On 2-$(v,3)$ trades of minimum volume*

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Abstract

In this paper, Steiner and non-Steiner 2-$(v,3)$ trades of minimum volume are considered. It is shown that these trades are composed of a union of some Pasch configurations and possibly some 2-$(v',3)$ trades with $6 \leq v' \leq 10$. We determine the number of non-isomorphic Steiner 2-$(v,3)$ trades of minimum volume. As for non-Steiner trades the same thing is done for all v, except for $v \equiv 5 \pmod 6$.

1. Introduction

For given v, k, and t, let $X = \{1, 2, \ldots, v\}$ and let $P_k(X)$ denote the set of all k-subsets of X. The elements of X and $P_k(X)$ are called points and blocks, respectively.

A t-(v,k) *trade* $T = \{T^+, T^-\}$, consists of two disjoint collections of blocks T^+ and T^- such that for every $A \in P_t(X)$, the number of blocks containing A is the same in both T^+ and T^-.

The *foundation* of a trade is the set of elements covered by T^+ and T^- and is denoted by $found(T)$. In a t-(v,k) trade, we take v to be the foundation size. The number of blocks in $T^+(T^-)$ is called the volume of the trade T and is denoted by $vol(T)$.

A t-(v,k) trade T is called *Steiner*, if each element $A \in P_t(X)$ occurs at most once in $T^+(T^-)$. T is called *simple*, if there are no repeated blocks in $T^+(T^-)$. Here, we are concerned only with simple 2-$(v,3)$ trades.

A trade T is called *fundamental*, if it contains no proper trade.

Two trades $T_1 = \{T_1^+, T_1^-\}$ and $T_2 = \{T_2^+, T_2^-\}$ are called *isomorphic*, if there exists a bijection $\sigma : \text{found}(T_1) \rightarrow \text{found}(T_2)$ such that $\sigma(T_1) = \{\sigma(T_1^+), \sigma(T_1^-)\} = \{T_2^+, T_2^-\} = T_2$.

Bryant [1] has determined the spectrum (the set of allowable volumes) of Steiner 2-$(v,3)$ trades. In Table 1, the minimum volume of such trades is given. In this paper, we determine the number of non-isomorphic Steiner 2-$(v,3)$ trades of minimum

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volume. When a 2-$(v,3)$ trade is not Steiner, we determine the possible minimum volume for all v, except for $v \equiv 5 \pmod 6$, and obtain the number of non-isomorphic non-Steiner 2-$(v,3)$ trades of minimum volume.

Table 1.
Minimum volume of Steiner 2-$(v,3)$ trades.

$v \pmod 6$	minimum volume
0	$\frac{2v}{3}$
1	$\frac{2v+4}{3}$
2	$\frac{2v+2}{3}$
3	$\frac{2v+3}{3}$
4	$\frac{2v+4}{3}$
5	$\frac{2v+2}{3}$

2. Preliminaries

We denote a trade T with foundation size f and volume s by $T = T(s,f)$. The number of occurrences of a point x in $T^+(T^-)$ is denoted by r_x. If $r_x = 2$, we call x a *regular point*, otherwise x is said to be an *irregular point*. Symbols $1, 2, \ldots$, and $A, B, \ldots$, are used for regular and irregular points respectively, and $x, y, \ldots$, to refer to either kind. The number of occurrences of the pair xy is denoted by λ_{xy}. If $\lambda_{xy} \le 1$, we call the pair Steiner, otherwise non-Steiner. Clearly for a non-Steiner pair xy, we have $r_x \ge 3$ and $r_y \ge 3$. The points x and y are said to be adjacent if $\lambda_{xy} \ne 0$, and the collection of all (not necessarily distinct) adjacencies of x is denoted by $N(x)$. The *elements occurrence sequence* of T (abbreviated to EOS(T)) is the non-decreasing sequence $a_1, a_2, \ldots, a_f$ where a_i denotes the number of occurrences of the ith element of the foundation of T.

A trade T may consist of two trades such as T_1 and T_2, then we use the notation $T = T_1 + T_2$. When the foundations of T_1 and T_2 are disjoint $T = T_1 \oplus T_2$ is used instead.

The unique trade $T(4,6)$ is commonly called a Pasch configuration or briefly a Pasch.

The special case of the following Lemma for $r = 2$ and its corollary has been proved in [1].

Lemma 2.1. Let T be a fundamental trade. Let $x \in \text{found}(T)$ with at most one irregular adjacency and let $r_x = r$. Then, $T = T(2r, 2r+2)$ and has a unique structure with EOS$(T) = 2, \ldots, 2, r, r$.

Proof. Let $N(x) = \{y_1, \dots, y_{2r}\}$. With no loss of generality, we can assume that the blocks of T containing x are:

T^+	T^-
xy_1y_2	xy_1y_3
xy_3y_4	xy_2y_5
xy_5y_6	xy_4y_7
$\vdots$	$\vdots$
$xy_{2r-1}y_{2r}$	$xy_{2r-2}y_{2r}$

T must also contain the following blocks:

y_1y_3-	y_1y_2-
y_2y_5-	y_3y_4-
y_4y_7-	y_5y_6-
$\vdots$	$\vdots$
$y_{2r-2}y_{2r}$-	$y_{2r-1}y_{2r}$-.

Clearly, the only possible way to fill the blanks is to use a fixed new point. □

Corollary 2.1. If $\mathrm{EOS}(T) = 2, \dots, 2, r$, then r is even and T is the union of disjoint Pasches, except possibly for some Pasches, which contain the irregular point.

3. Steiner trades of minimum volume

In this section, we are concerned with Steiner trades of minimum volume. Bryant [1] has determined the minimum volume of these trades (Table 1). Here, we investigate, up to isomorphism, the structure as well as the number of such trades. If $f \equiv 0 \pmod 6$, then by Table 1, the minimum volume is $2f/3$ and by Corollary 2.1, the trade has a unique structure. Thus we have:

Lemma 3.1 [1]. If $f \equiv 0 \pmod 6$, then the Steiner trade of minimum volume is the union of disjoint Pasches.

For the sake of simplicity in the statements of lemmas, we make the following note, which is also used in Section 4.

Note. Let T be a trade. By T_c, we denote the part of T which is the union of disjoint Pasches consisting only of regular points. By $P(A, B)$, we mean a Pasch containing irregular points A, B in which $\lambda_{AB} = 0$ and by $P(AB)$, we mean $\lambda_{AB} = 1$. The union of Pasches of T containing some irregular points is denoted by T_p. Clearly, if $A \in \mathrm{found}(T)$ and $r_A = 3$, then $A \notin \mathrm{found}(T_p)$.

Apart from T_c and T_p, we have in T a Pasch-free trade T'. Let $x \in \mathrm{found}(T')$ be an irregular point with $r_x = r$ such that $N(x)$ contains at most one irregular point. By Lemma 2.1, x appears in a $T(2r, 2r+2)$. We denote the union of such parts of T' by T_m. By $T_m(A)$, we mean $A \in \mathrm{found}(T_m)$. Therefore, we have $T = T_c \oplus (T_p + T_m + T_r)$ where $T_r = T(s_r, f_r)$ is a Pasch-free trade in which for each $x \in \mathrm{found}(T_r)$, $N(x)$

contains at least two irregular points. Hence, $f_r \leq R$ in which R is the sum of occurrences of irregular points of found(T_r).

In the following lemmas, we use the above notations and determine T_p, T_m and T_r to give a complete description of the structure of the trade T. T_c obviously has a unique structure and we omit it from our statements. $A, B, \ldots$ will denote irregular points in non-increasing occurrence order. The small trades which appear in the following lemmas are listed in Table 2 in the Appendix.

Lemma 3.2. Let $T = T\left(\frac{2f+2}{3}, f\right)$ be a Steiner trade. Then

(i) If $\mathrm{EOS}(T) = 2, \ldots, 2, 4$, then $f \equiv 5 \pmod 6$, $T_p = P(A) + P(A)$ and $T_m = T_r = \varnothing$.

(ii) If $\mathrm{EOS}(T) = 2, \ldots, 2, 3, 3$, then $f \equiv 2 \pmod 6$, $T_p = T_r = \varnothing$ and $T_m = T(6, 8)$.

Proof.

(i) This is just Corollary 2.1.

(ii) Clearly $T_p = \varnothing$. There are just two irregular points, hence $T_r = \varnothing$. By Lemma 2.1, $T_m = T(6, 8)$. □

Lemma 3.3. Let $T = T\left(\frac{2f+3}{3}, f\right)$ be a Steiner trade. Then

(i) $\mathrm{EOS}(T) \neq 2, \ldots, 2, 5;\ 2, \ldots, 2, 3, 4$.

(ii) If $\mathrm{EOS}(T) = 2, \ldots, 2, 3, 3, 3$, then $f \equiv 3 \pmod 6$, $T_p = T_m = \varnothing$ and $T_r = T(7, 9)$.

Proof. .

(i) By Lemma 2.1 and Corollary 2.1, both cases are impossible.

(ii) If $T_m \neq \varnothing$, then by Lemma 2.1, $T_m = T(6, 8)$ and $\mathrm{EOS}(T_r) = 2, \ldots, 2, 3$ which is by Corollary 2.1 impossible. So $T_m = \varnothing$. We have $f_r \leq 9$, hence $T_r = T(7, 9)$ which is unique by Table 2. □

Lemma 3.4. Let $T = T\left(\frac{2f+4}{3}, f\right)$ be a Steiner trade. Then

(i) If $\mathrm{EOS}(T) = 2, \ldots, 2, 6$, then $f \equiv 4 \pmod 6$, $T_p = P(A) + P(A) + P(A)$ and $T_m = T_r = \varnothing$.

(ii) If $\mathrm{EOS}(T) = 2, \ldots, 2, 3, 5$, then $f \equiv 1 \pmod 6$, $T_P = P(A)$, $T_m(A) = T(6, 8)$ and $T_r = \varnothing$.

(iii) If $\mathrm{EOS}(T) = 2, \dots, 2, 4, 4$, then $f \equiv 4 \pmod 6$, $T_r = \varnothing$ and for T_p and T_m one of the following occurs:

(a) $T_p = \varnothing$ and $T_m = T_4(8, 10)$;

(b) $T_p = (P(A) + P(A)) \oplus (P(B) + P(B))$ and $T_m = \varnothing$;

(c) $T_p = (P(A) \oplus P(B)) + P(AB)$ and $T_m = \varnothing$;

(d) $T_p = (P(A) \oplus P(B)) + P(A, B)$ and $T_m = \varnothing$;

(e) $T_p = P(A, B) + P(A, B)$ and $T_m = \varnothing$;

(f) $T_p = P(A, B) + P(AB)$ and $T_m = \varnothing$.

(iv) If $\mathrm{EOS}(T) = 2, \dots, 2, 3, 3, 4$, then $f \equiv 1 \pmod 6$, $T_r = \varnothing$ and for T_p and T_m one of the following occurs:

(a) $T_p = P(A) + P(A)$ and $T_m = T(6, 8)$;

(b) $T_p = P(A)$ and $T_m(A) = T(6, 8)$.

(v) If $\mathrm{EOS}(T) = 2, \dots, 2, 3, 3, 3, 3$, then $T_p = \varnothing$ and for T_m and T_r one of the following occurs:

(a) $f \equiv 1 \pmod 6$, $T_m = \varnothing$ and $T_r = T_1(6, 7)$;

(b) $f \equiv 4 \pmod 6$, $T_m = \varnothing$ and $T_r = T_1(8, 10)$;

(c) $f \equiv 4 \pmod 6$, $T_m = T(6, 8) \oplus T(6, 8)$ and $T_r = \varnothing$.

Proof.

(i) This is just Corollary 2.1.

(ii) There are just two irregular points, so $T_r = \varnothing$. By Lemma 2.1, $T_p = P(A)$ and $T_m(A) = T(6, 8)$.

(iii) There are just two irregular points, so $T_r = \varnothing$. If $T_p = \varnothing$, then by Lemma 2.1, $T_m = T_4(8, 10)$. Otherwise $T_m = \varnothing$ and T_p can be any of the forms (a)-(e).

(iv) We have $f_r \leq 10$. But $T_r = T(6, 7)$ and $T_r = T(8, 10)$ with $\mathrm{EOS}(T_r) = 2, \dots, 2, 3, 3, 4$ do not exist (cf. Table 2). So $T_r = \varnothing$. If $T_p = \varnothing$, then $\mathrm{EOS}(T_m) = 2, \dots, 2, 3, 3, 4$ which forces a Pasch containing an irregular point, a contradiction.

If $T_p = P(A)$, then $\mathrm{EOS}(T_m) = 2, \dots, 2, 3, 3$ and by Lemma 2.1, $T_m(A) = T(6, 8)$. If $T_p = P(A) + P(A)$, then $\mathrm{EOS}(T_m) = 2, \dots, 2, 3, 3$ and by Lemma 2.1, $T_m = T(6, 8)$.

(v) Clearly $T_p = \varnothing$ and $f_r \leq 12$. If $T_r = \varnothing$, then by Lemma 2.1, $T_m = T(6, 8) \oplus T(6, 8)$, otherwise we have $T_r = T_1(6, 7)$ or $T_r = T_1(8, 10)$, with $\mathrm{EOS}(T_r) = 2, \dots, 2, 3, 3, 3, 3$. Hence, $T_m = \varnothing$. □

4. Non-Steiner trades of minimum volume

In this section, we consider non-Steiner trades of minimum volume. In order to establish the main results, we need the following lemmas.

Lemma 4.1. Let T be a non-Steiner trade. If there is a point $A \in \text{found}(T)$ such that $r_A = 3$, then T must have at least three irregular points.
Proof. Suppose T has only two irregular points A and B. Clearly the pair AB is non-Steiner. Thus T contains the following blocks:

T^+	T^-
$AB1$	$AB3$
$AB2$	$AB4$
$A34$	$A12$
$B3$-	$B1$-
$B4$-	$B2$-
12-	34-

The only possible way to complete the blocks containing 3 and 4 is to use a fixed point z which is irregular, a contradiction. □

Lemma 4.2. Let $T = T(s, f)$ contain a non-Steiner pair AB with $r_A = r_B = 3$. Then $s \geq \frac{2f+6}{3}$. If equality holds, then $T = T_c \oplus T(6,6)$.
Proof. With no loss of generality, we can assume that T contains the following blocks:

T^+	T^-
ABx	ABz
ABy	ABt
Azt	Axy
Bzt	Bxy

Now, the pairs xy and zt are non-Steiner, hence $r_x, r_y, r_z, r_t \geq 3$. Therefore $s \geq \frac{2f+6}{3}$. If $s = \frac{2f+6}{3}$, then $xyz, xyt \in T^+$ and $ztx, zty \in T^-$ and we have a $T(6,6)$. □

For a non-Steiner trade $T = T(s, f)$, by Lemmas 4.1 and 4.2, we have $\text{EOS}(T) \neq 2, \ldots, 2, 3, 4;\ 2, \ldots, 2, 3, 3, 3$. Hence $s \geq \frac{2f+4}{3}$.

We make a modification to the note in Section 3.
Note (continued). When T_r is non-Steiner, we improve the upper bound for f_r to $R-2$ or $R-3$. We omit some details. If $f_r = R$, then each point has exactly two irregular adjacencies. Let n_x be the number of (not necessarily distinct) irregular adjacencies of x. To prove $f_r \leq R-3$, we show that there exists a set of points like Q such that $\sum_{x \in Q}(n_x - 2) \geq 6$.

Let AB be a non-Steiner pair in T_r. If $ABC \in T_r^+$, then each of A and B has three irregular adjacencies. Now, let $AB1 \in T_r^+$. We can assume, with no loss of generality, that T_r contains the following blocks:

T_r^+	T_r^-
$1AB$	$1Ax$
$1xy$	$1By$
Ax-	AB-
By-	xy-

If x is an irregular point (so we can let $x = D$), then 1 and A have at least three irregular adjacencies each and $AD1 \in T_r^-$. When both x and y are regular points (so let $x = 2$ and $y = 3$), then blanks in the blocks of T_r^+ are necessarily filled with an irregular fixed point. Thus A and B have at least three irregular adjacencies each, and we have $AE2, AE3 \in T_r^+$. Therefore, we have four "groups" of blocks each of which can be of three types: ABC (type 1), AD1 (type 2) and AE2, AE3 (type 3).

Since at least two groups are in T_r^+ or T_r^-, we have $f_r \leq R - 2$. Furthermore, in the following cases $f_r \leq R - 3$:

- At least three groups appear in $T_r^+ (T_r^-)$.
- Two groups appear in each of T_r^+ and T_r^- and one of these four groups is of type 2.
- There are three groups of type 1.

If two groups appear in each of T_r^+ and T_r^- (all of types 1 or 3), we have:

- If all groups are of type 3, then T_r contains $T(7, 9) + T(7, 9)$.
- If T_r^+ contains one group of type 1 and T_r^- contains one group of type 3, then T_r will contain a Pasch, a contradiction.

So we have the following lemma:

Lemma 4.3. If T_r is non-Steiner, then $f_r \leq R - 2$. If equality holds, T_r contains $T(7, 9) + T(7, 9)$.

In what follows, $A, B, \ldots$ will denote irregular points in non-increasing occurrence order. Table 3 (in the Appendix) consists of the small trades which appear in the following lemmas.

Lemma 4.4. Let $T = T\left(\frac{2f+4}{3}, f\right)$ be a non-Steiner trade. Then

(i) $\mathrm{EOS}(T) \neq 2, \ldots, 2, 3, 5;\ \ 2, \ldots, 2, 3, 3, 3, 3$.

(ii) If $\mathrm{EOS}(T) = 2, \ldots, 2, 4, 4$, then $f \equiv 4 \pmod 6$, $T_p = P(AB) + P(AB)$ and $T_m = T_r = \varnothing$.

(iii) If $\mathrm{EOS}(T) = 2, \dots, 2, 3, 3, 4$, then $f \equiv 1 \pmod 6$, $T_p = T_m = \varnothing$ and $T_r = T_2(6, 7)$.

Proof.

(i) This is clear by Lemmas 4.1 and 4.2.

(ii) There are only two irregular points and $\lambda_{AB} \geq 2$, so $T_m = \varnothing$. We have $f_r \leq 5$, so $T_r = \varnothing$ and consequently we have $T_p = P(AB) + P(AB)$.

(iii) By Lemma 4.1, $T_p = \varnothing$. If $T_m \neq \varnothing$, then $T_m = T(6, 8)$ and $\mathrm{EOS}(T_r) = 2, \dots, 2, 4$, so $T_m + T_r$ is a Steiner trade. Hence $T_m = \varnothing$. Since $f_r \leq 7$, we have $T_r = T_2(6, 7)$ (Table 3). □

Lemma 4.5. Let $T = T\left(\frac{2f+5}{3}, f\right)$ be a non-Steiner trade. Then

(i) $\mathrm{EOS}(T) \neq 2, \dots, 2, 3, 6$;

(ii) $\mathrm{EOS}(T) \neq 2, \dots, 2, 4, 5$;

(iii) $\mathrm{EOS}(T) \neq 2, \dots, 2, 3, 3, 5$;

(iv) $\mathrm{EOS}(T) \neq 2, \dots, 2, 3, 4, 4$;

(v) $\mathrm{EOS}(T) \neq 2, \dots, 2, 3, 3, 3, 3, 3$;

(vi) If $\mathrm{EOS}(T) = 2, \dots, 2, 3, 3, 3, 4$, then $f \equiv 2 \pmod 6$, $T_p = T_m = \varnothing$ and $T_r = T(7, 8)$.

Proof.

(i) It follows from Lemma 4.1.

(ii) By Corollary 2.1 and Lemma 4.1, $T_p = \varnothing$. As $\lambda_{AB} \geq 2$ and there are only two irregular points, we have $T_m = \varnothing$. By $f_r \leq 6$, it follows that $T_r = \varnothing$.

(iii) By similar arguments as in (ii), we have $T_p = \varnothing$. If $T_m \neq \varnothing$, then $T_m = T(6, 8)$ and $\mathrm{EOS}(T_r) = 2, \dots, 2, 5$ which is impossible. Since $f_r \leq 8$, we have $T_r = T(7, 8)$ with $\mathrm{EOS}(T_r) = 2, \dots, 2, 3, 3, 5$ and such a trade does not exist(cf. Table 3) .

(iv) By similar arguments as in (iii), $T_p = T_m = \varnothing$ and $T_r = T(7, 8)$ with $\mathrm{EOS}(T_r) = 2, \dots, 2, 3, 4, 4$, which does not exist (cf. Table 3).

(v) By Lemma 4.2, this is obvious.

(vi) By Lemma 4.2, $T_p = \varnothing$. If $T_m \neq \varnothing$, then $T_m = T(6, 8)$ and $\mathrm{EOS}(T_r) = 2, \dots, 2, 3, 4$, which is by Lemma 4.1 impossible. So $T_m = \varnothing$ and since $f_r \leq 10$, we have $T_r = T(7, 8)$. □

Lemma 4.6. Let $T = T\left(\frac{2f+6}{3}, f\right)$ be a non-Steiner trade. Then

(i) $\mathrm{EOS}(T) \neq 2, \ldots, 2, 3, 7;\ 2, \ldots, 2, 5, 5$.

(ii) If $\mathrm{EOS}(T) = 2, \ldots, 2, 4, 6$, then $f \equiv 3 \pmod 6$, $T_p = P(AB) + P(AB) + P(A)$ and $T_m = T_r = \varnothing$.

(iii) If $\mathrm{EOS}(T) = 2, \ldots, 2, 3, 3, 6$, then $f \equiv 0 \pmod 6$, $T_p = P(A)$, $T_m = \varnothing$ and $T_r(A) = T_2(6, 7)$.

(iv) If $\mathrm{EOS}(T) = 2, \ldots, 2, 3, 4, 5$, then $f \equiv 0 \pmod 6$, $T_p = P(A)$, $T_m = \varnothing$ and $T_r(A) = T_2(6, 7)$.

(v) If $\mathrm{EOS}(T) = 2, \ldots, 2, 4, 4, 4$, then $f \equiv 3 \pmod 6$, $T_m = \varnothing$. We have either $T_p = \varnothing$ and $T_r = T_2(8, 9)$, or $T_r = \varnothing$ and for T_p, we have one of the following:

   (a) $(P(A) + P(A)) \oplus (P(BC) + P(BC))$;

   (b) $P(A) + P(BC) + P(ABC)$;

   (c) $P(A) + P(BC) + P(AB, BC)$;

   (d) $P(ABC) + P(AB, AC)$;

   (e) $P(AB, AC) + P(AB, AC)$;

   (f) $P(AB, AC) + P(AB, BC)$.

(vi) If $\mathrm{EOS}(T) = 2, \ldots, 2, 3, 3, 3, 5$, then $f \equiv 3 \pmod 6$, $T_p = T_m = \varnothing$ and $T_r = T_6(8, 9)$.

(vii) If $\mathrm{EOS}(T) = 2, \ldots, 2, 3, 3, 4, 4$, then $f \equiv 0 \pmod 6$ and for T_p, T_m and T_r one of the following occurs:

   (a) $T_p = P(A) + P(A), T_m = \varnothing$ and $T_r = T_2(6, 7)$;

   (b) $T_p = P(A), T_m = \varnothing$ and $T_r(A) = T_2(6, 7)$;

   (c) $T_p = P(AB) + P(AB), T_m = T(6, 8)$ and $T_r = \varnothing$;

   (d) $T_p = P(AB), T_m(AB) = T(6, 8)$ and $T_r = \varnothing$.

(viii) If $\mathrm{EOS}(T) = 2, \ldots, 2, 3, 3, 3, 3, 4$, then $f \equiv 3 \pmod 6$, $T_p = \varnothing$ and for T_m and T_r one of the following occurs:

   (a) $T_m = T(6, 8)$ and $T_r = T_2(6, 7)$;

   (b) $T_m = \varnothing$ and $T_r = T_3(8, 9)$.

(ix) If $\mathrm{EOS}(T) = 2, \ldots, 2, 3, 3, 3, 3, 3, 3$, then $f \equiv 0 \pmod 6$, $T_p = T_m = \varnothing$ and $T_r = T(6, 6)$.

Proof.

(i) By Lemma 4.1, $\mathrm{EOS}(T) \neq 2, \ldots, 2, 3, 7$.

Let $\mathrm{EOS}(T) = 2, \ldots, 2, 5, 5$. By Lemmas 4.1 and 4.2, $T_p = T_m = \varnothing$.

Since $f_r \leq 7$, we must have $T_r = T(6,6)$. Since this trade has a different EOS(cf. Table 3), $T_r = \varnothing$.

(ii) Clearly $T_m = \varnothing$. Since $f_r \leq 7$, we must have $T_r = T(6,6)$, which has a different EOS(cf. Table 3), hence $T_r = \varnothing$. As $\lambda_{AB} \geq 2$, we must have $T_p = P(AB) + P(AB) + P(A)$.

(iii) Clearly $T_m = \varnothing$. If $T_p = P(A)$, then $\mathrm{EOS}(T - T_p) = 2, \ldots, 2, 3, 3, 4$ and by Lemma 4.4(iii) $T_m = \varnothing$ and $T_r = T_2(6,7)$. So let $T_p = \varnothing$. Since $f_r \leq 9$, we have $T_r = T(6,6)$ or $T_r = T(8,9)$. Neither of these two trades with the specified EOS exists(cf. Table 3).

(iv) By similar arguments as in (iii), the result follows.

(v) Clearly $T_m = \varnothing$. Since $f_r \leq 9$, we have $T_r = T(6,6)$, which has a different EOS (cf. Table 3), $T_r = T_3(8,9)$(cf. Table 3) or $T_r = \varnothing$. If $T_r = \varnothing$, then for T_p, one of the cases (a)-(f) occurs.

(vi) By Lemmas 4.1 and 4.2 $T_m = T_p = \varnothing$. As $f_r \leq 11$, $T_r = T(6,6)$, which has a different EOS, or $T_r = T_7(8,9)$ (cf. Table 3).

(vii) Suppose $T_p \neq \varnothing$. If $T_p = P(A)$ or $T_p = P(A) + P(A)$, then $\mathrm{EOS}(T - T_p) = 2, \ldots, 2, 3, 3, 4$. By Lemma 4.4(iii), $T_m = \varnothing$ and for T_r we have $T_r(A) = T_2(6,7)$ or $T_r = T_2(6,7)$. If $T_p = P(AB)$ or $T_p = P(AB) + P(AB)$, then $\mathrm{EOS}(T - T_p) = 2, \ldots, 2, 3, 3$. By Lemma 2.1, we have $T_r = \varnothing$ and $T_m(AB) = T(6,8)$ or $T_m = T(6,8)$. Now let $T_p = \varnothing$. If $T_m \neq \varnothing$, it must neccessarily be $T(6,8)$ and hence $\mathrm{EOS}(T_r) = 2, \ldots, 2, 4, 4$. But then, Lemma 4.4(ii) forces $T_p \neq \varnothing$. So $T_m = \varnothing$. Since $f_r \leq 11$, we have $T_r = T(6,6)$, or $T_r = T(8,9)$. None of these has the specified EOS (cf. Table 3).

(viii) By Lemma 4.2, $T_p = \varnothing$. If $T_m \neq \varnothing$, then $T_m = T(6,8)$ and $\mathrm{EOS}(T_r) = 2, \ldots, 2, 3, 3, 4$. Therefore, by Lemma 4.4(iii), $T_r = T_2(6,7)$ (Table 3). We now assume that $T_m = \varnothing$. Since $f_r \leq 13$, $T_r = T_3(8,9)$ (Table 3) or $T_r = T(10,12)$. We show that no such $T(10,12)$ exists. Suppose $r_A = 4$ and $r_B = 3$. With no loss of generality, we consider the following blocks:

T_r^+	T_r^-
$AB1$	ABy
ABx	ABz
Byz	$B1x$
Ayr	$A1r$
Azs	Axs
$1xr$	
xs-	

The three remaining blocks in T_r^+ must contain four new regular points and are of the form $234, 25-, 35-$, but then 2 has at most one irregular adjacency which is a contradiction.

(ix) By Lemma 4.2, the result follows. □

5. Conclusions

In the following theorems, we summarize the results of Sections 3 and 4 on the number of non-isomorphic trades of minimum volume.

Theorem 5.1. Up to isomorphism, the number of Steiner trades of minimum volume is as follows:

f (mod 6)	minimum volume	# trades	exceptions
0	$\frac{2f}{3}$	1	
1	$\frac{2f+4}{3}$	4	$\#T(6,7)=1$, $\#T(10,13)=3$
2	$\frac{2f+2}{3}$	1	
3	$\frac{2f+3}{3}$	1	
4	$\frac{2f+4}{3}$	9	$\#T(8,10)=4$, $\#T(12,16)=8$
5	$\frac{2f+2}{3}$	1	

Theorem 5.2. Up to isomorphism, the number of non-Steiner trades of minimum volume is as follows:

$f \pmod 6$	minimum volume	# trades	exceptions
0	$\frac{2f+6}{3}$	7	$\#T(6,6)=1$, $\#T(10,12)=5$
1	$\frac{2f+4}{3}$	1	
2	$\frac{2f+5}{3}$	1	
3	$\frac{2f+6}{3}$	10	$\#T(8,9)=6$,
4	$\frac{2f+4}{3}$	1	

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Appendix

Table 2 contains, up to isomorphism, all Steiner trades of minimum volume for $7 \leq f \leq 10$ [2, Table 2]. In Table 3, all non-isomorphic non-Steiner trades of minimum volume are given for $6 \leq f \leq 9$ [3,4].

Table 2.
Steiner trades of minimum volume for $7 \leq f \leq 10$.

$T_1(6,7)$	$T(6,8)$	$T(7,9)$	$T_1(8,10)$	$T_2(8,10)$	$T_3(8,10)$	$T_4(8,10)$
123	123	123	127	123	123	123
167	145	145	138	145	145	145
247	167	167	28A	167	167	167
256	248	248	379	189	189	189
346	368	358	459	24A	24A	24A
357	578	369	46A	268	35A	36A
		579	57A	279	68A	58A
			689	35A	79A	79A
127	124	124	128	124	124	124
136	136	136	137	135	135	136
235	157	157	27A	168	168	158
246	238	238	389	179	179	179
347	458	359	45A	23A	23A	23A
567	678	458	469	267	45A	45A
		679	579	289	67A	67A
			68A	45A	89A	89A

Table 3.
Non-Steiner trades of minimum volume for $6 \leq f \leq 9$.

$T(6,6)$	$T_2(6,7)$	$T(7,8)$	$T_1(8,9)$	$T_2(8,9)$	$T_3(8,9)$	$T_4(8,9)$	$T_5(8,9)$	$T_6(8,9)$
123	123	123	123	123	123	123	123	123
124	124	147	145	145	145	145	145	145
156	156	148	247	248	248	246	239	248
256	157	156	268	249	249	248	247	249
345	267	245	356	267	347	239	268	347
346	345	358	357	345	356	347	346	356
		678	239	568	589	356	357	467
			589	579	679	789	489	789
125	126	124	124	124	124	124	124	124
126	127	138	135	135	135	135	135	135
134	135	145	236	234	234	234	236	234
234	145	167	237	268	289	236	237	289
356	234	235	289	279	367	289	289	367
456	567	478	359	458	458	379	349	456
		568	457	459	479	456	457	478
			568	567	569	478	468	479

References

1. D. E. Bryant, *On the volume of trades in the triple systems,* Australas. J. Combin. **15** (1997), 161–176.

2. G. B. Khosrovshahi and H. R. Maimani, *On Steiner* 2-$(v, 3)$ *trades of small volumes*, Ars Combin., to appear.

3. G. B. Khosrovshahi, H. R. Maimani and R. Torabi, *Classification of* 2-(6,3) *and* 2-(7,3) *trades*, Australas. J. Combin., **19** (1999), 55–72.

4. Z. Eslami, G. B. Khosrovshahi and B. Tayfeh-Rezaie, *On Classification of* 2-(8,3) *and* 2-(9,3) *trades,* submitted.

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