Excess Graphs and Bicoverings

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Abstract. The classical bicovering problem seeks to cover all pairs from a v-set by a family F of k-sets so that every pair occurs at least twice and the cardinality of F is minimal. A weight function is introduced for blocks in such a design, and its use in constructing bicoverings is illustrated.

1. Introduction.

A covering is a collection of k-sets (blocks) chosen from elements of a v-set so that each pair from the v elements occurs at least once. The cardinality of a minimal covering is written $N_1(2,k,v)$, or simply $N(2,k,v)$. A λ-covering is a covering where each pair appears at least λ times; if $\lambda = 2$, we have a *bicovering*. The cardinality of a minimal bicovering is denoted by $N_\lambda(2,k,v)$.

It is well known that

$$N_\lambda(2,k,v) \geq v\, N_\lambda(1,k-1,v-1)/k = v\lceil \lambda(v-1)/(k-1) \rceil/k.$$

Henceforth, we shall use the term bicovering to denote a minimal bicovering.

2. The Weight Function

For any λ-covering consisting of b blocks, take a block B. Let x_i be the number of blocks meeting B in exactly i elements ($0 \leq i \leq k$). Then the number of blocks in the covering, excluding block B, is:

$$\Sigma x_i = b - 1.$$

If r_i is the frequency of element i, and λ_{ij} is the number of pairs (i,j), then counting the number of other occurrences of elements from B and the other occurrences of element pairs from B gives:

$$\Sigma\, ix_i = \Sigma\,(r_i - 1),$$

$$\Sigma\, i(i-1)x_i/2 = \Sigma\,(\lambda_{ij} - 1).$$

We now define a weight function w(B) for the block B. Clearly, this function should be non-negative, and we choose the definition

$$w(B) = \Sigma\, a_i x_i$$

where the a_i are non-negative integers. We may select the a_i in any way that we choose; however, for applications to the case k = 5, we shall use the definition

$$w(B) = x_0 + x_3 + 3x_4 + 6x_5.$$

Direct computation then establishes

Lemma 1. $w(B) = (b-1) - \Sigma\,(r_i - 1) + \Sigma\,(\lambda_{ij} - 1)$.

3. Excess Graphs and Excess Pairs

In a bicovering, each pair of elements occurs at least twice. If we represent pairs by edges of a graph on v nodes, then each edge occurs with multiplicity at least 2. Removing 2 copies of K_v from this graph leaves the *excess graph* of the bicovering. For example, consider a (2,5,19) bicovering; then $N_2(2,5,19) \geq \lceil 171/5 \rceil = 35$. Suppose that there is a bicovering in 35 blocks; these 35 blocks contain 350 pairs, and the bicovering requires 342 pairs. Hence there are 8 excess pairs and the excess graph contains 8 edges.

The frequency of each element in this bicovering is at least $N_2(1,4,18) = 9$. Hence, in the excess graph, any node of frequency 9 is an isolated point; any node of frequency 10 has degree 4; any node of frequency 11 has degree 8; etc. Since there are 175 elements in the bicovering, and at least 171 are required, there is an excess of 4 in the frequency counts. We refer to the points of frequency greater than 9 as excess points; they are the only points of positive degree in the excess graph (henceforth, we shall delete the isolated points).

If only one excess point appears in a (2,5,19) bicovering in 35 blocks, then 8 excess pairs are formed. This requires 8 loops in the excess graph (Figure 1). Since loops are not permitted, this case is trivially excluded.

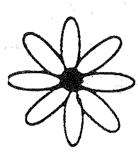

Figure 1: One Point in the Excess Graph

If two points A and B appear in the excess graph (Figure 2), then each must have valence 8. Now, the pair AB appears 8 times in the excess graph and twice normally; so AB occurs 10 times in the bicovering. However, A and B each appear exactly 11 times. Thus, there are 10 blocks of the form ABxxx, a block Axxxx, and a block Bxxxx. Since each of A and B appears twice with the other 17 elements, it follows that we must take these blocks as A1234 and B1234. From Lemma 1, we have $w(A1234) = 34 - (8 * 4 + 10) + 10 = 2$. But $w(B) = x_0 + x_3 + 3x_4 + 6x_5 \geq 3$, since A1234 has a quadruple intersection with B1234. This contradiction rules out the case of two exceptional points.

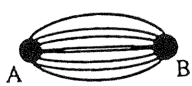

Figure 2: Two Points in the Excess Graph

Figure 3 illustrates the excess graph on three points; element A has frequency 11, while B and C each have frequency 10.

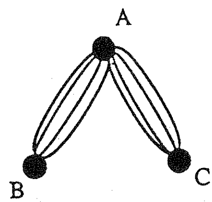

Figure 3: Three Points in the Excess Graph

Figure 4 illustrates the four excess graphs on four points. All four points in these graphs would have a frequency of 10 in a bicovering.

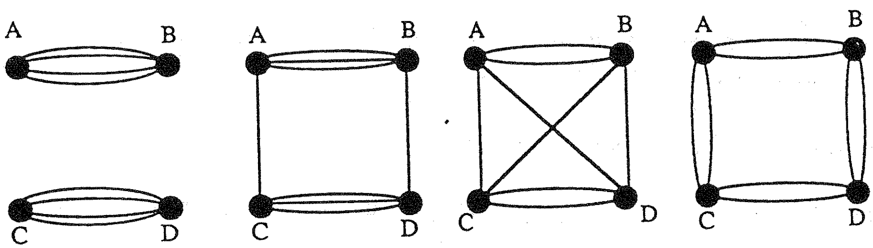

Figure 4: Excess Graphs with Four Excess Points

4. Upper Bounds.

There is little point in discussing the excess graphs for a possible bicovering in 35 blocks unless a good upper bound on the number of blocks in a (2,5,19) bicovering is known. Now $N(2,5,19) \geq 19$, and the bound 19 is easily achieved by cycling on the block (1 2 3 7 10). Hence, duplication of this covering results in a (2,5,19) bicovering in 38 blocks. (If we want a bicovering without repeated blocks, we could cycle the blocks (0 1 5 11 13) and (0 1 4 7 9).)

However, an upper bound of 36 can be achieved. Take ordinary elements 1 through 9, barred elements 1 through 9, and a fixed point P. Take four initial blocks and cycle on the 9 ordinary and 9 barred elements (leaving P fixed). Since P must appear twice with each element, P occurs in one initial block with two ordinary and two barred elements. Since there are ordinary (and barred) differences 1,2,3,4, and 16 mixed differences, we see that the other three initial blocks contribute at least 7 ordinary differences, 7 barred differences, and 14 mixed differences. There are six possible block types (x,y), where x is the number of ordinary elements, y the number of barred elements.

A: (5,0). B: (4,1) C: (3,2) D: (2,3) E (1,4) F (0,5)

The contributions to the ordinary, barred, and mixed differences (in that order) are as follows.

A:(10,0,0) B:(6,0,4) C:(3,1,6) D: (1,3,6) E: (0,6,4) F: (0,10,0)

If there are no type C or D blocks, then at most 12 mixed differences appear. Since at least 14 mixed differences are required, we may , by symmetry, take a block of type D. The two other blocks must give at least 6 ordinary differences, 4 barred differences, and 8 mixed differences; hence there must one block of type B and one of type E. Once the patterns of ordinary and barred elements in the four blocks have been determined, it is easy to write down a set of four initial blocks as (P14$\underline{67}$), (15$\underline{689}$), (1238$\underline{1}$), ($\underline{1257}$1).

It is easily verified that these 4 initial blocks generate a bicovering in 36 blocks; thus $N_2(2,5,19) \leq 36$. This establishes

Lemma 2. $N_2(2,5,19)$ is either 35 or 36.

Determination of the exact value of $N_2(2,5,19)$ will require discussion of the excess graphs shown in Figures 3 and 4.