \end{aligned}$$

we define the graph $C_3[2K_1](5,2)$ by

$$\begin{aligned} V(\Gamma) &= \{(i,k,x) \mid i \in Z_3, k \in Z_2, x \in Z_5\}, \\ E(\Gamma) &= \{((i,k,x),(j,l,y)) \mid i,j \in Z_3, i \neq j, \ k,l \in Z_2, x - y \in H_2\}; \end{aligned}$$

and we define the graph $C_3[2K_1](5,2)'$ by

$$\begin{aligned} V(\Gamma) &= \{(i,k,x) \mid i \in Z_3, k \in Z_2, x \in Z_5\}, \\ E(\Gamma) &= \{((i,k,x),(j,l,y)) \mid i,j \in Z_3, i \neq j, \ k,l \in Z_2, x - y \in t^{k-l} H_2\}. \end{aligned}$$

Lemma 3.1 *Let K act on B_i simply primitively. Assume that the block graph $\Sigma \cong K_6$ and $\Gamma \neq K_6[5K_1]$. Then Γ is isomorphic to $K_6(5,2)$.*

Proof Since $\Sigma \cong K_6$, we have $\overline{A} = A/K \cong S_6$ or A_6 by Lemma 3.1. Let $C = C_A(K)$ so that C is a normal subgroup of A. If $\overline{A} \cong S_6$, then the socle T ($\cong A_6$) of $\overline{A}$ is simple and Γ is T-symmetric. So in any case we may assume that A/K is a simple group. Thus either $C \leq K$ or $CK = A$. If $C \leq K$ then A/C is a group of automorphisms of K and has A_6 or S_6 as a factor group. Clearly, $|K| > 10$, so we have $CK = A$. Since $Z(K) = 1$, we have $K \cap C = 1$. It follows that $C \cong S_6$ or $C \cong A_6$ and $A = K \times C$. Now we distinguish two cases: (a) $C \cong S_6$, and (b) $C \cong A_6$.

(a) $C \cong S_6$: Then there exists $\rho \in C$ such that $\rho^6 = 1$ and the action of ρ on Σ is

$$(B_0, B_1, B_2, B_3, B_4, B_5)^\rho = (B_1, B_2, B_3, B_4, B_5, B_0).$$

Let $P = \langle \pi \rangle$. So $R = \langle \rho, \pi \rangle \cong Z_{30}$ is a regular subgroup of A. Then Γ is a Cayley graph of R. Now we determine the graph Γ. Assume that $v = (0,0) \in B_0$ is the unique fixed point of H in B_0. Renaming the vertices of the graph when necessary, we may assume that for any $x \in Z_5$, $(0,x)^\pi = (0, x+1)$, $(i,x)^\rho = (i+1, x)$. Now we have

$$(i,x)^\pi = (0,x)^{\rho^i \pi} = (0,x)^{\pi \rho^i} = (0, x+1)^{\rho^i} = (i, x+1),\ i = 1,2,3,4,5.$$

Since Γ is a Cayley graph, to determine Γ it suffices to determine the neighbourhood $\Gamma_1(v)$ of $v = (0,0)$ in Γ. Set $\Gamma_1^{B_i}(v) = \Gamma_1(v) \cap B_i$ for $i = 1,2,3,4,5$. Then

$$\Gamma_1(v) = \Gamma_1^{B_1}(v) \cup \Gamma_1^{B_2}(v) \cup \Gamma_1^{B_3}(v) \cup \Gamma_1^{B_4}(v) \cup \Gamma_1^{B_5}(v).$$

Since $v = (0,0)$ is the only fixed point of H in B_0, $\Gamma_1^{B_1}(v) = \{(1,x) | x \in L\}$ where L is a coset of H_r. Replacing π by one of its powers we may assume that $L = H_r$. To determine $\Gamma_1^{B_i}(v), i = 2,3,4,5$, we should consider other automorphisms of Γ. Take $\lambda \in C$ such that $o(\lambda) = 5$ and the action of λ on Σ is $B_0^\lambda = B_0$ and $(B_1, B_2, B_3, B_4, B_5)^\lambda = (B_2, B_3, B_4, B_5, B_1)$. Thus we have

$$\Gamma_1^{B_i}(v) = (\Gamma_1^{B_1}(v))^{\lambda^{i-1}} = \{(i,x) | x \in H_r\}, i = 2,3,4,5.$$

Since Γ is undirected, $(3,x) \in \Gamma_1^{B_3}(v)$ if and only if $(3,-x) \in \Gamma_1^{B_3}(v)$. So we have $(3, H_r) = (3, -H_r)$. It follows that $H_r = -H_r$, and hence that r is even, that is $r = 2$. This proves that $\Gamma \cong K_6(5,2)$.

(b) $C \cong A_6$:

Since C is the doubly-transitive permutation group of $\Sigma = \{B_0, B_1, B_2, B_3, B_4, B_5\}$, we can choose $\rho \in C$ such that $(0,x)^\rho = (i,x)$ for any given $i \in \{2,3,4,5\}$. Renaming the vertices of graph when necessary, we may assume that for any $x \in Z_5$, $(0,x)^\pi = (0, x+1)$. Since $\rho \in C$, we have $\rho\pi = \pi\rho$ and hence

$$(i,x)^\pi = (0,x)^{\rho\pi} = (0,x)^{\pi\rho} = (0,x+1)^\rho = (i,x).$$

Since $C \cong A_6$ is a doubly-transitive group of Σ, we can take $\sigma \in C$ such that the action of σ on Σ is $\sigma = (B_1, B_2, B_3, B_4, B_5)$. As in case (a), we may assume that $\Gamma_1^{B_1}(v) = \{(1,x) | x \in H_r\}$. Thus

$$\Gamma_1^{B_i}(v) = (\Gamma_1^{B_1}(v))^{\sigma^{i-1}} = \{(i,x) | x \in H_r\}, i = 2,3,4,5$$

as in case (a). So we get the same graph as in case (a). $\square$

Lemma 3.2 *Let K act on B_i simply primitively. In case (2), (3), (4) of Lemma 2.2, K contains a normal subgroup P of A such that*

(1) $K = P \cdot H$ and $H \cong H_r$, where $r = 1$, or $r = 2$;

(2) P has a complement M in A and M is an extension of H by $\overline{A}$.

(3) Let $C = C_A(P)$. Then A/C is isomorphic to a subgroup of $AutP \cong Z_4$.

Proof Noting that K is a Frobenius group with Frobenius kernel P of order $p = 5$, we may assume that $K = P \cdot H$ and $H \cong Z_r$ for a divisor r of 4. Since K does not act doubly transitively on B_i, we have $|H| \neq 4$ and thus $|H| = 2$ or $|H| = 1$. Since $P char K \lhd A$, it follows that $P \lhd A$. Next, in case (2),(3),(4) of Lemma 2.2 we claim that P has a complement M in A. To see this we use the fact that the order of P is prime to that of $\overline{A}$, so by the Schur-Zassenhaus theorem P has a complement M in A and the claim is established. Also M is an extension of H by $\overline{A}$. Statement (3) is obvious. □

In what follows we will use this information frequently. Fix $C = C_A(P)$, $P = \langle \pi \rangle$ and $o(\pi) = 5$ throughout this section.

Lemma 3.3 *Let K act on B_i simply primitively. Assume that the block graph $\Sigma \cong K_{3,3}$ and $\Gamma \neq K_{3,3}[5K_1]$. Then Γ is isomorphic to $K_{3,3}(5, 2)$.*

Proof It is easy to show that if $\Sigma \cong K_{3,3}$, then $\overline{A} \cong S_3^2 \cdot Z_2$. Thus M is an extension of H by $S_3^2 \cdot Z_2$. Renaming the block graph Σ, we assume that

$$\{B_{0,0}, B_{0,1}, B_{0,2}, B_{1,0}, B_{1,1}, B_{1,2}\}$$

is the complete block system of A. Take $\sigma, \tau \in M$ such that

$$\bar{\sigma} = \sigma K, \bar{\tau} = \tau K \in \overline{A}$$

and $o(\bar{\sigma}) = 3$, $o(\bar{\tau}) = 2$ and the actions of σ, and τ on Σ are

$$\begin{aligned} \sigma &= (B_{0,0}, B_{0,1}, B_{0,2})(B_{1,0}, B_{1,1}, B_{1,2}) \\ \tau &= (B_{0,0}, B_{1,0})(B_{0,1}, B_{1,2})(B_{0,2}, B_{1,1}). \end{aligned}$$

It is easy to see that $\sigma \in C$. Since $\sigma^3 \in K \cap M = H$ and $H \cap C = 1$, $\sigma^3 = 1$. Now we distinguish two cases: (a) $\tau \in CH$, and (b) $\tau \notin CH$.

(a) $\tau \in CH$: Replacing τ by τh for some suitable $h \in H$ we may assume $\tau \in C$. Since $\rho^2 \in C \cap K \cap M = 1$, so $\tau^2 = 1$.

So $R = \langle \pi, \sigma, \tau \rangle \cong Z_{30}$ is a regular subgroup of A and thus Γ is a Cayley graph of R. Now we determine the graph Γ. Assume that $(0, 0, 0) \in B_{0,0}$ is the unique fixed point of H. Renaming the vertices of the graph when necessary, we assume that for any $x \in Z_5$,

$$\begin{aligned} &(0,0,x)^{\pi} = (0,0,x+1), &&(0,0,x)^{\sigma} = (0,1,x), \\ &(0,0,x)^{\sigma^2} = (0,2,x), &&(0,0,x)^{\tau} = (1,0,x), \\ &(0,0,x)^{\tau\sigma} = (1,1,x), &&(0,0,x)^{\tau\sigma^2} = (1,2,x). \end{aligned}$$

Now we have

$$
\begin{array}{lllll}
(0,1,x)^\pi & = (0,0,x)^{\sigma\pi} & = (0,0,x)^{\pi\sigma} & = & (0,0,x+1)^\sigma = (0,1,x+1), \\
(0,2,x)^\pi & = (0,0,x)^{\sigma^2\pi} & = (0,0,x)^{\pi\sigma^2} & = & (0,0,x+1)^{\sigma^2} = (0,2,x+1), \\
(1,0,x)^\pi & = (0,0,x)^{\tau\pi} & = (0,0,x)^{\pi\tau} & = & (0,0,x+1)^\tau = (1,0,x+1), \\
(1,1,x)^\pi & = (0,0,x)^{\tau\sigma\pi} & = (0,0,x)^{\pi\tau\sigma} & = & (0,0,x+1)^{\tau\sigma} = (1,1,x+1), \\
(1,2,x)^\pi & = (0,0,x)^{\tau\sigma^2\pi} & = (0,0,x)^{\pi\tau\sigma^2} & = & (0,0,x+1)^{\tau\sigma^2} = (1,2,x+1).
\end{array}
$$

Since Γ is a Cayley graph, to determine Γ it suffices to determine the neighbourhood $\Gamma_1(v)$ of $v = (0,0,0)$ in Γ. Set $\Gamma_1^{B_{1,i}} = \Gamma_1(v) \cap B_{1,i}$ for $i = 0,1,2$. Then

$$\Gamma_1(v) = \Gamma_1^{B_{1,0}}(v) \cup \Gamma_1^{B_{1,1}}(v) \cup \Gamma_1^{B_{1,2}}(v).$$

First we need to determine $\Gamma_1^{B_{1,0}}(v)$. Since $v = (0,0,0)$ is the only fixed point of H, as in Lemma 3.5, we may assume that $\Gamma_1^{B_{1,0}}(v) = \{(1,0,x) | x \in H_r\}$.

Under the action of σ, we have

$$
\begin{array}{ll}
\Gamma_1^{B_{1,1}}(v) & = (\Gamma_1^{B_{1,0}}(v))^\sigma = (1,1,H_r), \\
\Gamma_1^{B_{1,2}}(v) & = (\Gamma_1^{B_{1,0}}(v))^{\sigma^2} = (1,2,H_r),
\end{array}
$$

Since Γ is undirected, $(1,0,x) \in \Gamma_1^{B_{1,0}}(v)$ if and only if $(1,0,-x) \in \Gamma_1^{B_{1,0}}(v)$. So we have $(1,0,H_r) = (1,0,-H_r)$. It follows that $H_r = -H_r$ and hence that r is even. So $r = 2$. It follows that $\Gamma \cong K_{3,3}(5,2)$.

(*b*) $\tau \notin CH$:

Since A/C is cyclic, both $\langle\rho\rangle C/C$ and HC/C are cyclic. Now $C \cap K = P$ and hence $CK/C \cong K/C \cap K \cong H = H_2$; but $CK = CH$ so A/C has order $2|A/CH|$. Hence $A/C \cong Z_4$. We could have chosen $\langle\tau C\rangle = A/C = MC/C$ with $\tau \in M$; then τ is such that τC still generates A/C and $\tau^2 \in H$. Since $\tau \notin CH$ and $\tau^2 \in CH$, HC/C is a subgroup of $\langle\tau\rangle C/C$ of index 2. Noting that $HC/C \cong H_2$, the automorphism of P induced by τ is of order 4. Writing $H_2 = \langle s\rangle$ and $\pi^\tau = \pi^t$ for some $s,t \in Z_4$, we have $|\langle t\rangle : \langle s\rangle| = 2$; so we may assume $s = t^2$. Let $H = \langle\gamma\rangle$. We may assume that $\pi^\gamma = \pi^s$. Since ρ^2 induces an automorphism of P of order 2, we may assume that ρ^2 and γ induce the same automorphism $\pi \mapsto \pi^s$ on $P = \langle\pi\rangle$.

Now we determine the graph Γ. Assume again $V(\Gamma) = \{(i,k,x) | i \in Z_2, k \in Z_3, x \in Z_5\}$. Renaming the vertices of the graph when necessary, we may assume that for any $x \in Z_5$, $(0,0,x)^\pi = (0,0,x+1)$, $(0,0,x)^\tau = (1,0,tx)$, $(0,1,x)^\tau = (1,2,tx)$ and $(0,2,x)^\tau = (1,1,tx)$. $(1,0,x)^\tau = (0,0,tx)$, $(1,1.x)^\tau = (0,2,tx)$, $(1,2,x)^\tau = (0,1,tx)$. As in case (a), we may assume $(0,0,0) \sim (1,0,1)$. Acting by H, we have $(0,0,0) \sim (1,0,y)$ for $y \in H_2$; acting by P, we have $(0,0,x) \sim (1,0,y)$ for $y-x \in H_2$. Acting by τ, we have $(1,0,tx) \sim (0,0,ty)$ for $y-x \in H_2$ i.e $(1,0,x) \sim (0,0,y)$ for $y-x \in tH_2$. So we have $(1,0,x) \sim (0,0,0)$ for $-x \in tH$ and this implies that $x \in -tH_2 = tH_2$ which contradicts $\Gamma_1^{B_{1,0}} = (1,0,H_2)$. So (*b*) $\tau \notin CH$ does not happen.

At last, $K_{3,3}(5,2) \cong G(2\cdot 5)[3K_1]$. It follows that the map Φ from $K_{3,3}(5,2)$ to $G(2\cdot 5)[3K_1]$ is a graph isomorphism, where

$$\Phi : (i,k,x) \longrightarrow (i,x,k), i \in Z_2, k \in Z_3, x \in Z_5,$$

since $(i,k,x) \sim (j,l,y)$ in $K_{3,3}(5,2)$ implies that $(i,x,k) \sim (j,y,l)$ in $G(2\cdot 5)[3K_1]$. $\square$

Lemma 3.4 *Let K act on B_i simply primitively. Assume that the block graph $\Sigma \cong C_3[2K_1]$ and $\Gamma \neq C_3[2K_1][5K_1]$. Then Γ is isomorphic to $C_3[2K_1](5,2)$.*

Proof Since $\Sigma \cong C_3[2K_1]$ and $\overline{A} \cong Z_2 \mathrm{wr} S_3$, $\overline{A}$ contains a normal subgroup $\overline{N} \cong Z_2^3$ and a subgroup $\overline{T} \cong S_3$ such that $\overline{A} = \overline{N} \cdot \overline{T}$.

Since $P \lhd A$, we may assume that M is an extension of $H \cong Z_r$ by $\overline{A}$.

Now we deal with the two situations (i) $H \cong H_2$ and (ii) $H = 1$ separately in the following:

(i) $H \cong H_2$: It is convenient to rename the vertices of Σ as

$$\Sigma = \{B_{(i,k)} | i = 0,1,2;\ k = 0,1\}$$

since Σ is the lexicograph product $C_3[2K_1]$. Hence we can take $\rho, \tau, \tau_0, \tau_1, \tau_2 \in M$ such that their actions on Σ are

$$\begin{aligned} \rho = &\ (B_{(0,0)}, B_{(1,0)}, B_{(2,0)})(B_{(0,1)}, B_{(1,1)}, B_{(2,1)}), \\ \tau = &\ (B_{(0,0)}, B_{(0,1)})(B_{(1,0)}, B_{(1,1)})(B_{(2,0)} B_{(2,1)}), \\ \tau_0 = &\ (B_{(0,0)}, B_{(0,1)}),\ \tau_1 = (B_{(1,0)}, B_{(1,1)}),\ \tau_2 = (B_{(2,0)} B_{(2,1)}). \end{aligned}$$

Since $\rho K \in S_3 K \leq \overline{A}$, so $\rho K \in \overline{A}' = A'K/K$ and thus $\rho \in A'K \leq CK = CH$. Replacing ρ by ρh for some suitable $h \in H$ we may assume that $\rho \in C$ and thus $\rho^3 \in K \cap M \cap C = 1$.

Now we distinguish two cases:(1) $\tau \in CH$, (2) $\tau \notin CH$.

(a) $\tau \in CH$: As above, we may assume $\tau \in C$ and thus $\tau^2 = 1$. Since $\rho\tau = \tau\rho$, it follows that $\langle \rho, \tau \rangle \cong Z_6$ and thus $R = \langle \rho, \tau, \pi \rangle \cong Z_{30}$ is a regular subgroup of A. So we conclude that Γ is a Cayley graph of R.

Assume $V(\Gamma) = \{(i,j,x) | i \in Z_3, j \in Z_2, x \in Z_5\}$. Renaming the vertices of the graph when necessary, we may assume that for any $x \in Z_5$,
$(0,0,x)^\pi = (0,0,x+1)$, $(0,0,x)^\rho = (1,0,x)$, $(1,0,x)^\rho = (2,0,x)$,
$(0,1,x)^\rho = (1,1,x)$, and $(1,1,x)^\rho = (2,1,x)$.

Since $\rho^3 \in K, \tau^2 \in K$, $\rho^3, \tau^2 \in K \cap C \cap M = 1$ and hence $\rho^3 = \tau^2 = 1$. Now we determine the graph Γ. Assume that $v = (0,0,0) \in B_{(0,0)}$ is the unique fixed point of H in $B_{(0,0)}$. Renaming the vertices of the graph when necessary, we may assume that for any $x \in Z_5$, $(0,0,x)^\pi = (0,0,x+1)$. Now we have

$$\begin{aligned} (1,0,x)^\pi = (0,0,x)^{\rho\pi} &= (0,0,x)^{\pi\rho} = (0,0,x+1)^\rho = (1,0,x+1), \\ (2,0,x)^\pi = (0,0,x)^{\rho^2\pi} &= (0,0,x)^{\pi\rho^2} = (0,0,x+1)^{\rho^2} = (2,0,x+1), \\ (1,1,x)^\pi = (0,0,x)^{\tau\rho\pi} &= (0,0,x)^{\pi\tau\rho} = (0,0,x+1)^{\tau\rho} = (1,1,x+1), \\ (2,1,x)^\pi = (0,0,x)^{\tau\rho^2\pi} &= (0,0,x)^{\pi\tau\rho^2} = (0,0,x+1)^{\tau\rho^2} = (2,1,x+1). \end{aligned}$$

Since Γ is a Cayley graph, to determine Γ it suffices to determine the neighbourhood $\Gamma_1(v)$ of $v = (0,0,0)$ in Γ. Set $\Gamma_1^{B_{(i,j)}}(v) = \Gamma_1(v) \cap B_{(i,j)}$ for $i = 1,2,\ j = 0,1$. Then

$$\Gamma_1(v) = \Gamma_1^{B_{(1,0)}}(v) \cup \Gamma_1^{B_{(1,1)}}(v) \cup \Gamma_1^{B_{(2,0)}}(v) \cup \Gamma_1^{B_{(2,1)}}(v).$$

Since $v = (0,0,0)$ is the only fixed point of H in $B_{0,0}$, as before we may assume that $\Gamma_1^{B_{(1,0)}}(v) = \{(1,0,x) | x \in H_2\}$. Acting by τ_1, we have

$$\Gamma_1^{B_{(1,1)}}(v) = \Gamma_1^{B_{(1,0)}}(v)^{\tau_1} = \{(1,1,x) | x \in H_2\}.$$

Acting by π, we have

$$(0,0,x) \sim (1,0,y),\ (0,0,x) \sim (1,1,y),\ y-x \in H_2.$$

Thus acting by ρ, we get

$$(1,0,x) \sim (2,0,y), (2,0,x) \sim (0,0,y),\ for\ y-x \in H_2.$$

So we have $(0,0,0) \sim (2,0,x)$, $x \in -H_2 = H_2$. Acting by τ_2, we get $(0,0,0) \sim (2,1,x)$, $x \in H_2$. Now it is easy to check that $\Gamma \cong C_3[2K_1](5,2)$.

(b) $\tau \notin CH$: As before we may assume that $\tau^2 \neq 1$ and $\tau^2 = \gamma \in H$. Thus $\tau\pi\tau^{-1} = \pi^t$. Since $\tau \mid_{B_{(1,i)}} = \tau_1$, it follows that τ_1 induces the same action as τ on P. Similarly, we have $\tau \mid_{B_{(2,i)}} = \tau_2$. As before, we may suppose that $\Gamma_1^{B_{(1,0)}}(v) = \{(1,0,x)|x \in H_2\}$. Acting by τ_1, we have $\Gamma_1^{B_{(1,1)}}(v) = \{(1,1,tx)|x \in H_2\}$. So acting by ρ, it follows that

$$\Gamma_1^{B_{(2,0)}}(v) = \{(2,0,x)|x \in H_2\},$$
$$\Gamma_1^{B_{(2,1)}}(v) = \{(2,1,x)|x \in tH_2\}.$$

Now it is easy to check that $\Gamma \cong C_3[2K_1](5,2)'$.

(ii) $H = 1$: In this case we consider that $B_{(0,0)}$, $B_{(1,0)}$ and $B_{(2,0)}$ are the three adjacent blocks in Σ. We claim that the induced graph $\Gamma(B_{(0,0)}, B_{(1,0)}, B_{(2,0)})$ is isomorphic to $5K_3$. Set $B_{(0,0)} = \{a_i\}$, $B_{(1,0)} = \{b_i\}$ and $B_{(2,0)} = \{c_i\}$, $i = 1,2,3,4,5$. Without loss of generality we may assume that $a_i \sim b_i$, and $a_i \sim c_i$ for $i = 1,2,3,4,5$. Since $\Sigma \cong C_3[2K_1]$, there exists an element in A/K which fixes $B_{(0,0)}$ setwise and interchanges $B_{(1,0)}$ and $B_{(2,0)}$. Since K is transitive on each $B_{(i,j)}$, we can find λ which fixes $B_{(0,0)}$ pointwise and interchanges $B_{(1,0)}$ and $B_{(2,0)}$. So λ will interchange b_i and c_i for $i = 1,2,3,4,5$. By the remark of Lemma 4.2, we have

$$\Gamma(B_{(0,0)}, B_{(1,0)}, B_{(2,0)}) \cong 5K_3.$$

Acting by τ_0, τ_1, τ_2 and ρ, it is easy to prove that $\Gamma \cong 5C_3[2K_1]$. Since $5C_3[2K_1]$ is not connected, no new graphs occur here.

Since $C_3[2K_1](5,2)$ is a Cayley graph of an Abelian group and $C_3[2K_1](5,2)'$ is a Cayley graph of a non-abelian group, it is easy to prove that they are non-isomorphic symmetric graphs.

Finally $C_3[2K_1](5,2) \cong G(3\cdot 5,2)[2K_1]$. It follows that the map Φ from $C_3[2K_1](5,2)$ to $G(3\cdot 5,2)[2K_1]$ is a graph isomorphism, where

$$\Phi : (i,k,x) \longrightarrow (i,x,k), i \in Z_3, k \in Z_2, x \in Z_5,$$

since $(i,k,x) \sim (j,l,y)$ in $C_3[2K_1](5,2)$ implies that $(i,x,k) \sim (j,y,l)$ in $G(3\cdot 5,2)[2K_1]$. $\square$

Lemma 3.5 *Let K act on B_i simply primitively. Assume that the block graph $\Sigma \cong C_6$ and $\Gamma \neq C_6[5K_1]$. Then Γ is isomorphic to $C_6(5,2)$, $C_6(5,2)'$ or C_{30}.*

Proof The first two graphs have been determined in [7], where

$$C^{\pm 1}(5;6,1) = C_6(5,2); \quad C^{\pm\epsilon}(5;6,1) = C_6(5,2)'.$$

When $H_r = 1$, it is easy to prove that $\Gamma \cong C_{30}$. $\square$

Summarizing the result obtained in this section, we have

Theorem 4 *If Γ is a symmetric graph of order 30 and A acts on $V(\Gamma)$ imprimitively, and if A has a block of length 5, then Γ is isomorphic one of the graphs $C_6(5,2)$, $C_6(5,2)'$, $C_3[2K_1](5,2)'$ or $K_6(5,2)$.*

4 A has a block of length r=3

In this section we assume that $\Sigma = \{B_1, \cdots, B_{10}\}$ is a complete block system of A, and that K is the kernel of the action of A on Σ. Then $1 \leq K \leq S_3$. Set $\overline{A} = A/K$. We also use Σ to denote the corresponding block graph.

We distinguish the following two subcases: 1. $|K| = 3$; 2. $K = 1$.

4.1 $|K| = 3$

Since K acts on B_i simply-primitively and faithfully, $K \cong K^{B_i} \cong Z_3$ and for any adjacent blocks B_i and B_j and any $v \in B_i$, $|\Gamma_1(v) \cap B_j| = 1$ and so if there is a new graph occuring in this section, then it must be a covering graph Γ of Σ and we use $\Sigma \cdot \mathbb{Z}_3$ to denote it.

Definition 4.1 *A k-fold covering graph Γ of Σ is a graph whose vertex set is $V(\Sigma \times W)$ and if $uv \in E(\Gamma)$ then the induced graph between two blocks $\{(v,w) | \; w \in W\}$ and $\{(u,w) | \; w \in W\}$ is a perfect matching, where $|W| = k$.*

We quote a basic lemma about covering graphs from [3].

Lemma 4.2 *Let K act on B_i simply primitively. Assume $\Sigma \cong K_{10}$ that B_i, B_j, and B_k are any three blocks in Σ. If there exists $\overline{\tau} \in A/K$ such that $B_1^{\overline{\tau}} = B_1$, $B_2^{\overline{\tau}} = B_3^{\overline{\tau}}$ and $B_3^{\overline{\tau}} = B_2$, then the induced block graph $\Gamma(B_i, B_j, B_k)$ of (B_i, B_j, B_k) is isomorphic to $3K_3$, and $\Gamma \cong 3K_{10}$.*

Actually, Lemma 4.2 would still be true under the hypothesis that $|B_i|$ is odd. We have used it for proving Lemma 4.3.

Lemma 4.3 *Let K act on B_i simply primitively. Assume that the block graph $\Sigma \cong K_{10}$ and $\Gamma \neq K_{10}[3K_1]$. In this case there are no new graphs occuring.*

Proof If $\Sigma \cong K_{10}$ then $A/K = \overline{A} \cong A_{10}$ or S_{10}. Thus the action of $\overline{A}$ on Σ is at least 3-transitive. By the therom of Gaschötz, it easy to prove that $A = S \times K$, where $S \cong A_{10}$ or S_{10}. Set $K = \langle \pi \rangle$. If $B_1 = \{u, v, w\}$, then

$$S_u = S_u^{\pi} = (S \cap A_u)^{\pi} = S \cap A_{u^{\pi}} = S \cap A_v = S_v.$$

So we have $S_u = S_v = S_w$. As Σ is at least 3-transitive there exists $\tau \in S$ such that $B_1^\tau = B_1$, $B_2^\tau = B_3^\tau$ and $B_3^\tau = B_2$. So τ acting on B_1 either has a fixed point and hence fixes all the points in B_1 or $\tau = (u, v, w)$. In the latter case, replacing τ by $\tau\pi^2$, also fixes all the points in B_1.

By the above discussion, if $\Gamma(B_i, B_j) = 3K_2$, then τ must fixes one edge of this three and interchanges two endpoints of this edge. By Lemma 4.2, $\Gamma(B_i, B_j, B_k) \cong 3K_3$ and then $\Gamma = 3K_{10}$ and is not connected and hence is excluded. □

Checking $\overline{A}$ in the Lemma 2.2 (b) except for the case $\overline{A} \cong S_{10}$, K has a complement M in A such that $M \cong \overline{A}$ by the Schur-Zassenhaus theorem. We fix such an M in the rest of this section.

Lemma 4.4 *Let K act on B_i simply primitively. Assume that the block graph $\Sigma \cong K_5[2K_1]$ and $\Gamma \neq (K_5[2K_1])[3K_1]$. Then $\Gamma \cong 3K_5[2K_1]$.*

Proof Since $\Sigma \cong K_5[2K_1]$ and $\overline{A} \cong Z_2\mathrm{wr}T(5,2)$, by the Schur-Zassenhaus theorem K has a complement M in A such that $M \cong \overline{A} \cong Z_2\mathrm{wr}T(5,2)$. Assume that B_j and B_k are any pair of blocks which are not adjacent in Σ, and $B_i \sim B_j$, $B_i \sim B_k$. By the structure of M, there exists τ such that $B_i^\tau = B_i$, $B_j^\tau = B_k$ and $B_k^\tau = B_j$. Add some edges to Σ such that $\Sigma^* = K_{10} = \Sigma \cup K_5[2K_1]^c$. Then τ is still an automorphism of Σ^* since τ is an automorphism of $K_5[2K_1]^c$. Let Γ^* be the graph whose block graph is Σ^*. By Lemma 4.3, $\Gamma^* \cong 3K_{10}$ and then removing these extra edges we have $\Gamma \cong 3K_5[2K_1]$. □

Similarly we have

Lemma 4.5 *Let K act on B_i simply primitively. Assume that the block graph $\Sigma \cong C_5[2K_1]$ and $\Gamma \neq C_5[2K_1][3K_1]$. Then $\Gamma \cong 3C_5[2K_1]$.*

Lemma 4.6 *Let K act on B_i simply primitively. Assume that the block graph $\Sigma \cong O_3$ or O_3^c, and $\Gamma \neq O_3[3K_1]$ or $\Gamma \neq O_3^c[3K_1]$. In this case there are no new graphs occurring.*

Proof In this case $\Sigma \cong O_3$ or O_3^c, and $A/K \cong A_5$ or S_5. Since $K = \langle h \rangle \cong A_3$, again using the Gaschötz theorem we have $A = S \times K$, where $S \cong A_5$ or S_5. Since $\sigma = (12)(45) \in A_5$ and $(1,2), (3,4), (3,5) \in O_3$, we have $(1,2)^\sigma = (1,2)$, $(3,4)^\sigma = (3,5)$ and $(3,5)^\sigma = (3,4)$. So there exists $\tau \in S$ such that $B_0^\tau = B_0$ while $B_1^\tau = B_2^\tau$ and $B_2^\tau = B_1$ and thus Γ is isomorphic to $3O_3$ or $3O_3^c$. □

Lemma 4.7 *Let K act on B_i simply primitively. Assume that the block graph $\Sigma \cong K_{5,5}$ and $\Gamma \neq K_{5,5}[3K_1]$. Then $\Gamma \cong 3K_{5,5}$.*

Proof Since $\Sigma \cong K_{5,5}$ and $\overline{A} \cong S_5^2 \cdot Z_2$, by the Schur-Zassenhaus theorem K has a complement M in A such that $M \cong \overline{A}$. Suppose that

$$\begin{aligned} V(\Sigma) &= \{B_0, B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9\}, \\ E(\Sigma) &= \{B_iB_j | i - j = odd\}. \end{aligned}$$

Then there exists $\rho_1, \rho_2 \in M$ such that the action of ρ_1 and ρ_2 on Σ are $\rho_1 = (B_1, B_3, B_5, B_7, B_9)$ and $\rho_2 = (B_0, B_2, B_4, B_6, B_8)$. Assume that $(i, x) \in B_i$, $x =$

0, 1, 2. Since K is transitive on each B_i, we can find λ which fixes B_0 pointwise and $(1,x)^\lambda = (3,x)$, $(3,x)^\lambda = (5,x)$, $(5,x)^\lambda = (7,x)$, $(7,x)^\lambda = (9,x)$, $(9,x)^\lambda = (1,x)$. As before, assume that $(0,0) \sim (1,1)$. Acting by λ, we have $(0,0) \sim (3,1)$ and $(0,0) \sim (5,1)$. Since K is transitive on each B_i, acting by K, we have

$$\begin{array}{ll} (0,1) \sim (1,2), & (0,1) \sim (3,2), (0,1) \sim (5,2), (0,1) \sim (7,2), (0,1) \sim (9,2), \\ (0,2) \sim (1,0), & (0,2) \sim (3,0), (0,2) \sim (5,0), (0,2) \sim (7,0), (0,2) \sim (9,0). \end{array}$$

Similarly acting by ρ_2, we have

$$\begin{array}{ll} \Gamma_1(0,0) = \Gamma_1(2,0) = & \Gamma_1(4,0) = \Gamma_1(6,0) = \Gamma_1(8,0) \\ \Gamma_1(0,1) = \Gamma_1(2,1) = & \Gamma_1(4,1) = \Gamma_1(6,1) = \Gamma_1(8,1) \\ \Gamma_1(0,2) = \Gamma_1(2,2) = & \Gamma_1(4,2) = \Gamma_1(6,2) = \Gamma_1(8,0). \end{array}$$

So it is easy to check that Γ is isomorphic to $3K_{5,5}$. $\square$

Similarly we have the following

Lemma 4.8 *Let K act on B_i simply primitively. Assume that the block graph $\Sigma \cong G(2\cdot 5, 4)$ and $\Gamma \neq G(2\cdot 5,4)[3K_1]$. Then $\Gamma \cong 3G(2\cdot 5, 4)$.*

Lemma 4.9 *Let K act on B_i simply primitively. Assume that the block graph $\Sigma \cong C_{10}$ and $\Gamma \neq C_{10}[3K_1]$. Then $\Gamma \cong C_{30}$.*

Proof Since $\Sigma \cong C_{10}$ and $\overline{A} \cong D_{20}$, then $M \cong \overline{A} \cong D_{20}$. Take $\rho \in M$ such that $\bar{\rho} = \rho K \in \overline{A}$ and $o(\bar{\rho}) = 10$. So $\rho^{10} \in K \cap M = 1$ and $o(\rho) = 10$. Let $C = C_A(K)$ so that $1 \leq A/C \leq Z_2$ and hence $C \geq A'$. Since ρ^2 is an element of order 5, $\rho^2 \in C$. Now we consider the two cases (a) $\rho \in C$ and (b) $\rho \notin C$ separately.

(a) $\rho \in C$: Then $R = \langle \rho, \pi \rangle \cong Z_{30}$ is a regular subgroup of A and thus Γ is a Cayley graph of R. So to determine the graph Γ it suffices to determine the neighbourhood of $v = (0,0)$. If $v \sim (1,0)$, acting by ρ it easy to check that $\Gamma \cong 3C_{10}$. If $v \sim (1,1)$ or $(1,2)$ then $\Gamma \cong C_{30}$.

(b) $\rho \notin C$: In this case we can check the that graph Γ is isomorphic to $3C_{10}$. $\square$

4.2 $K = 1$

In this case A and $\overline{A}$ are isomorphic as abstract groups. But as permutation groups, $\overline{A}$ is a group of degree 10 and A is of degree 30. Since $A \cong \overline{A}$, A is a transitive representation of degree 30 of $\overline{A}$. Then the one point stabilizer of $\overline{A}$ must have a subgroup of index 3.

Lemma 4.10 *Let $\overline{A}$ be a group of degree 10 which has a transitive representation of degree 30. Then $\overline{A} = A_5$, and hence $A_{\{B\}} = S_3$, $A_v = Z_2$. In this case Γ is not a symmetric graph and hence there is no new graph occurring.*

Proof Checking $\overline{A}$ in Lemma 4.1, the only possibility is $\overline{A} \cong A_5$ and $\Sigma \cong O_3$ or O_3^c. In this case the one block stabilizer $A_{\{B\}} = S_3$ and the one vertex stabilizer $A_v \cong Z_2$ in Γ. Since the length of suborbits of Z_2 is 1 or 2, this contradicts that the degree of the block graph O_3 or O_3^c is 3 or 6. $\square$

Summarizing the result obtained in this section, we have

Theorem 5 *If A has a block of length 3 and K acts on it simply-primitively, then there is no new graph occuring.*

5 A has a block of length r=2

In this section we assume that $\Sigma = \{B_i | i \in Z_{15}\}$ is a complete block system of A, and that K is the kernel of the action of A on Σ. Set $\overline{A} = A/K$. We also use Σ to denote the corresponding block graph. Then Σ is $\overline{A}$-symmetric.

We may assume that Γ is not isomorphic to $\Sigma[2K_1]$ and K acts on B_i faithfully. We distinguish the following two subcases: 1. $|K| = 2$; 2. $K = 1$.

5.1 $|K| = 2$

In this case the transformation of exchanging the two vertices in all blocks simultaneously is an automorphism of Γ.

Now we define the graph $\Sigma \cdot \mathbb{Z}_2'$ by

$$\begin{aligned} V(\Sigma \cdot \mathbb{Z}_2') &= \{(u, x) \mid i \in \Sigma, x \in Z_2\}, \\ E(\Sigma \cdot \mathbb{Z}_2') &= \{((u, x), (v, y)) \mid uv \in E(\Sigma), y - x = 0\}; \end{aligned}$$

and define the graph $\Sigma \cdot \mathbb{Z}_2$ by

$$\begin{aligned} V(\Sigma \cdot \mathbb{Z}_2) &= \{(u, x) \mid u \in \Sigma, x \in Z_2\}, \\ E(\Sigma \cdot \mathbb{Z}_2) &= \{((u, x), (v, y)) \mid uv \in \Sigma, y - x = 1\}. \end{aligned}$$

We shall prove that Γ is isomorphic to one of these two graphs. It is clear that $\Sigma \cdot \mathbb{Z}_2' \cong 2\Sigma$. The following lemma is from [6].

Lemma 5.1 *The covering graph $\Gamma = \Sigma(K, \phi)$ of Σ can be produced as follows. Since K acts on B_i simply-primitively and faithfully, The vertex set of Γ is $V(\Gamma) \times K$, and two vertices (v_1, k_1), (v_2, k_2) are adjacent if and only if $v_1 v_2 \in \Sigma$ and $k_2 = k_1 \phi(v_1, v_2)$, where $\phi : E(\Sigma) \to K$ is such that $\phi(uv) = (\phi(vu))^{-1}$ for all $uv \in E(\Sigma)$.*

Let $K = \{1, \pi\}$ so that $\pi^2 = 1$. By virtue of this lemma, if we define that $\phi(uv) = 1$, then $\Gamma = \Sigma(K, \phi) \cong \Sigma \cdot \mathbb{Z}_2' \cong 2\Sigma$; if we define that $\phi(uv) = \pi$, then $\Gamma = \Sigma(K, \phi) \cong \Sigma \cdot \mathbb{Z}_2$, for all $uv \in E(\Sigma)$. The following lemma is also from [6].

Lemma 5.2 *The double covering graph $\Sigma \cdot \mathbb{Z}_2$ is connected if and only if Σ is not bipartite.*

Since none of the Σ in Lemma 5.1 are bipartite, we have

Lemma 5.3 *If K acts on Σ faithfully and $|K| = 2$, then $\Gamma \cong 2\Sigma$ or $\Gamma \cong \Sigma \cdot \mathbb{Z}_2$.*

5.2 $K = 1$

In this case A and $\overline{A}$ are isomorphic as abstract groups. But as permutation groups, $\overline{A}$ is a group of degree 10 and A is of degree 30. Since $A \cong \overline{A}$, A is transitive representation of degree 30 of $\overline{A}$. Then the one point stabilizer of $\overline{A}$ must have a subgroup of index 3 and the valency of Σ must divide that of Γ or equivalently, the size of orbit of one point in Σ must divide that of Γ. We denote a one point stabilizer of $\overline{A}$ in Σ by $A_{\{B\}}$, and a one point stabilizer A in Γ by A_v. We will run through the list of possibilities given in Lemma 2.1 to see whether or not they satisfy the above condition.

Case $\Sigma = G(15,2)$ and $\overline{A} = (Z_5 \cdot H_2) \cdot S_5$. Since $A_{\{B\}} = H_2 \cdot Z_2$ and $A_v = H_2$, hence the valency of Σ does not divide that of Γ which is $|H_2|$. So this cannot occur.

Case $\Sigma = C_{15}$ and $\overline{A} = D_{15}$. Here $A_{\{B\}} = Z_2$ and $A_v = 1$. Hence this cannot occur.

Case $\Sigma = F'(1)$ and $\overline{A} = P\Gamma L(2,4)$. Since $A_{\{B\}} = Z_2^2 \cdot Z_2$ and $A_v = Z_2^2$ and valency of Σ is 8, the size of orbit of a one point stabilizer of Σ does not divide that of Γ.

Case $\Sigma = G(15,4)$ and $\overline{A} = S_5 \times S_3$. Here $A_{\{B\}} = S_4 \times Z_2$ and $A_v = S_4$. Since the valency of Σ is 8, it follows that Γ is a double cover of Σ. Thus no new graph occurs.

Case $\Sigma = K_5[3K_1]$, $C_5[3K_1]$ or $K_3[5K_1]$. In all cases, we assume that $B_i \sim B_j$ and $v \in B_i$. Since $A_{\{B_i\}}$ is not a normal subgroup and $A_{\{B_i\}}$ acts on B_i as an involution, when it interchanges two vertics in B_i, it does not exchanges the two vertices in all blocks simultaneously. This shows that $\Gamma = \Sigma[2K_1]$ and hence no new graph occurs.

Case $\Sigma = T_6$ or T_6^c and $\overline{A} = S_6$. Here $A_{\{B\}} = S_4$ and $A_v = A_4$. Since the valency of T_6, T_6^c are 8, 6 respectively, and only 6 divides $|A_4|$, this forces that Γ is isomorphic to a double cover of T_6^c. Thus no new graph occurs.

Finally we discuss the case $\Sigma = K_{15}$ in the following

Lemma 5.4 *If $\Sigma = K_{15}$, then the graphs Γ are known.*

Proof Since $A \cong \overline{A}$ (as abstract groups), so as permutation groups, $\overline{A}$ is doubly transitive and of degree 15, and A is transitive and of degree 30.

Let π and $\bar{\pi}$ be the permutation characters of A and $\overline{A}$, respectively. Since $\overline{G}$ is 2-transitive, $\bar{\pi} = 1 + \chi$, where χ is an irreducible character of $\overline{G}$ of degree 14. Since A is transitive, $\langle \pi, 1 \rangle = 1$; so we may assume that $\pi = 1 + k\chi + \gamma$, where $k = \langle \pi, \chi \rangle$ and γ is a character of A. For π degree 30, we have $k = 1$ or 2. If $k = 2$, then γ is a non-principal linear character of A. Let N be the kernel of γ. Then N is intransitive as the restriction of π on N contains the principal character of N at least twice, and $N \geq A'$. By the classification of 2-transitive groups of degree 15, A/N must be a 2-group. So N has 2 orbits in $V(\Gamma)$; and N acts on each block doubly-transitively. We shall determine such graphs in section 7, we can exclude the case $k = 2$; so we have $k = 1$, and then $\langle \pi, \bar{\pi} \rangle = 2$. Assume $S = A_{\{B_i\}}$ is the setwise stabilizer of B_i in A. Then $\bar{\pi}$ is the induced character of the principal character 1_S of S to A. By Frobenius reciprocity we have

$$\langle \pi, \bar{\pi} \rangle_A = \langle \pi, (1_S)^G \rangle_A = \langle \pi|_S, 1_S \rangle_S = 2,$$

so S has exactly 2 orbits on $V(\Gamma)$, one of which is the block B_i itself. Now it is easy to see that if B_i and B_j are adjacent in Σ, then $\Gamma(B_i, B_j) \cong 2K_2$. Hence the transformation of exchanging the two vertices in all blocks simultaneously is an automorphism of Γ. So the graphs are contained in the case $|K| = 2$. $\square$

Summarizing the result obtained in this section, we have

Theorem 6 *If A has a block of length 2 and K acts on it simply-primitively, then Γ is isomorphic one of the graph in following*

(1) $(K_5[3K_1]) \cdot \mathbb{Z}_2$; *(2)* $C_{15} \cdot \mathbb{Z}_2$; *(3)* $T_6^c \cdot \mathbb{Z}_2$; *(4)* $T_6 \cdot \mathbb{Z}_2$; *(5)* $G(15, 2) \cdot \mathbb{Z}_2$; *(6)* $F'(1) \cdot \mathbb{Z}_2$; *(7)* $K_{5,5,5} \cdot \mathbb{Z}_2$; *(8)* $(C_5[3K_1]) \cdot \mathbb{Z}_2$; *(9)* $(K_3[5K_1]) \cdot \mathbb{Z}_2$.

6 A has a block of length $2p = 6$ and A acts on this block primitively

In this case $\Sigma = \{B_0, B_1 \cdots B_4\}$ is a complete block system of A, and $\Sigma \cong K_5$ or C_5 and K acts on B_i simply-primitively as well. Since there is no simply-primitive representation of degree 6, it suffices to consider the following.

$K = 1$

In this case $\overline{A} \cong A$ as abstract groups. So $\overline{A}$ has a transitive representation of degree 30, and then the stabilizer of one point in $\overline{G}$ has a subgroup of index 6.

If $\Sigma \cong C_5$, then $\overline{A} \cong D_{10}$ and hence the stabilizer of one point in $\overline{A}$ has no subgroup of index 6.

If $\Sigma \cong K_5$, then $\overline{A} \cong A_5$ or S_5. If $\overline{A} \cong A_5$, then the stabilizer of one point in Σ is A_4 which has subgroup Z_2 of index 6. But this contradicts the valency of K_5 being 4.

If $\Sigma \cong K_5$ and $\overline{A} \cong S_5$, the stabilizer of one point in Σ is S_4 which has a subgroup $W_4 \cong \langle (12)(34), (14)(23) \rangle$ of index 6. In this case $A_v = W_4$ and hence W_4 has orbit-length 4 in Γ which implies Γ is a 6-fold cover of K_5. Let $D = H(45)H$, where $H = W_4$. Then $A = \langle D \rangle = \langle (45), W_4 \rangle$ and it follows that Γ is connected. Since $W_4^{(45)} \cap W_4 = 1$, it follows that the valency of Γ is 4. We denote this graph by $K_5 \cdot \mathbb{Z}_6$.

Summarizing the results in this section we have

Theorem 7 *If A has a block of length 6 and K acts on it simply-primitively, then* $\Gamma \cong K_5 \cdot \mathbb{Z}_6$.

7 A has a block of length 15 and A acts on this block primitively

In this case Γ is bipartite with a bipartition $V(\Gamma) = B_1 \cup B_2$ and K acts on B_1 and B_2 primitively and faithfully as well. We distinguish the following two subcases.

Subcase 7.1. K acts on B_i doubly-transitively.

In this case $K^{B_i} \cong K$ is a doubly-transitive group of degree 15. By the classification of doubly-transitive groups, K^{B_i} has at most two nonequivalent 2-transitive representations of degree 15. How one of two cases will happen: (1) K^{B_1} and K^{B_2} are equivalent, (2) K^{B_1} and K^{B_2} are not equivalent.

(1) In this case, for any vertex $v \in B_1$, K_v fixes exactly one vertex u in B_2 and acts transitively on $B_2 - \{u\}$. If for any $w \in B_2 - \{u\}$, (v, w) is an edge of Γ, then $(v, x) \in E(\Gamma)$ for all $x \in B_2 - \{u\}$. It follows that Γ is isomorphic to $2pK_2$ or $K_{2p,2p}$ minus a 1-factor. But $2pK_2$ is not connected, so we have proved that $\Gamma \cong K_{15,15}$.

(2) If K^{B_1} and K^{B_2} are not equivalent, then $K \cong A_7$. There is a design $PG(3,2)$ with $\lambda = 1$ admitting the automorphism group A_7 on points and in this situation A_7 has two representations. This is because there are two different 2-transitive permutation representations of degree 15, and these are interchanged by the outer automorphisms of A_7. So if the induced graph by B_i has edges, then they are not isomorphic by A_7 but by A and this graph is not connected. We denote this graph by $2PG(3,2)$.

Subcase 7.2. K acts on B_i simply-primitively and faithfully.

First we define the *bipartite square* $\Gamma^{(2)}$ of a graph $\Gamma = (V, E)$. Take another graph $\Gamma' = (V', E')$ which is isomorphic to Γ with $V \cap V' = \emptyset$. Then $\Gamma^{(2)}$ is defined by

$$\begin{aligned} V(\Gamma^{(2)}) &= V \cup V', \\ E(\Gamma^{(2)}) &= \{uv', u'v | u, v \in V, u', v' \in V', uv \in E\}. \end{aligned}$$

Now we can determine the graphs in this subcase. By [3] the only simply-primitive groups of order 30 are S_6 acting on unordered pairs of a 6-element set, so the group is of degree 15 and rank 3. We label the vertices of B_1 with (12), (13), (14), (15), (16), (23), (24), (25), (26), (34), (35), (36), (45), (46), and (56); and label the vertices of B_2 with (12)′, (13)′, (14)′, (15)′, (16)′, (23)′, (24)′, (25)′, (26)′, (34)′, (35)′ (36)′, (45)′, (46)′ and (56)′. Then the stabilizer $K_{(12)}$ of (12) in K has three orbits in B_1, that is $\{(12)\}, \{(34), (35), (36), (45), (46,)(56)\}$ and $\{(13), (14), (15), (16), (23), (24), (25), (26)\}$. Since K acts on B_i faithfully, $K_{(12)}$ must fix one vertex in B_2. Without loss of generality, we may assume that this vertex is (12)′; hence the orbits of K in B_2 are quite similar to those in B_1, but adding a prime to each vertex. Since Γ is symmetric, the vertices adjacent to (12) form an orbit of $K_{(12)}$ in B_2. So Γ must be isomorphic to $3pK_2$, or the bipartite square of T_6 or T_6^c. But $3pK_2$ is not connected, so we have proved the following

Theorem 8 *If K acts on B_i simply primitively and faithfully, then Γ is isomorphic to the bipartite square of T_6 or T_6^c.*

8 A has a block of length $2p = 10$ and A acts on this block primitively

In this section we deal with the case where A has a block of length 10.

In this case $\Sigma = B_1 \cup B_2 \cup B_3$ is a complete block system of A, and K acts on B_i primitively as well. We also use Σ to denote the corresponding block graph, and then $\Sigma \cong K_3$ and $Z_3 \leq A/K \leq S_3$. It is obvious that $K \neq 1$. First we prove the following.

In the rest of this section assume K_i^B is faithful and simply-primitive. By the classification of primitive group of degree $2p$, the only simply-primitive graphs of degree 10 are A_5 and S_5. So $K \cong K_i^B = A_5$ or S_5. It acts on unordered pairs of the set $\{1,2,3,4,5\}$, that is $\{(12),(13),(14),(15),(23),(24),(25),(34),(35),(45)\}$.

1. Consider when $A/K = S_3, K = A_5$. Suppose that

$$A/K = \langle \alpha, \beta \mid \alpha^3 = \bar{\beta}^2 = 1, \beta^{-1}\alpha\beta = \alpha^{-1} \rangle$$

Let $\bar{\tau}$ be τ acting on K by conjugation, for $\tau \in A$. Let $\phi : A \to \mathrm{Aut}\, K \cong S_5$ be defined by $\phi : \tau \to \bar{\tau}, \tau \in A$. Then ϕ is a homomorphism. Since $K = A_5 \cong \mathrm{Inn}\, A_5$, $A_5 \leq \mathrm{Im}\,\phi \leq S_5$.

(a). If ϕ is not surjective, then $A/\mathrm{Ker}\,\phi \cong A_5$. It follows that $|\mathrm{Ker}\,\phi| = 6$ by comparing the order as above. However $K \cap \mathrm{Ker}\,\phi \lhd K$, so $K \cap \mathrm{Ker}\,\phi = 1$. Therefore $A = K \times \mathrm{Ker}\,\phi$ and hence $\mathrm{Ker}\,\phi \cong S_3$. So we can suppose that $\langle \alpha, \beta \mid \alpha^3 = \beta^2 = 1, \beta^{-1}\alpha\beta = \alpha^{-1} \rangle$. As $\mathrm{Ker}\,\phi \lhd A$, the orbits which $\mathrm{Ker}\,\phi$ acts on $V(\Gamma)$ are A-imprimitive blocks $C_i, (i = 1, 2, \cdots)$. Assume that $\overline{A}_{B_0} = \langle \bar{\beta} \rangle$. Then $A_{B_0} = \langle \beta, K \rangle = \langle \beta \rangle \times K$.

Since K acts on B_0 primitively, so does A_{B_0} on B_0. As $\langle \beta \rangle \lhd A_{B_0}$, $\langle \beta \rangle$ has orbits of length 1 on B_0. Thus β fixes every point of B_0 and interchanges B_1 and B_2. It shows β is not a regular element and hence $\mathrm{Ker}\,\phi$ acts on $V(\Gamma)$ unregularly. So $\mathrm{Ker}\,\phi$ has blocks of length 3. Let Δ be a complete block system of $\mathrm{Ker}\,\phi$. Thus $A/\mathrm{Ker}\,\phi \cong K$, and $\Delta \cong O_3$ or O_3^c. We have discussed this in section 4.

(b) If ϕ is surjective, then $A/\mathrm{Ker}\,\phi \cong S_5$. It follows that $\mathrm{Ker}\,\phi = Z_3$ by comparing the order as above. Also $K \cap \mathrm{Ker}\,\phi = 1$. Therefore $\langle K, \mathrm{Ker}\,\phi \rangle / K \cong Z_3$ and hence $\langle \alpha, K \rangle = \langle \mathrm{Ker}\,\phi, K \rangle$. So without loss of generality we can suppose that $\mathrm{Ker}\,\phi = \langle \alpha \rangle$. Then it is obvious that

$$A/\mathrm{Ker}\,\phi = \langle \beta, K \rangle \langle \alpha \rangle / \langle \alpha \rangle \cong \langle \beta, K \rangle / \langle \beta, K \rangle \cap \langle \alpha \rangle = \langle \beta, K \rangle.$$

Thus $\langle \beta, K \rangle \cong S_5$. Hence $A = \langle \alpha \rangle \cdot \langle \beta, K \rangle = \langle \alpha \rangle \cdot S_5$. For $\langle \beta, K \rangle$, without loss of generality, suppose that $\beta = (45)$. So $\langle \alpha, \beta \mid \alpha^3 = \beta^2 = 1; \beta^{-1}\alpha\beta = \alpha^{-1} \rangle \cong S_3$. Since $\langle \alpha \rangle \lhd A$, the orbits which $\langle \alpha \rangle$ acts on $V(\Gamma)$ are A-imprimitive blocks of length 3. Denotes its complete block system by $\Delta = \{C_i\}, (i = 1, 2, \cdots)$. Since $A/\mathrm{Ker}\,\phi \cong K$, so $\Delta \cong O_3$ or O_3^c. These have been discussed in section 4.

2. Case $A/K = S_3$ and $K = S_5$. Then $|A| = 720$. Define $\phi : A \to \mathrm{Aut}\, K \cong S_5$ by $\phi : \tau \to \bar{\tau}, \tau \in A$. Then $A/\ker\phi = S_5$ and thus $|\mathrm{Ker}\,\phi| = 12$. Since $K \cap \mathrm{Ker}\,\phi \lhd K$, it follows that $K \cap \mathrm{Ker}\,\phi = A_5$ or 1. If $K \cap \mathrm{Ker}\,\phi = A_5$, then $K \cap \mathrm{Ker}\,\phi \lhd \mathrm{Ker}\,\phi$ and this contradicts $|\ker\phi| = 12$. So we have that $K \cap \ker\phi = 1$ and hence $A = S_5 \times S_3$. Let $G = A_5 \times S_3$ so that Γ is a G-symmetric graph. It is the same as the case $K = A_5$.

Summarizing our results in this section, we have

Theorem 9 *If G has a block of length 10 and acts on it simply-primitively, then there is no new graph occuring.*

Summarizing the main result obtained in this paper, we have

Theorem 10 *The connected symmetric graphs of order 30 are (1) K_{30}; (2) the lexicographic product graphs as in Theorem 2; (3) deleted lexicographic product graphs as in Theorem 3; (4) the covering graphs as in Theorem 6; (5) $K_5 \cdot \mathbb{Z}_6$; (6) C_{30}; (7) $T_6^{(2)}$ or $T_6^{c(2)}$; (8) $C_6(5,2)$ or $C_6(5,2)'$; (9) $C_3[2K_1](5,2)'$; (10) $K_6(5,2)$.*

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