Existence of six incomplete MOLS

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Abstract

Six mutually orthogonal Latin squares of order v (briefly 6 MOLS(v)) missing a subdesign of 6 MOLS(n) are denoted by 6 IMOLS(v, n). The necessary condition for their existence is $v \geq 7n$. For small v and n where $1 \leq v \leq 1000$ and $0 \leq n \leq 50$, the existence has been investigated by Colbourn and Dinitz, who give a table of possible exceptions in the CRC Handbook of Combinatorial Designs. In this paper, we study the existence of 6 IMOLS(v, n) when v and n are beyond the above range. First, we show that 6 IMOLS(v, n) always exist when $v > 1000$ and $0 \leq n \leq 50$. Second, we extend the table to list the possible exceptions when $51 \leq n \leq 97$. For $n \geq 98$, we are able to show that 6 IMOLS(v, n) exist whenever $v \geq 8n + 139$. We improve this in several ways, including that for all $n \geq 23$ when n is a prime power, and for all $n \geq 781$, the necessary condition $v \geq 7n$ is also sufficient. Some results on 4 IMOLS(v, n) are also mentioned.

1. Introduction

Let S be a set and $\boldsymbol{H} = \{S_1, S_2, ..., S_n\}$ be a set of subsets of S. A *holey Latin square* having *hole set* $\boldsymbol{H}$ is an $|S| \times |S|$ array L, indexed by S, satisfying the following properties:

(1) every cell of L either contains a symbol of S or is empty,

(2) every symbol of S occurs at most once in any row or column of L,

(3) the subarrays indexed by $S_i \times S_i$ are empty for $1 \leq i \leq n$ (these subarrays are referred to as *holes*),

(4) symbol $s \in S$ occurs in row or column t if and only if $(s, t) \in (S \times S) \setminus \cup_{1 \leq i \leq n}(S_i \times S_i)$.

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Australasian Journal of Combinatorics 12(1995), pp.175-191

The *order* of L is |S|. If the holes are pairwise disjoint, the holey Latin square is denoted by ILS(s; s_1, ..., s_n), where "I" stands for incomplete. Two holey Latin squares L and M on symbol set S and hole set $\boldsymbol{H}$ are said to be *orthogonal* if their superposition yields every ordered pair in $(S\times S)\setminus\cup_{1\le i\le n}(S_i\times S_i)$. We use the notation k IMOLS(s; s_1, ..., s_n) to denote a set of ILS(s; s_1, ..., s_n) where any two of them are orthogonal. If $\boldsymbol{H} = \emptyset$, we obtain k MOLS(s). If $\boldsymbol{H} = \{S_1\}$, we simply write k IMOLS(s, s_1). When k = 2, we further simplify the notation by IMOLS(s, s_1).

Let N(s) denote the largest value of k for which k MOLS(s) exist. It is well known that k MOLS(s) are equivalent to a TD(k + 2, s); see [2] for example. k IMOLS(v, n) are equivalent to an incomplete transversal design TD(k + 2, v) - TD(k + 2, n).

k MOLS and k IMOLS have played an important role in the construction of various kinds of combinatorial designs. In [11], Horton started to look at the existence of k IMOLS. Simple counting shows the following.

Lemma 1.1 If there exist k IMOLS(v, n), then $v \geq (k + 1)n$.

When k = 2, the existence has been completely solved in [10].

Theorem 1.2 [10] For any integer $n \geq 1$, IMOLS(v, n) exist if and only if $v \geq 3n$, except (v, n) = (6, 1).

For k = 3, the existence was solved in [12] when $n \geq 154$. Du [7], [8] has lowered the bound and listed 109 pairs of (v, n) as possible exceptions. Recently, Abel, Colbourn and Yin have further reduced the list to 24 possible exceptions, which we state as follows.

Theorem 1.3 [1] For any integer $n \geq 1$, there exist 3 IMOLS(v, n) if and only if $v \geq 4n$ except (v, n) = (6, 1) and possibly excepting the 24 pairs of (v, n) shown in Table 1.1.

n	possible exceptions v
1	10
2	16, 20, 31
3	21, 25, 29, 33, 36, 37, 41, 44
4	38
5	23, 27, 38
6	26, 33, 40, 44, 48, 52, 53
7	41

Table 1.1 24 possible exceptions of 3 IMOLS(v, n)

When $n \geq 4$, Drake and Lenz [6] have shown the following which has been slightly improved as pointed out in [13].

Theorem 1.4 There exist 5 IMOLS(v, n) if $v \geq 7n + 7$ and $n \geq 781$.

For n = 7, 8, 9 and k = 4, 5, some results on k IMOLS(v, n) were presented in [13]. Most recently, Colbourn and Dinitz [4] have investigated the existence of k IMOLS(v, n) for $1 \leq v \leq 1000$ and $0 \leq n \leq 50$. A table of possible exceptions for $k \leq 6$ has been included in the CRC Handbook of Combinatorial Designs. In this paper, we study the existence of 6 IMOLS(v, n) when v and n are beyond the above range. We first show that 6 IMOLS(v, n) always exist when $v > 1000$ and $0 \leq n \leq 50$. Second, we extend the table to list the possible exceptions for 6 IMOLS(v, n) when $51 \leq n \leq 97$. For $n \geq 98$, we are able to show that 6 IMOLS(v, n) exist whenever $v \geq 8n + 139$. We improve this in different ways, including that for all $n \geq 23$ when n is a prime power, and for all $n \geq 781$, the necessary condition $v \geq 7n$ is also sufficient. Some results on 4 IMOLS(v, n) are also mentioned.

2. Constructions

The constructions used are mainly the special cases or variations of the working corollaries in [3].

Lemma 2.1 [3] Suppose there exist 7 MOLS(t) and 6 IMOLS($m + u_i$; s, u_i) for $1 \leq i \leq t$. Then there exist 6 IMOLS(mt + u; st, u), where $u = \Sigma_{1 \leq i \leq t} u_i$. Moreover, (1) if there exist 6 MOLS(u), then there are 6 IMOLS(mt + u, st); (2) if s = 0 or there exist 6 MOLS(st), then there exist 6 IMOLS(mt + u, u). (3) if $s = u_1 = ... = u_{t-1} = 0$ and there exist 6 MOLS($m + u_t$), then there exist 6 IMOLS($mt + u_t$, m).

Lemma 2.2 [3] Suppose there exist 8 MOLS(t) and 6 IMOLS($m + u_i + v_j$; s, u_i, v_j) for $1 \leq i \leq t$ and $1 \leq j \leq t$. Then there exist 6 IMOLS(mt + u + v; st, u, v), where $u = \Sigma_{1 \leq i \leq t} u_i$ and $v = \Sigma_{1 \leq j \leq t} v_j$. Moreover, (1) if s = 0 and there exist 6 MOLS(u), then there exist 6 IMOLS(mt + u + v, v); (2) if there exist both 6 MOLS(u) and 6 MOLS(v), then there exist 6 IMOLS(mt + u + v, st).

Lemma 2.3 [3] Suppose there exist 6 + e MOLS(t), 6 MOLS(m) and 6 IMOLS(m + w_i; 1, w_i) for $1 \le i \le e$. Then there exist 6 IMOLS(mt + w, m + w), where $w = \Sigma_{1 \le i \le e} w_i$. Moreover, if there exist 6 MOLS(m + w), then there exist 6 IMOLS(mt + w, t).

Lemma 2.4 [3] Suppose there exist 7 + e MOLS(t), 6 MOLS(m), 6 IMOLS(m + v_j; 1, v_j) and 6 IMOLS(m + w_i + v_j; 1, w_i, v_j) for $1 \le i \le e$ and $2 \le j \le t$. Then there are 6 IMOLS(mt + w + v; m + w, v), where $w = \Sigma_{1 \le i \le e} w_i$ and $v = \Sigma_{2 \le j \le t} v_j$. Moreover, (1) if there exist 6 MOLS(v), then there exist 6 IMOLS(mt + w + v, m + w); (2) if there exist 6 MOLS(m + w), then there exist 6 IMOLS(mt + w + v, v); (3) if there exist both 6 MOLS(m + w) and 6 MOLS(v), then there exist 6 IMOLS(mt + w + v, t) and 6 IMOLS(mt + w + v, m).

We shall need generalizations of Lemmas 2.3 and 2.4 regarding holey common transversals in 6 IMOLS. Suppose k IMOLS(v, n) are based on set X and hole H. A set of |X| - |H| cells is called a *holey common transversal* if it intersects each row and each column with index from $X \setminus H$ exactly once and contains in each square every element from $X \setminus H$ exactly once. Two holey common transversals are *disjoint* if they have no cells in common.

Lemma 2.5 Suppose there exist 6 IMOLS(t, s) with k disjoint holey common transversals missing the hole of size s. Suppose there exist 6 MOLS(m) and 6 IMOLS(m + w_i; 1, w_i) for $1 \le i \le k$. Then there exist 6 IMOLS(mt + w, ms + w), where $w = \Sigma_{1 \le i \le k} w_i$. Moreover, if there exist 6 IMOLS(ms + w, s), then there exist 6 IMOLS(mt + w, t).

To get 6 IMOLS(n, s) with k disjoint holey common transversals missing the hole of size s, we employ the following construction.

Lemma 2.6 Suppose $N(t) \ge 8$ and $N(k) \ge 7$, where $0 \le k \le t$. Then there are 6 IMOLS(7t + u + k; u) with k disjoint holey common transversals missing the hole of size u, where $0 \le u \le t$.

Proof. In Lemma 2.2, take m = 7, s = 0, v = k and u_i, v_j = 0 or 1. Since $N(8) \ge 7$, $N(9) \ge 7$ and $N(k) \ge 7$, we get 6 IMOLS(7t + u + k; u) with k disjoint holey common transversals missing the hole of size u. •

We state a special case of Lemma 2.6 when k = 0.

Lemma 2.7 Suppose $N(t) \geq 7$. Then there are 6 IMOLS$(7t + u, u)$ with u disjoint holey common transversals missing the hole of size u, where $0 \leq u \leq t$.

To apply the above constructions, the following known results on k MOLS(v) are useful.

Lemma 2.8 [4] For any integer $v \geq b$ there exist k MOLS(v), where

k	4	5	6	7	8	12
b	43	63	76	781	2775	7317

3. 6 IMOLS(v, n), $0 \leq n \leq 97$

In this section, we discuss the existence of 6 IMOLS(v, n), where $0 \leq n \leq 97$.

Lemma 3.1 [9, Lemma 3.19] For any integer n, at least one of the integers n, n + 1, n + 2, ..., n + 9 is not divisible by any of the numbers 2, 3, 5 and 7.

Lemma 3.2 There exists a series of integers $\{q_i\}_{i=1,2,...}$ such that $N(q_i) \geq 10$, $0 < q_{i+1} - q_i \leq 10$ and $7q_{i+1} + 75 \leq 8q_i$.

Proof. From Lemma 3.1, any 10 consecutive integers contain at least one number n such that $N(n) \geq 10$. So, there is an infinite series: $a_1 = 149, a_2, ...$ such that for any $i \geq 1$, $N(a_i) \geq 10$ and $0 < a_{i+1} - a_i \leq 10$.

Since $149 \leq a_i$, $70 \leq a_i - 75$ and then $a_{i+1} \leq a_i + 10 \leq (8a_i - 75)/7$. We further have $7a_{i+1} + 75 \leq 8a_i$.

Define a series $\{q_i\}_{i=1,2,...}$ as follows:

$q_1 = 97, q_2 = 100, q_3 = 103, q_4 = 107, q_5 = 109, q_6 = 113,$

$q_7 = 117, q_8 = 121, q_9 = 127, q_{10} = 131, q_{11} = 137, q_{12} = 144,$

$q_{i+12} = a_i$ for $i \geq 1$.

It is readily checked that for any $i \geq 1$, we have $N(q_i) \geq 10$, $0 < q_{i+1} - q_i \leq 10$ and $7q_{i+1} + 75 \leq 8q_i$. •

Lemma 3.3 For any integer $n \geq 755 + s$ and $0 \leq s \leq 97$, there exist 6 IMOLS(n, s).

Proof. Apply Lemma 2.2 (1) with $t = q_i$, $m = 7$, u_i, $v_j = 0$ or 1. Since 6 IMOLS$(m + u_i + v_j; u_i, v_j)$ and 8 MOLS(t) all exist [4], we obtain 6 IMOLS$(7q_i + u + v; u, v)$. Let $u \geq 76$,

$v = s$ and $n = 7q_i + u + v$. Since 6 MOLS(u) exist [4], we have 6 IMOLS(n, s) for $n \in [7q_i + 76 + s, 8q_i + s]$, $i = 1, 2, \ldots$.

Since $7q_{i+1} + 76 \le 8q_i + 1$ from Lemma 3.2, we get 6 IMOLS(n, s) for $n \ge 7q_1 + 76 + s = 755 + s$. •

Theorem 3.4 For any integer s, $0 \le s \le 50$, there exist 6 IMOLS(n, s) except those pairs (n, s) listed in [4, Table 3.10].
Proof. In Table 3.10 of [4], the possible exceptions of (n, s) are listed for $0 \le s \le 50$ and $1 \le n \le 1000$. Apply Lemma 3.3 when $n > 1000$. •

Theorem 3.5 For any integer s, $51 \le s \le 97$, there exist 6 IMOLS(n, s) except those pairs (n, s) listed in the Appendix.
Proof. In the Appendix, the possible exceptions (n, s) for $n \le 755 + 97 = 852$ and $51 \le s \le 97$ are listed; they were generated by a computer program which is illustrated in [5]. The list is an extension of [4, Table 3.10]. The conclusion then follows from Lemma 3.3. •

To close this section, we give a general bound on the order of 6 IMOLS when the hole size is greater than 97.

Theorem 3.6. For any integer $n \ge 98$, there exist 6 IMOLS(v, n) whenever $v \ge 8n + 139$.
Proof. For any integer $n \ge 98$, let $n^* = \min\{q_i : q_i \ge n, i \ge 4\}$. It is easy to see that $n \le n^* \le n + 9$. Let $n^* = q_j$. For any $i \ge j$, we have 6 IMOLS(v, n) where $v = 7q_i + k + n$, $76 \le k \le q_i$, this is similar to the proof of Lemma 3.3. This gives an interval $[7q_i + 76 + n, 8q_i + n]$. Lemma 3.2 guarantees that between the consecutive intervals $[7q_i + 76 + n, 8q_i + n]$ and $[7q_{i+1} + 76 + n, 8q_{i+1} + n]$, there is no gap. We have proved that 6 IMOLS(v, n) exist for any $v \ge 7q_j + 76 + n = 7n^* + 76 + n$. If $v \ge 8n + 139$, then $v \ge 7(n + 9) + 76 + n \ge 7n^* + 76 + n$, and therefore 6 IMOLS(v, n) exist. The proof is complete. •

4. 6 IMOLS(v, n), $n \ge 98$

In this section, we improve the bound $v \ge 8n + 139$ when $n \ge 98$. We show among other things that the necessary condition $v \ge 7n$ is also sufficient if $n \ge 781$, or $n \ge 98$ and $N(n) \ge 12$.

Lemma 4.1 Suppose $N(n) \geq 7$ and $n \geq 76$. Then there exist 6 IMOLS$(7n + k, n)$ and 6 IMOLS$(8n + k, n)$ for $76 \leq k \leq n$.

Proof. Apply Lemma 2.1 (1) with $t = n$, $m = 7$ or 8, $s = 1$, $u = k$ and $u_i = 0$ or 1. •

Lemma 4.2 If $n \geq 43$ and $N(n) \geq 8$, then there exist 6 IMOLS(v, n) for $v \in [7n + 7, 7n + 75]$.

Proof. Write $v = 7n + a + b$ such that there are both 6 MOLS(a) and 6 MOLS(b), where $a, b \leq 43$. In fact, if $a = 0$ or 1 and $N(b) \geq 6$, then the requirement is met. Otherwise, we list a and b below.

$15 = 7 + 8$,	$21 = 8 + 13$,	$22 = 9 + 13$,	$34 = 7 + 27$,	$35 = 8 + 27$
$36 = 9 + 27$,	$39 = 7 + 32$,	$45 = 8 + 37$,	$46 = 9 + 37$,	$47 = 16 + 31$
$48 = 16 + 32$,	$49 = 8 + 41$,	$50 = 9 + 41$,	$51 = 8 + 43$,	$52 = 9 + 43$
$53 = 16 + 37$,	$54 = 17 + 37$,	$55 = 23 + 32$,	$56 = 16 + 40$,	$57 = 16 + 41$
$58 = 17 + 41$,	$59 = 16 + 43$,	$60 = 17 + 43$,	$61 = 32 + 29$,	$62 = 25 + 37$
$63 = 23 + 40$,	$64 = 27 + 37$,	$65 = 25 + 40$,	$66 = 25 + 41$,	$67 = 27 + 40$
$68 = 27 + 41$,	$69 = 29 + 40$,	$70 = 43 + 27$,	$71 = 31 + 40$,	$72 = 32 + 40$
$73 = 32 + 41$,	$74 = 31 + 43$,	$75 = 32 + 43$.		

Apply Lemma 2.2 (2) with $t = n$, $m = 7$, $s = 1$, u_i, $v_j = 0$ or 1. Since 6 IMOLS$(m + u_i + v_j; 1, u_i, v_j)$ and 8 MOLS(t) all exist [4], we obtain 6 IMOLS(v; n, a, b). The existence of both 6 MOLS(a) and 6 MOLS(b) implies the existence of 6 IMOLS(v, n). The proof is complete. •

Lemma 4.3 Suppose $N(n) \geq 8$ and $n \geq 98$. Then there exist 6 IMOLS$(8n + k, n)$ for $1 \leq k \leq 75$.

Proof. When $98 \leq n \leq 150$, it is readily checked that there are integers a and b such that $8n + k = 7n + a + b$, $N(a) \geq 6$, $N(b) \geq 6$ and $7 \leq a, b \leq n$. When $n \geq 151$, take $a = k + 75$ and $b = n - 75$ so that $k + 75 \geq 76$ and $n - 75 \geq 76$. Apply Lemma 2.2 (2) with $t = n$, $m = 7$, $s = 1$, $u = a$, $v = b$ and u_i, $v_j = 0$ or 1. •

Before we prove the next proposition, we make two claims as follows.

Claim 1 For integers N and s satisfying $N \geq 109$ and $N - 1 \leq s \leq 2N$, there are integers b and c such that $b + c = s$, $N(b) \geq 6$, $N(c) \geq 6$ and $0 \leq b, c \leq N$.

Proof. If $N \geq 153$, then we take b and c such that $b + c = s$ and $76 \leq b, c \leq N$. By Lemma 2.8, we have $N(b) \geq 6$ and $N(c) \geq 6$. For $109 \leq N \leq 152$, the conclusion is readily checked by a simple program. •

Claim 2 Let n be a positive integer. Let a be an integer such that $n + a$ is prime and $n + a \geq 109$. Let s satisfy $n + 2a - 1 \leq s \leq 2n + 2a$. Then there exist 6 IMOLS($7n + 6a + s$, n).

Proof. Let $N = n + a$. We have $N \geq 109$ and $N - 1 \leq s \leq 2N$. By Claim 1, we write $s = b + c$ such that $N(b) \geq 6$, $N(c) \geq 6$ and $0 \leq b, c \leq N$. Since N is prime and $n + s \geq 2N - 1$, Colbourn and Dinitz [5, Section 5.7] established that 6 IMOLS($7n + 6a + s$; n, b, c) exist. Filling the holes of sizes b and c gives the required result. •

Proposition 4.4 Suppose $N(n) \geq 7$ and $n \geq 98$. Then there exist 6 IMOLS(v, n) for any $v \geq 8n + 76$.

Proof. When $n \geq 138$, combine Theorem 3.6 and Lemma 4.1. We need further deal with the case $98 \leq n \leq 137$ and show the existence of 6 IMOLS(v, n) for $9n \leq v \leq 8n + 138$. In this case, we apply Claim 2 with N = 109, 113, 127, 131, 137 and 139. We can take suitable a to get the required result. For example, when $98 \leq n \leq 104$ we take $N = 109$ and $11 \geq a \geq 5$. Notice that for n = 110, 111 and 112 we take both N = 113 and N = 127.•

Proposition 4.5 Suppose $N(n) \geq 8$ and $n \geq 98$. Then there exist 6 IMOLS(v, n) for any $v \geq 7n$ except possibly $v = 7n + k$ and k = 3, 4, 5, or 6.

Proof. Combining Lemmas 4.1 - 4.3 and Proposition 4.4 we need only deal with the case $v = 7n + k$, k = 0, 1, 2. Write $v = 7n + a + b$ such that a, b = 0 or 1. Apply Lemma 2.2 (2) with $t = n$, $m = 7$, $s = 1$, u_i, $v_j = 0$ or 1. •

Proposition 4.6 Suppose $N(n) \geq 9$ and $n \geq 98$. Then there exist 6 IMOLS(v, n) for any $v \geq 7n$ except possibly $v = 7n + k$ and k = 4, 5, or 6.

Proof. By Proposition 4.5 we need only deal with the case $v = 7n + 3$. Apply Lemma 2.4 (3) with $t = n$, $m = 7$, $s = 1$, $w_1 = w_2 = 1$, $v_2 = 1$, $v_3 = ... = v_t = 0$. •

Proposition 4.7 Suppose $N(n) \geq 10$ and $n \geq 98$. Then there exist 6 IMOLS(v, n) for any $v \geq 7n$ except possibly $v = 7n + k$ and k = 5 or 6.

Proof. By Proposition 4.6 we need only deal with the case $v = 7n + 4$. Apply Lemma 2.3 with $t = n$, $m = 7$, $s = w_1 = w_2 = w_3 = w_4 = 1$. •

Proposition 4.8 Suppose $N(n) \geq 11$ and $n \geq 98$. Then there exist 6 IMOLS(v, n) for any $v \geq 7n$ except possibly $v = 7n + k$ and $k = 6$.

Proof. By Proposition 4.7 we need only deal with the case $v = 7n + 5$. Apply Lemma 2.4 (3) with $t = n$, $m = 7$, $s = w_1 = w_2 = w_3 = w_4 = 1$, $v_2 = 1$, $v_3 = ... = v_t = 0$. •

Theorem 4.9 Suppose $N(n) \geq 12$ and $n \geq 98$. Then there exist 6 IMOLS(v, n) if and only if $v \geq 7n$.

Proof. By Proposition 4.8 we need only deal with the case $v = 7n + 6$. Apply Lemma 2.3 with $t = n$, $m = 7$, $s = w_1 = ... = w_6 = 1$. •

Remark In Propositions 4.5 - 4.8 and Theorem 4.9, the conditions on N(n) can be weakened if we consider idempotent MOLS. In fact, in each case $N(n) \geq e$ can be replaced by (e - 1) idempotent MOLS(n).

Corollary 4.10 Suppose $n \geq 7317$. Then there exist 6 IMOLS(v, n) if and only if $v \geq 7n$.

Proof. When $n \geq 7317$, by Lemma 2.8 we have $N(n) \geq 12$. Apply Theorem 4.9. •

In what follows, we lower the bound $n \geq 7317$ to $n \geq 781$. To ease the notation, denote $S_{a,b} = \{n\text{: 6 IMOLS}(7n + k, n) \text{ exist for all } k, a \leq k \leq b\}$.

Lemma 4.11 Suppose there exist 6 IMOLS(n, s) with 6 disjoint holey common transversals missing the hole of size s. Then $n \in S_{0,6}$ if $s \in S_{0,6}$.

Proof. Apply Lemma 2.5 with $t = n$, $k = 6$, $m = 7$ and $w_i = 0$ or 1 for $1 \leq i \leq 6$. •

Lemma 4.12 For $700 \leq n \leq 7316$, there exist 6 IMOLS(7n + k, n) for $0 \leq k \leq 6$.

Proof. From [4, Table 2.63], any six consecutive integers not less than 97 contain at least one integer q such that $N(q) \geq 8$. Let $\{q_i\}_{i = 0, 1, ...}$ be a series such that $q_i \geq 97$, $N(q_i) \geq 8$ and $0 \leq q_{i+1} - q_i \leq 6$.

For $700 \leq n \leq 7316$, we may write $n = 7q_i + u + k$ such that $21 \leq u + k \leq 63$, $N(u) \geq 12$ and $N(k) \geq 7$ or $k = 0$. We list the decomposition as follows, where the case $k = 0$ is ommitted.

21 = 13 + 8,	22 = 13 + 9,	24 = 16 + 8,	26 = 17 + 9,	28 = 19 + 9
30 = 13 + 17,	33 = 16 + 17,	34 = 17 + 17,	35 = 16 + 19,	36 = 13 + 23
38 = 19 + 19,	39 = 31 + 8,	40 = 31 + 9,	42 = 31 + 11,	44 = 17 + 27

$45 = 37 + 8$, $46 = 37 + 9$, $48 = 23 + 25$, $50 = 23 + 27$, $51 = 19 + 32$
$52 = 25 + 27$, $54 = 27 + 27$, $55 = 23 + 32$, $56 = 25 + 31$, $57 = 25 + 32$
$58 = 27 + 31$, $60 = 19 + 41$, $62 = 13 + 49$, $63 = 31 + 32$.

Applying Lemma 2.6 or 2.7 with $t = q_i$ gives 6 IMOLS(n, u) with at least 6 disjoint holey common transversals missing the hole of size u, where $u \geq 13$. Since $N(u) \geq 12$, we may apply Lemmas 2.3 - 2.4 to get $u \in S_{0,6}$. This is similar to the proofs in Propositions 4.5 - 4.8 and Theorem 4.9. Further apply Lemma 4.11, we get $n \in S_{0,6}$. •

Lemma 4.13 Suppose there exist 6 IMOLS(n, s) with 75 disjoint holey common transversals missing the hole of size s. Then $n \in S_{7,75}$ if $s \in S_{7,75}$.

Proof. Apply Lemma 2.5 with $t = n$, $k = 75$, $m = 7$ and $w_i = 0$ or 1 for $1 \leq i \leq 75$, where $w_1 = \ldots = w_7 = 1$. •

Lemma 4.14 For $749 \leq n \leq 7316$, there exist 6 IMOLS(7n + k, n) for $7 \leq k \leq 75$.

Proof. When $n \geq 2775$, $N(n) \geq 8$ from Lemma 2.8 and then the conclusion holds by Lemma 4.4. So, we may focus on the interval $749 \leq n \leq 2774$.

Suppose n can be written as $n = 7q + u + k$ such that $N(q) \geq 8$, $q \geq 75$, $N(k) \geq 7$ and $75 \leq k \leq t$. Applying Lemma 2.6 with $t = q$ gives 6 IMOLS(n, u) with at least 75 disjoint holey common transversals missing the hole of size u, where $0 \leq u \leq q$. From Lemma 4.13, we have $n \in S_{7,75}$ if $u \in S_{7,75}$. To get $u \in S_{7,75}$, we may take $u \geq 43$, $N(u) \geq 8$ and apply Lemma 4.4.

Simple computation shows that for $749 \leq n \leq 2774$ and $n \neq 780, 796, 797, 798, 799$, the above decomposition of n is always possible and thus $n \in S_{7,75}$, where $q, u \in$ **E** and $k \in$ **E** $\cup$ **S**,

E = {43, 47, 49, 53, 59, 61, 64, 67, 71, 73, 79, 80, 81, 82, 83, 89, 97, 99, 100, 101, 103, 107, 109, 112, 113, 117, 121, 125, 127, 128, 131, 137, 139, 143, 144, 149, 151, 153, 154, 157, 160, 163, 167, 169, 171, 173, 176, 177, 179, 181, 185, 187, 189, 191, 193, 197, 199, 205, 211, 217, 221, 227, 233, 239, 243, 247, 251, 253, 256, 259, 261, 265, 269, 275, 281, 293, 297, 304, 307, 311, 313, 315, 317, 319},

S = {88, 91, 96, 104, 105, 115, 120, 129, 133, 135, 136, 141, 145, 147, 152, 155, 158, 161, 165, 168}.

For $n = 797, 798, 799$, since $N(n) \geq 8$, we get $n \in S_{7,75}$ from Lemma 4.4. Apply Lemma 2.7 with $n = 7t + u$, where

$$n = 780, \ t = 100, \ u = 80;$$

$n = 796, \; t = 101, \; u = 89.$

We get 6 IMOLS(n, u) with at leat 75 disjoint holey common transversals missing the hole of size u. Therefore, $n \in S_{7,75}$ since $u \in S_{7,75}$. The proof is complete. •

We are now in a position to prove the main results of this section.

Theorem 4.15 Suppose $n \geq 781$. Then there exist 6 IMOLS(v, n) if and only if $v \geq 7n$.

Proof. By Corollary 4.10, Lemmas 4.12 and 4.14, we need only deal with the case when $v \geq 7n + 76$. By Lemma 2.8, $N(n) \geq 7$. From Lemma 4.1 we have 6 IMOLS(v, n) for $7n + 76 \leq v \leq 8n$. From Proposition 4.2 we have 6 IMOLS(v, n) for $v \geq 8n + 76$. This leaves the interval $8n + 1 \leq v \leq 8n + 75$.

For $8n + 1 \leq v \leq 8n + 75$, apply Lemma 2.1 (3) with $t = 8$, $m = n$ and $u_t = v - 8n$. Since 6 IMOLS($n + u_t$, u_t) exist by Lemma 3.3 and 6 MOLS($n + u_t$) also exist from Lemma 2.8, we obtain 6 IMOLS(v, n). The proof is complete. •

Theorem 4.16 Suppose n is a prime power and $n \geq 23$. Then there exist 6 IMOLS(v, n) if and only if $v \geq 7n$.

Proof. When $n \geq 98$, the conclusion comes from Theorem 4.9. Below 98, it comes from [4, Table 3.10] and the Appendix. •

5. Results on 4 IMOLS

From Theorem 4.15, we know that 4 IMOLS(v, n) exist if $n \geq 781$ and $v \geq 7n$. In this section, we improve the bound $n \geq 781$ to $n \geq 98$ while keeping $v \geq 7n$. We need the following lemmas, where the first comes from [4, Table 2.64] and the other two are working lemmas for 4 IMOLS similar to Lemmas 2.2 - 2.3.

Lemma 5.1 There exist 4 MOLS(n) for any integer $n > 1$ and $n \notin E_4 = \{2, 3, 4, 6, 10, 14, 18, 22, 34, 42\}$.

Lemma 5.2 [3] Suppose there exist 6 MOLS(t) and 4 IMOLS($m + u_i + v_j$; s, u_i, v_j) for $1 \leq i \leq t$ and $1 \leq j \leq t$. Then there exist 4 IMOLS(mt + u + v; st, u, v), where $u = \Sigma_{1 \leq i \leq t} u_i$ and $v = \Sigma_{1 \leq j \leq t} v_j$. Moreover, (1) if $s = 0$ and there exist 4 MOLS(u), then there exist 4 IMOLS(mt + u + v, v); (2) if there exist both 4 MOLS(u) and 4 MOLS(v), then there exist 4 IMOLS(mt + u + v, st).

Lemma 5.3 [3] Suppose there exist 4 + e MOLS(t), 4 MOLS(m) and 4 IMOLS($m + w_i$; 1, w_i) for $1 \le i \le e$. Then there exist 4 IMOLS(mt + w, m + w), where $w = \Sigma_{1 \le i \le e} w_i$. Moreover, if there exist 4 MOLS(m + w), then there exist 4 IMOLS(mt + w, t).

Lemma 5.4 There exist 4 IMOLS(v, n) if $n \ge 98$ and $8n + 98 \le v \le 8n + 138$.

Proof. Write $v = 7(n + 7) + u + n$, where $49 \le u \le 89$. From Lemma 2.8 we have $N(u) \ge 4$. Since $N(n + 7) \ge 6$ and there are 4 IMOLS($7 + u_i + v_j$; 1, u_i, v_j) for = 0 or 1, we apply Lemma 5.2 (1) to obtain 4 IMOLS(v, n). •

Lemma 5.5 There exist 4 IMOLS(v, n) if $n \ge 98$ and $8n \le v \le 8n + 97$.

Proof. Write $v = 7n + u + n$, where $0 \le u \le 97$. From Lemma 2.8 we have $N(u) \ge 4$ when $u \notin E_4$. Since $N(n) \ge 6$ and there are 4 IMOLS($7 + u_i + v_j$; 1, u_i, v_j) for = 0 or 1, we apply Lemma 5.2 (1) to obtain 4 IMOLS(v, n) for $v = 8n + u$, $u \notin E_4$. Rewrite $v = 7n + (u + 5) + (n - 5)$, and apply Lemma 5.2 (2) with s = 1 to get 4 IMOLS(v, n) for $v = 8n + u$, $u \in E_4$. •

Lemma 5.6 There exist 4 IMOLS(v, n) if $n \ge 98$ and $7n \le v \le 8n - 1$.

Proof. Write $v = 7n + a + b$, such that $N(a) \ge 4$, $N(b) \ge 4$ and $0 \le a + b \le n - 1$. This can always be done except $v = 7n + 3$ and $v = 7n + 4$. So, we can use Lemma 5.2 (2) with s = 1 to get 4 IMOLS(v, n) for $v \ne 7n + 3, 7n + 4$.

For $v = 7n + 3$ or $7n + 4$, we apply Lemma 5.3 with e = 2, t = 7, m = n and $w_1 = w_2 = 1$ or 2. Since 4 IMOLS($n + w_i$, w_i) exist from [4, Table 3.12], we get the desired 4 IMOLS(v, n). •

We can now summarize the above results in the following.

Theorem 5.7 If $n \ge 98$, then 4 IMOLS(v, n) exist whenever $v \ge 7n$.

6. Concluding remarks

We have shown that the necessary condition $v \ge 7n$ for the existence of 6 IMOLS(v, n) is sufficient for $n \ge 781$ (Theorem 4.15), or for $n \ge 21$ and n is a prime power (Theorem 4.16), or for $n \ge 98$ and $N(n) \ge 12$ (Theorem 4.9). For $n \ge 98$ and $N(n) \ge 8$, the existence results are also fairly conclusive (Propositions 4.5 - 4.8). Since 6 IMOLS(v, n) exist for pairs (v, n) where $v \ge 755 + n$ and $0 \le n \le 97$ (Lemma 3.3), or $v \ge 8n + 139$ and

$n \geq 98$ (Theorem 3.9), the unknown pairs (v, n) are finite in number, which have been explicitly listed in [4, Table 3.10] when $0 \leq n \leq 50$, and in the Appendix when $51 \leq n \leq 97$.

As a consequence of the results for 6 IMOLS(v, n), we have also shown that 4 IMOLS(v, n) exist whenever $v \geq 7n$ and $n \geq 98$. However, the necessary condition in this case is $v \geq 5n$. It seems difficult to cover the gap between 5n and 7n.

Acknowledgments The first author acknowledges the support by NSERC under grant A0579. Both authors express their appreciation to NSERC Canada for a Foreign Researcher Award to support the visit of the second author to Waterloo.

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Appendix

Unknown pairs (v, n) for the existence of 6 IMOLS(v, n), $51 \leq n \leq 97$

The following list of unknown pairs (v, n) for the existence of 6 IMOLS(v, n), $51 \leq n \leq 97$, serves as supplementary material to the Table 3.10 in [4], where the unknown pairs (v, n) for $0 \leq n \leq 50$ are given. The present list is generated by a computer program which uses various constructions available and is described in [5].

n	v (6 IMOLS(v, n) are not known)
51	357 359 360 362 363 364 365 366 367 368 369 370 371 372 373 374 375 376 377 378 379 380 381 382 383 384 385 386 387 388 389 390 391 393 394 396 397 398 399 400 401 402 403 404 405 406 407 408 409 410 411 412 413 414
52	364 365 366 367 368 369 370 371 372 373 374 375 376 377 378 379 380 381 382 383 384 386 387 388 389 390 391 392 393 394 395 397 398 399 400 401 402 403 404 405 406 407 408 409 410 411 412 413 414 415
53	
54	378 379 380 381 382 383 384 385 386 388 389 390 391 392 393 394 395 396 397 399 400 401 402 403 404 405 406 407 408 409 410 411 412 413 414 415 416 417 418 419 420 421 422 423 424 425 426 427 428 429 430 431 432 434 436 437 438 439 440 442 443 444 445 447 448 449 450 451 452 455 456 457 458 459
55	385 387 388 389 390 391 392 393 394 395 397 398 400 401 402 403 404 405 406 407 408 409 410 411 412 413 414 415 416 417 418 419 420 421 422 423 424 426 427 428 429 430 431 432 433 434 435 436 437 438 439 440 441 443 444 445 446 448 449 450 451 452 453 456 457 459 460
56	394 395 396 397 398 402 404 406 407 410 413 414 416 418 420 422 425 426 427 430 431 434 436 437 438 440 444 446 447 451 453
57	402 403 404 405 409 411 413 414 417 419 420 421 423 425 432 433 434 437 438 443 445 447 451 453 454 458
58	406 408 409 410 411 412 413 414 415 416 417 418 419 420 421 422 423 424 425 426 427 428 429 430 431 432 433 434 435 436 437 438 439 440 441 442 443 444 446 447 448 449 451 452 453 454 455 456 458 459 460 461 462 463 469

59	
60	420 421 422 423 424 425 426 427 428 429 430 431 432 433 434 435 436 437 438 439 440 441 442 443 444 445 446 448 449 450 451 453 454 455 456 457 458 460 461 462 464 465 466 468 469 470 471 472 473 474 475 476 477 478 479
61	
62	434 436 437 438 439 440 441 442 443 444 445 446 447 448 450 451 452 453 455 456 457 458 459 460 462 463 464 465 466 467 468 470 471 472 473 474 475 476 477 478 479 480 481 482 483 484 485 486 487 488 489 490 493 494 495 496 499 500 501
63	443 444 445 446 447 451 453 463 469 474 476 481 482 483 486 487 492 494 496 500 502
64	
65	458 459 460 461 465 467 469 470 473 475 476 477 479 481 483 485 488 489 490 491 493 494 499 500 501 503 507 510 515 522 524
66	462 464 465 466 467 468 469 470 471 472 474 475 476 477 478 479 480 481 482 483 484 485 486 487 488 489 490 491 492 493 494 495 496 497 498 499 500 501 502 503 504 505 508 509 510 511 512 513 515 516 517 518 524
67	
68	476 477 478 479 480 481 482 483 484 485 486 487 488 489 490 491 492 493 494 495 496 497 499 500 501 502 503 504 505 506 507 510 511 512 513 514 515 517 518 522 526 530 532 534 538 542 546 548 550 554
69	483 484 485 486 487 488 489 490 491 492 493 494 495 497 498 500 501 502 503 504 505 506 507 508 511 512 514 515 516 518 522 524 526 530 532 534 538 540 550
70	490 492 493 494 495 496 498 499 501 502 503 504 505 506 507 508 509 512 513 514 515 516 517 522 524 530 532 534 538 540 542 546 548 550 556 564
71	
72	507 508 509 514 516 518 522 524 526 530 532 534
73	
74	518 519 520 521 523 524 525 526 527 528 529 530 531 532 533 534 535 536 537 538 539 540 541 542 543 544 545 546 548 549 550 551 552 553 554 556 557 558 559 560 561 563 565 566 567 568 572 573 574 575 579 580 581 582

75	525 526 527 528 529 530 531 532 533 534 535 536 537 538 539 540 541 542 543 544 545 546 547 549 550 551 552 553 554 555 556 557 558 559 560 561 562 564 566 567 568 569 573 574 575 576 580 581 582
76	533 534 535 536 537 538 539 540 542 543 544 545 546 547 548 550 551 552 553 554 555 556 558 559 560 561 562 563 565 567 568 569 570 574 575 576 577 581 582
77	541 542 543 544 545 549 551 553 554 557 559 561 563 569 575 577 578
78	546 548 549 550 551 552 556 558 560 564 567 570 572 576 579
79	
80	564 565
81	
82	577 578
83	
84	588 589 590 591 592 593 594 597 598 599 601 602 603 604 605 606 607 608 609 610 611 613 614 615 617 618 619 620 621 622 624 625 626 627 629 631 632 634 635 638 639 640 641 642 645 646 647 648 650 651 653 654 656 657 658 659 674 684
85	595 596 597 598 599 600 602 603 604 605 606 607 608 609 610 611 613 614 615 616 618 619 620 621 622 623 625 626 627 628 629 630 632 633 635 636 639 641 642 643 646 647 648 649 650 651 652 654 656 657 658 659 660 662 663 664 665 666 667 668 670 682
86	602 603 604 605 606 607 608 609 610 611 612 613 614 615 616 617 619 620 621 622 623 624 626 627 628 629 631 633 634 636 637 640 641 642 643 647 648 649 650 651 652 653 655 657 658 659 660 661 663 664 665 666 667 668 669 671 672 673 676 677 684
87	609 611 612 613 614 615 616 617 618 620 621 622 623 624 625 627 628 629 630 631 632 634 635 637 638 641 643 644 648 649 650 651 652 653 656 658 659 660 662 664 665 666 667 669 670 672 673 674 677 678 680 683 684
88	622 628 630 636 638 642 644 646 650 651 660 661 668 670 671
89	
90	630 632 633 635 636 642 644 645 648 651 652 654 656 660 663 665 668 676
91	652 655 657 661 663 670 671 676 681 688 689 692

92	644 646 647 648 649 650 651 653 654 655 656 657 658 661 662 663 664 665 667 669 670 671 672 674 675 677 678 679 681 682 683 685 686 688 689 690 691 693 695 696 699 700 701 702 703 707
93	651 654 655 656 657 663 665 666 669 671 672 673 675 679 684 686 687 689 690 693 696 697 702
94	658 660 662 663 664 668 670 672 673 676 678 679 680 684 688 691 692 693 694 697 702 709
95	667 668 670 671 675 677 680 685 686 691 693 695 698 699 703 710
96	686 687 690 692 693 694 696 700 707 711 716
97	

(Received 4/1/95)