On the Periodicity of Graph Games

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Abstract

Starting with the empty graph on p vertices, two players alternately add edges until the graph has some property P. The last player to move loses the *P-avoidance game* (there is also a *P-achievement game*, where the last player to move wins). If P is a property enjoyed by the complete graph on p vertices then one player must win after a finite number of moves. Assuming both players play optimally, the winner will depend only on p. All the games solved in the literature have followed a certain pattern. Namely, for sufficiently large p, the winner has been determined by the residue of p mod 4. We say such a game has a *period* of 4 (or in some cases a proper divisor of 4). In this paper we exhibit avoidance games with arbitrarily large period. We also prove that the equivalent games of 4-cycle achievement and 3-path avoidance have period 7.

§1. Introduction

We are interested in graph games of the following form. Two players agree on a graph property P, and on whether the aim of the game will be to create a graph with property P (the P-achievement game) or to avoid creating such a graph (the P-avoidance game). The *game graph* G starts off as $\overline{K}_p$ (that is, p isolated vertices). The first player to move is designated Player A, and the other player is known as Player B. These two players take turns to add a single undistinguished edge to G, with play halting when G first achieves property P. The player who made the last move wins the achievement game, but loses if the aim was P-avoidance. The possible end positions in both games are related to extremal graph theory.

If we choose P to be a property of the complete graph K_p but not of $\overline{K}_p$, then the game must finish (after at most $\binom{p}{2}$ moves) in a win to one of the players. In game theoretic terms, this makes it a zero-sum, two-person game with complete information, a finite game tree and no draws. It is a well known theorem of von Neumann, that in such games one player has a winning strategy. Call this player W and call W's opponent L. The statement that W has a winning strategy means that

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whatever moves L makes, W can always choose moves in reply which eventually lead to W being victorious. For an explanation of von Neumann's theorem and other game theoretic notions, see for example [2, p. 69].

Of course, there is a big gap between knowing that W exists and knowing which player W is, or what a winning strategy for W is. A number of games have hitherto been studied in the literature. Harary and Robinson looked at connectedness achievement and avoidance games in [5]. In [1] Buckley and Harary solved diameter-2 avoidance, but were unable to solve diameter-2 achievement for $p > 6$. Games where the goal is a graph containing a vertex of degree Δ were studied in [4]. The related games where P is the property that every vertex has degree at least δ appeared in [3]. The avoidance game was solved but the achievement game only proved tractable for $\delta \leq 3$. The results of these papers are summarised below.

$$\text{Connectedness achievement} \begin{cases} \text{A wins if } p = 2;\ \text{ or } p \geq 4, p \equiv 0, 3 \bmod 4 \\ \text{B wins if } p = 3;\ \text{ or } p \geq 4, p \equiv 1, 2 \bmod 4 \end{cases}$$

$$\text{Connectedness avoidance} \begin{cases} \text{A wins if } p \geq 3 \text{ and } p \equiv 2, 3 \bmod 4 \\ \text{B wins if } p = 2 \text{ or } p \equiv 0, 1 \bmod 4 \end{cases}$$

$$\text{Diameter-2 avoidance} \begin{cases} \text{A wins if } p \equiv 0, 3 \bmod 4 \\ \text{B wins if } p \equiv 1, 2 \bmod 4 \end{cases}$$

$$\text{Maximum degree-3 avoidance} \begin{cases} \text{A wins if } p = 7;\ \text{ or } p \geq 6 \text{ and even} \\ \text{B wins if } p = 4, 5;\ \text{ or } p \geq 9 \text{ and odd} \end{cases}$$

$$\text{Minimum degree-}\delta \text{ avoidance (odd } \delta) \begin{cases} \text{A wins if } p \equiv 0, 3 \bmod 4 \\ \text{B wins if } p \equiv 1, 2 \bmod 4 \end{cases}$$

$$\text{Minimum degree-}\delta \text{ avoidance (even } \delta) \begin{cases} \text{A wins if } p \equiv 1, 2 \bmod 4 \\ \text{B wins if } p \equiv 0, 3 \bmod 4 \end{cases}$$

$$\text{Minimum degree=1 achievement} \begin{cases} \text{A wins if } p = 2;\ \text{ or } p \geq 7, p \equiv 0, 3 \bmod 4 \\ \text{B wins if } p = 3, 4;\ \text{ or } p \geq 5, p \equiv 1, 2 \bmod 4 \end{cases}$$

$$\text{Minimum degree=2 achievement} \begin{cases} \text{A wins if } p = 3, 6;\ \text{ or } p \geq 7, p \equiv 0, 3 \bmod 4 \\ \text{B wins if } p = 4, 5;\ \text{ or } p \geq 9, p \equiv 1, 2 \bmod 4 \end{cases}$$

$$\text{Minimum degree=3 achievement} \begin{cases} \text{A wins if } p \geq 6 \text{ and } p \equiv 1, 2 \bmod 4 \\ \text{B wins if } p = 4, 5;\ \text{ or } p \geq 7, p \equiv 0, 3 \bmod 4 \end{cases}$$

The above results display a striking pattern. In all cases the winner is determined by the residue of p modulo 4; possibly barring a few small exceptional values of p. This observation motivates the following definition.

Definition. *The period of the P-achievement game is the smallest integer τ with the following property: There exists an integer N such that for all $\rho \geq N$, Player A wins P-achievement on $p = \rho + \tau$ vertices if and only if Player A wins P-achievement* *when $p = \rho$. If no such τ exists then we say the period is infinite or that the game is aperiodic. The period of the P-avoidance game is defined analogously.*

It seems common for a graph game to have period 4, posing one obvious question. What other periods (if any) are possible? This paper will partially answer that question. To make our analyses easier we assume throughout that both players use one step look-ahead. That is, no player will consider a move which immediately loses the game (or allows the opponent to win immediately) unless there is no option available. The reader is also warned that in our descriptions of a winning strategy for W we will often attribute a definite move to W without regard to other successful choices W could have made at the same stage. This practice is followed for the sake of clarity, given that in general W will have a multitude of winning strategies available.

§2. Large component avoidance

We call our first game *large component avoidance*. The aim is to avoid creating a component containing more than n vertices. By changing our definition of "large" (i.e. varying n) we will be able to generate a family of games with different periods. The game will sometimes be called *n^+-component avoidance* to emphasize the role of n. Note that large component avoidance is a generalisation of the connectedness avoidance game studied in [5] under the name of "Don't Connect It".

In what follows we will assume that $p > n \geq 1$, otherwise the game is not properly defined since neither player can win nor lose. The winner of large component avoidance is shown by the following result.

Theorem 1. *Let $p > n \geq 1$ be given. Define m and r to be the unique integers satisfying $p = mn + r$ and $0 \leq r < n$. Then player A wins n^+-component avoidance on p vertices in the following cases (with player B winning in all other cases).*

$$(1)\ n \text{ is even and } p \equiv 2, 3 \bmod 4.$$

$$(2)\ n \equiv 1 \bmod 4 \text{ and } \begin{cases} r = 1 \text{ or} \\ r \equiv 2, 3 \bmod 4. \end{cases}$$

$$(3)\ n \equiv 3 \bmod 4 \text{ and } \begin{cases} m \text{ even, } r = 1. \\ m \text{ even, } r \equiv 2, 3 \bmod 4. \\ m \text{ odd, } r \equiv 0, 1 \bmod 4, r \neq 1. \end{cases}$$

We shall prove Theorem 1 in a moment, but first observe what it tells us about the period of n^+-component avoidance. If n is even then clearly the period is 4, so there is nothing new here. However, when $n \equiv 1 \bmod 4$ the winner depends only on the residue of p modulo n, so the period is n. (To be sure the period is not a proper divisor of n we observe that within the cycle the only three consecutive wins for A are $p \equiv 1, 2, 3 \bmod n$.) Similarly, when $n \equiv 3 \bmod 4$ the period is $2n$, with the only three consecutive wins for A being when $p \equiv 1, 2, 3 \bmod 2n$. Thus, there is a graph game of period τ for every positive integer $\tau \equiv 1, 5$ or $6 \bmod 8$.

In order to prove Theorem 1 we introduce some terminology. For each component C of G, let $|C|$ be the *size* of the component (the number of vertices contained in C). We say that C is a *fragment* if $1 < |C| < n$. The vertices in fragments, together with the isolated vertices (vertices of degree 0) form the set of *live vertices.* We use the symbols f, l and e respectively to denote the number of fragments, live vertices and isolated edges (components of size 2), at a given stage of the game. Also, when $l \geq 1$ let s denote the size of the largest component among the live vertices.

There are four types of move possible.

M_1	Joins two isolated vertices,
M_2	Joins an isolated vertex to a fragment,
M_3	Joins two fragments,
M_4	Joins two vertices within the same component.

Following the terminology of [5], M_4 will be called a *conservative move*, because it does not change the number of components or their sizes.

To establish who wins large component avoidance it is instructive to study the penultimate positions (the positions from which the next player to move cannot avoid losing immediately). Clearly, in the penultimate position every component will be a complete subgraph of G and the sum of the sizes of the two smallest components will be greater than n. To win large component avoidance, players guide play to a penultimate position which suits them. Player A wants the penultimate position to have an odd number of edges, while B wants an even number.

Proof of Theorem 1.

We first note that when $n = 1$, Theorem 1 is trivially satisfied because Player A loses on the first turn of every game. Henceforth assuming that $n > 1$, we prove Theorem 1 by providing a winning strategy for W. This strategy consists of two phases, with the basic idea being to minimise f.

Phase 1:

We show inductively that W can ensure that

(i) $f \leq 1$ when it is L's turn to move and

(ii) $f \leq 2$ and $f - e \leq 1$ when it is W's turn to move,

and that after a finite number of moves the game will reach a position for which $l \leq n + 1$, at which stage Phase 1 is completed.

Clearly, before the first move of the game $f = 0$ and the appropriate condition is satisfied. On later moves we may assume that W's strategy has worked for preceding moves. Thus, assume that $f \leq 1$ and it is L's move. Now, the only move which increases f is M_1, which increases both f and e by 1. Hence it follows that (ii) must hold after L has moved. Alternately, suppose that (ii) holds on W's turn and that $l > n + 1$. Then there must be enough isolated vertices available for W to play

$$\begin{cases} M_1 & \text{if } f = 0, \\ M_2 & \text{if } f = 1 \text{ or } s = n-1 \text{ (playing on the larger fragment if } f = 2), \\ M_3 & \text{if } f = 2 \text{ and } s < n-1. \end{cases}$$

It is an easy matter to check that after W's move $f \leq 1$ and the game graph contains no large component. Moreover, every move W makes decreases the integral quantity $l - s$ and L's moves never increase this quantity. Since $s < n$, the terminating condition for Phase 1 must eventually be satisfied. Note that if need be L can imitate W's strategy, so play will not halt before Phase 1 is completed.

Phase 2:

We observe that every move of the game either leaves l unchanged or decreases it by n. Since $l = p$ at the start of the game it follows that at the beginning of Phase 2, $l \equiv p \bmod n$ and $2 \leq l \leq n + 1$. In addition, the move which completed Phase 1 must have decreased l and hence also decreased f. It follows that at the start of Phase 2, $e = f \leq 1$. W's strategy now splits into two cases.

Case 1: $p \not\equiv 1 \bmod n$.

In this case W has done the hard work; the result of the game is determined. Since $l \leq n$ there is only one penultimate position which can now be reached, namely the disjoint union of K_r with m copies of K_n. We denote this position by $mK_n \cup K_r$.

Case 2: $p \equiv 1 \bmod n$.

This case reduces to the connectedness avoidance game studied in [5]. The two players play "Don't Connect It" on the $l = n + 1$ live vertices, the only difference being that there may be conservative moves available on the non-live vertices. Examining the strategy given in [5] shows these surplus moves do not interfere with the winning strategy, although if there are an odd number of them the identity of W will change. The only requirement for Harary and Robinson's strategy to be applicable is that if n is odd then W must have a turn before the number of components (in the live vertices) is reduced below 3. By our comments at the start of this phase, at W's first turn in Phase 2 there will be at most 2 edges played between live vertices. Hence, there will certainly be at least 3 components unless $l = n + 1 \leq 4$. Thus the only special case we need examine further is $n = 3$.

When $n = 3$ and $p \equiv 1 \bmod n$ the strategy given above must be slightly modified. In Phase 1, W plays exactly the same moves, except that in a position where $l = 7$ and $f = e = 2$, W plays M_1 rather than M_2. The only penultimate position which can then be reached is $(m-1)K_3 \cup 2K_2$. We claim this is the same penultimate position reached as when the above exception is not encountered. To see this, study the possible positions at the end of Phase 1. We know the move which completed Phase 1 must have reduced l by 3 and reduced f by 1. However, prior to this last move $f \leq 1$ because of the inductive hypothesis of Phase 1 and the fact that W did not encounter the exception. Hence at the start of Phase 2, $f = 0$. This means that there are enough components of live vertices for W to play "Don't Connect It". W's strategy will be to play conservatively until L eventually has to play M_1, at which point W also plays M_1. This ensures that the penultimate position reached when $n = 3$ and $p \equiv 1 \mod n$ is always $(m-1)K_3 \cup 2K_2$.

This completes the description of a winning strategy for W. Now it is time to establish the identity of W (i.e. which player may choose to ensure victory by using this strategy). This depends only on the parity of the number of edges in the penultimate position. The complete graph K_a has $\binom{a}{2}$ edges, which is an odd number if and only if $a \equiv 2, 3 \mod 4$. It follows from our description of Phase 2, that when $p \not\equiv 1 \mod n$ Player A wins if and only if $m\binom{n}{2} + \binom{r}{2}$ is odd. Specifically, Player A wins if $p \not\equiv 1 \mod n$ and

(1) $n \equiv 0, 1 \mod 4$ and $r \equiv 2, 3 \mod 4$,
(2) $n \equiv 2, 3 \mod 4$ and m is even and $r \equiv 2, 3 \mod 4$ or
(3) $n \equiv 2, 3 \mod 4$ and m is odd and $r \equiv 0, 1 \mod 4$.

Player B wins all other games for which $p \not\equiv 1 \mod n$.

When $p \equiv 1 \mod n$ we rely on the analysis of "Don't Connect It" in [5], as summarised in §1. Player A wins "Don't Connect It" on $n+1$ vertices if and only if $n \equiv 1, 2 \mod 4$. However, we must also account for the $(m-1)\binom{n}{2}$ moves made in the non-live components. Hence if $p \equiv 1 \mod n$ then Player A wins n^+-component avoidance precisely when

(1) $n \equiv 1 \mod 4$,
(2) $n \equiv 2 \mod 4$ and m is odd or
(3) $n \equiv 3 \mod 4$ and m is even

Compiling the results presented above completes the proof of Theorem 1. ⊙

§3. 4-cycle achievement and 3-path avoidance

It is easily seen that the winner of the 4-cycle achievement game is the winner of the 3-path avoidance game, as a 3-path can always be turned into a 4-cycle by adding an edge between the end-points of the path, and this is the only way a 4-cycle can be created. Hence the two games are equivalent, and we need only consider the 3-path avoidance game in detail.

The components of a graph which contains no 3-paths can only be 3-cycles or stars ($K_{1,n}$ for some $n \geq 0$). Let the quadruple $S = (v, e, k, s)$ be known as the *game state*, where v, e, k are the number of components of G which are isolated vertices, isolated edges and $K_{1,2}$'s respectively; and $s = 0$ or ∞ depending on whether G respectively does not or does contain a non-trivial star ($K_{1,n}$ with $n \geq 3$). We will show that the winner of the game is determined by the game state. The winning strategy we will exhibit depends heavily on the value of the function $\phi(S) = v - 4e - 2k$.

When either player adds an edge to G they alter the game state. We think of a *move* as a quadruple $M = (m_1, m_2, m_3, m_4)$ such that if $S = S_0$ before a player's turn, then it is possible that $S = S_0 + M$ afterwards (here addition of quadruples is defined component-wise). Making the assumption that no player will ever construct a 3-path if they have an option not to, the only moves we need consider are:

$M_1 = (-2, 1, 0, 0)$	corresponding to joining two isolated vertices,
$M_2 = (-1, -1, 1, 0)$	forming a $K_{1,2}$ from an isolated edge,
$M_3 = (-1, 0, -1, \infty)$	forming a $K_{1,3}$ from a $K_{1,2}$,
$M_4 = (-1, 0, 0, -1)$	enlarging a non-trivial star,
$M_5 = (0, 0, -1, 0)$	forming a 3-cycle from a $K_{1,2}$.

We note that these moves change ϕ by $-6, 1, 1, -1$ and 2 respectively. Also, a particular move can be made if and only if the resulting game state has no negative components. We call such moves *legal* moves.

We define $\mathcal{L}$ to be the set of all game states from which the opponent of the player whose turn it is has a winning strategy. Since the game must end after a finite number of moves we have an alternate (recursive) definition. A given state is in $\mathcal{L}$ if and only if every legal move from that state reaches a state (not in $\mathcal{L}$) from which there is a move to a state in $\mathcal{L}$. Specifically, a game state S_0 is in $\mathcal{L}$ if and only if there is a *tactic*, being a quintuple $[R_1, R_2, R_3, R_4, R_5]$ of moves (called *replies*) with the following property holding for $1 \leq i \leq 5$: If M_i is a legal move from S_0 then R_i is a legal move from $S_0 + M_i$ and $S_0 + M_i + R_i \in \mathcal{L}$. If we know that a particular move is not legal from S_0 we may write $\bullet$ in place of the appropriate reply. So, for example, $[R_1, R_2, \bullet, R_4, \bullet]$ is a typical tactic when $k = 0$ in S_0 and M_3, M_5 are not legal moves.

To determine the winner of 3-path avoidance we prove a series of propositions that establish which states are in $\mathcal{L}$. The tactics required to prove these propositions will be numerous, so we leave to the reader the tedious business of checking that the legality of each reply R_i is implied by the legality of M_i.

In essence we play the game 'backwards'; first working out how the game might end.

Proposition 1. *If k is even and $v = 0$ then $S \in \mathcal{L}$.*

Proof: Repeated use of the tactic $[\bullet, \bullet, \bullet, \bullet, M_5]$ eventually reduces the game state to $(0, e, 0, s)$, from which the next player to move has no option but to create a 3-path and lose the game. ⊙

Once a non-trivial star is created in G (i.e. $s = \infty$), the game changes irreversibly because of the availability of a new move (M_4). The next two propositions study this phase of play.

Proposition 2. *If $s = \infty$, $\phi \leq 0$ and $\phi \equiv 0 \bmod 4$ then $S \in \mathcal{L}$.*

Proof: We induct on v, which must be even. The base case ($v = 0$) is handled by Proposition 1. For $v \geq 2$ there are two subcases. If $k > 0$ then W plays $[M_5, M_4, M_4, M_3, M_1]$. On the other hand if $k = 0$ then $v \geq 4$ so $[M_1, M_4, \bullet, M_2, \bullet]$ is a legal tactic. In each case W has reduced v, whilst preserving $\phi \leq 0$ and $\phi \equiv 0 \bmod 4$. ⊙

Proposition 3. *Let $s = \infty$ and $\phi \geq 0$. Then $S \in \mathcal{L}$ if any of the following are true*
(i) $\phi \equiv 0 \bmod 7$,
(ii) $\phi \equiv 3 \bmod 7$ *and* $k \neq 1$,
(iii) $\phi \equiv 5 \bmod 7$ *and* $k = 0$.

Proof: Suppose that $\phi > 0$ and that one of (i),(ii) or (iii) holds. Then $v \geq \phi \geq 3$ so by playing the tactic, Σ, defined by

$$\Sigma = \begin{cases} [M_4, M_4, M_4, M_1, M_1] & \text{if } \phi \equiv 0 \bmod 7 \text{ and } k \neq 2 \\ [M_4, M_4, M_4, M_1, M_3] & \text{if } \phi \equiv 0 \bmod 7 \text{ and } k = 2 \\ [M_4, M_3, \bullet, M_1, \bullet] & \text{if } \phi \equiv 3 \bmod 7 \text{ and } k = 0 \\ [M_4, M_4, M_3, M_1, M_5] & \text{if } \phi \equiv 3 \bmod 7 \text{ and } k = 2 \\ [M_4, M_4, M_4, M_1, M_5] & \text{if } \phi \equiv 3 \bmod 7 \text{ and } k > 2 \\ [M_2, M_3, \bullet, M_4, \bullet] & \text{if } \phi \equiv 5 \bmod 7 \end{cases}$$

W preserves the fact that one of (i),(ii) or (iii) holds. Moreover, after playing Σ either $\phi > 0$ (in which case we proceed by induction) or $\phi \in \{0, -4\}$. The base cases when $\phi \in \{0, -4\}$ are handled by Proposition 2. Note that one of these base cases must be reached after a finite number of turns because we know the game is finite and it cannot end while $v \geq \phi \geq 3$ because a move of M_1 is still available. ⊙

Although we will not prove it, the reader may care to verify that Propositions 2 and 3 represent a complete analysis of the game once a non-trivial star has been introduced (in the sense that the only states in $\mathcal{L}$ are the ones already mentioned). Hence the game is fairly simple once $s = \infty$. An obvious strategy when playing the game with $s = 0$ is to aim to construct a star at an opportune stage. Unfortunately, an unwilling opponent can always thwart such a plan. In our analysis for $s = 0$ that follows, whenever either player plays M_3 the reader should verify that the game reduces to Proposition 2 or 3. As these situations are well understood, our future comments will be restricted to the positions reached in which $s = 0$.

Proposition 4. *If $s = 0$ then $S \in \mathcal{L}$ for each of the following small cases. It is a simple matter to check for (i) to (vi) that W's tactic reduces the game to a case already solved.*

(i) $v = 1$, k even	W plays $[\bullet, M_5, M_5, \bullet, M_3]$
(ii) $v = 3$, k odd	W plays $[M_3, M_1, M_1, \bullet, M_1]$
(iii) $v = 4$, k odd, $e \geq 1$	W plays $[M_2, M_5, M_2, \bullet, M_2]$
(iv) $v = 4$, $e = k = 0$	W plays $[M_1, \bullet, \bullet, \bullet, \bullet]$
(v) $v = 5$, $e \leq 1$, $k = 0$	W plays $[M_1, M_5, \bullet, \bullet, \bullet]$
(vi) $v = 6$, k even, ($e \geq 1$ or $k = 0$)	W plays $[M_2, M_1, M_3, \bullet, M_1]$

Proposition 5. *If $s = 0$, $v \geq 4$, $e \geq 1$, $\phi \leq 0$ and $\phi \equiv 2 \bmod 4$ then $S \in \mathcal{L}$.*

Proof: We induct on v, which we note must be even. The base cases are handled by Proposition 4 (iii) and (vi). If $v \geq 8$ then W plays $[M_1, M_3, M_2, \bullet, M_1]$, thereby reducing v by either 2 or 4 whilst preserving $\phi \equiv 2 \bmod 4$, increasing e and decreasing ϕ. ⊙

Proposition 6. *If $s = k = 0$, $v \geq 7$, $e \geq 1$, $\phi \leq 3$ and $\phi \equiv 3 \bmod 4$ then $S \in \mathcal{L}$.*

Proof: W plays $[M_2, M_1, \bullet, \bullet, \bullet]$ to reach the situation handled by Proposition 5. ⊙

Proposition 7. *If $s = 0$ then $S \in \mathcal{L}$ in the following cases. In (i) to (iv) W's tactic reduces the game to Proposition 5, Proposition 6 or an earlier part of this proposition.*

(i) $\phi = 3$, $k \leq 1$, $e \geq 1$	W plays $[M_2, M_1, M_4, \bullet, M_1]$
(ii) $\phi = 8$, $k = 0$, $e \geq 1$	W plays $[M_2, M_1, \bullet, \bullet, \bullet]$
(iii) $\phi = 10$, $k = 0$	W plays $[M_1, M_3, \bullet, \bullet, \bullet]$
(iv) $\phi = 7$, $k = 0$, $e \geq 1$	W plays $[M_1, M_5, \bullet, \bullet, \bullet]$

Proposition 8. *Let $s = 0$ and $\phi \geq 12$. Then $S \in \mathcal{L}$ if either $k = 0$ and $\phi \equiv 1, 3 \bmod 7$, or $k = 1$ and $\phi \equiv 5 \bmod 7$.*

Proof: We induct on ϕ with W playing the tactic T, defined by

$$T = \begin{cases} [M_1, M_3, \bullet, \bullet, \bullet] & \text{if } \phi \equiv 1 \bmod 7 \\ [M_2, M_3, \bullet, \bullet, \bullet] & \text{if } \phi \equiv 3 \bmod 7 \\ [M_3, M_3, M_4, \bullet, M_1] & \text{if } \phi \equiv 5 \bmod 7 \end{cases}$$

Note that playing T strictly decreases ϕ and either preserves the inductive hypothesis or arrives at one of the base cases treated in Proposition 7. ⊙

Proposition 9. *Player A wins the game whenever $p \equiv 0, 2 \bmod 7$ or $p \in \{8, 13\}$. Player B wins all other games except possibly when $p \equiv 4 \bmod 7$.*

Proof: We analyse the game according to p's congruence class modulo 7. In each game where Player A is victorious, the play of the first edge reaches a state in $\mathcal{L}$.

$p \equiv 0 \bmod 7$: Player A wins each of the subcases $p = 7$, $p = 14$ and $p \geq 21$ as shown by Proposition 4(v), Proposition 7(ii) and Proposition 8 respectively.

$p \equiv 1 \bmod 7$: When $p = 8$, Player A wins by Proposition 4(vi). Otherwise $p \geq 15$ and Proposition 8 implies that Player A loses.

$p \equiv 2 \bmod 7$: Player A wins each of the subcases $p = 9$, $p = 16$ and $p \geq 23$ as shown by Proposition 7(i), Proposition 7(iii) and Proposition 8 respectively.

$p \equiv 3 \bmod 7$: Proposition 7(iii) and Proposition 8 imply that A loses.

$p \equiv 5 \bmod 7$: By Proposition 4(v), Player B wins when $p = 5$. For $p \geq 12$, B wins by playing $[M_2, \bullet, \bullet, \bullet, \bullet]$ followed by $[M_5, \bullet, M_4, \bullet, M_1]$. Since the vertices in a 3-cycle have effectively been eliminated, the game is now reduced to the $p \equiv 2 \bmod 7$ case, with the first edge already played. As Player A always wins when $p \equiv 2 \bmod 7$, Player B must win this game.

$p \equiv 6 \bmod 7$: By Proposition 4(vi), B wins when $p = 6$ and by Proposition 7(iv), A wins when $p = 13$. Otherwise $p \geq 20$ so after B plays $[M_1, \bullet, \bullet, \bullet, \bullet]$, Proposition 7(ii) and Proposition 8 tell us that player A must lose. ⊙

Proposition 10. *Player A wins when $p = 25$; Player B wins for all other $p \equiv 4 \bmod 7$.*

Proof: Player A wins when $p = 25$, by playing $[M_1, M_3, \bullet, \bullet, \bullet]$ from (23,1,0,0); which works by Proposition 7(iv). If $p \equiv 4 \bmod 7$ and $p \neq 25$ then Player B starts by playing $[M_1, \bullet, \bullet, \bullet, \bullet]$. If $p = 4$ or $p = 11$ then B now wins by Proposition 1 and Proposition 6 respectively. Otherwise the game state is $(p-4, 2, 0, 0)$ and play proceeds as follows.

(a) B plays $[M_1, M_2, \bullet, \bullet, \bullet]$ which reaches $(p-6, 0, 2, 0)$ or wins by Proposition 5 ($p = 18$), Proposition 7(ii) ($p = 32$), or Proposition 8 ($p \geq 39$).

(b) From $(p-6, 0, 2, 0)$ B plays $[M_2, \bullet, M_3, \bullet, M_5]$ and reaches $(p-6, 0, 0, 0)$ or $(p-9, 0, 3, 0)$. The former is equivalent to a game starting with $p \equiv 5 \bmod 7$, which Player B wins by Proposition 9.

(c) From $(p-9, 0, 3, 0)$ B plays $[M_2, \bullet, M_4, \bullet, M_5]$ to reach $(p-12, 0, 4, 0)$ or $(p-9, 0, 1, 0)$. The latter is equivalent to the position two moves into a game which started with $p \equiv 5 \bmod 7$. Again, by Proposition 9 we know Player B wins.

(d) From $(p-12, 0, 4, 0)$ for $p = 18$, B plays $[M_5, \bullet, M_3, \bullet, M_1]$ and wins by Proposition 5. For $p \geq 32$, B plays $[M_3, \bullet, M_1, \bullet, M_1]$ to reach $(p-14, 1, 3, 0)$.

(e) From $(p-14, 1, 3, 0)$ B plays $[M_3, M_3, M_1, \bullet, M_5]$ to win by Proposition 8. ⊙

This completes the analysis of 3-path avoidance. Combining Propositions 9 and 10, we have the following result.

Theorem 3. *The 3-path avoidance game has period 7. Player A wins on the exceptional values $p \in \{8, 13, 25\}$ and when $p \equiv 0, 2 \bmod 7$. Player B wins for all other values of p.*

§4. Concluding remarks

A number of graph games have so far eluded analysis. For example, the c-cycle avoidance games remain unsolved for all meaningful c. The 'easiest' case; that of 3-cycle avoidance, is known as Hajnal's triangle-free game and has only been solved for small values of p or under restricted conditions [6]. What has been demonstrated by this work is that such unsolved games need to be approached with broader possibilities in mind than previous evidence suggested. The preponderance of period 4 games in the literature arises because such games are often just difficult enough to present a challenge yet yield a solution. It seems unlikely that they represent an overwhelming trend within the class of graph games.

It would be interesting to discover a "natural" graph game which is aperiodic. A pathological example is not hard to create. Consider, say, a game where the goal is to achieve a graph with a prime number of components.

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