Nearly-acyclically pushable tournaments

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Abstract

Let D be a digraph and $X \subseteq V(D)$. By *pushing* X we mean reversing the orientation of each arc of D with exactly one end in X. Klostermeyer proved that it is NP-complete to decide if a given digraph can be made acyclic using the push operation. By contrast, Huang, MacGillivray, and Wood showed that the problem of deciding if a given multipartite tournament can be made acyclic using the push operation is solvable in polynomial time. We define a digraph to be *nearly-acyclic* if it is obtained from an acyclic digraph by substituting a (directed) triangle or a single vertex for each vertex of the acyclic digraph. It is shown that it is NP-complete to decide if a given digraph can be made nearly-acyclic using the push operation. In this paper, we characterize, in terms of forbidden subtournaments, the tournaments which can be made nearly-acyclic by pushing. This implies that the problem of deciding if a given tournament can be made nearly-acyclic using the push operation is solvable in polynomial time.

1 Introduction

Let D be a digraph and $X \subseteq V(D)$. We define D^X to be the digraph obtained from D by reversing the orientation of each arc with exactly one endvertex in X. We say the vertices of X are *pushed* and that D^X is the result of *pushing* X in D. Note that $D^X = D^{V(D)-X}$ and, when $X = \emptyset$ or $V(D)$, $D^X = D$. For any $X, Y \subseteq V(D)$, $(D^X)^Y = D^{X \triangle Y}$ where $X \triangle Y$ is the symmetric difference of X and Y.

The operation of pushing vertices was first introduced by Fisher and Ryan [2] in the context of tournaments. They computed the number of positive tournaments using an equivalence relation among all tournaments induced by the push operation.

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The study of this push operation has recently been focused on the problems of deciding if a given digraph can be pushed so that the resulting digraph satisfies certain properties, see [3, 5, 6, 7, 8, 9]. Klostermeyer [5] proved that the problems of deciding whether a given digraph can be made acyclic, strongly connected, Hamiltonian, or semi-connected, using the push operation are NP-complete. Huang, MacGillivray, and Yeo [4] proved that the problem of deciding if a given digraph can be made acyclic using the push operation remains NP-complete for the class of digraphs whose underlying graphs are bipartite.

By contrast, Klostermeyer [5] showed that every tournament on at least three vertices, except the two tournaments in Fig. 1, can be transformed into a Hamiltonian tournament by pushing. MacGillivray and Wood [8] proved that the two tournaments in Fig. 1 are the only forbidden substructures for the tournaments which can be made acyclic using the push operation. Thus, the problem of deciding if a tournament can be made Hamiltonian or acyclic using the push operation is solvable in polynomial time.

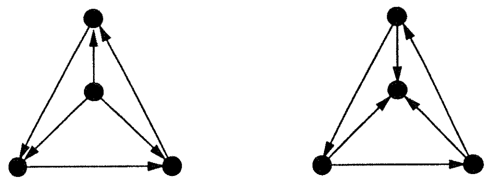


Figure 1: The forbidden subtournaments for acyclically pushable tournaments

Huang, MacGillivray, and Wood [3] characterized in terms of forbidden substructures the multipartite tournaments which can be made acyclic (resp. ordinary, unidirectional) using the push operation. The characterization implies that the problem of deciding if a given multipartite tournament can be made acyclic (resp. ordinary, unidirectional) using the push operation is solvable in polynomial time.

A digraph D is *nearly-acyclic* if it is obtained from an acyclic digraph D' by substituting either a triangle or a single vertex for each vertex of D'. A result in [4] implies that it is NP-complete to decite if a given digraph can be made nearly-acyclic using the push operation. In this paper, we present a characterization in terms of forbidden subtournaments of the tournaments which can be made nearly-acyclic using the push operation. Our characterization implies that the problems of deciding if a given tournament can be made nearly-acyclic using the push operation is solvable in polynomial time.

2 Terminology and preliminaries

All digraphs considered are finite and contain no loops or multiple arcs. If a digraph contains no cycles of length two, then it is an *oriented graph.* Terminology not defined in this paper follows [1] or [10].

Let D be a digraph. We use $V(D)$ (or simply V) and $A(D)$ (or simply A) to denote the vertex set and the arc set of D. The arc from a vertex x to a vertex y will be denoted by xy. If xy is an arc, then we say that x *dominates* y or y *is dominated by* x, and denote this by $x \to y$. The *in-neighbourhood* $I(x)$ (resp. *out-neighbourhood* $O(x)$) of a vertex x consists of all vertices y such that y dominates x (resp. x dominates y).

Let D be a digraph. A digraph H is called a *subdigraph* of D if $V(H) \subseteq V(D)$ and $A(H) \subseteq A(D)$. An *induced subdigraph* S of D is a subdigraph of D such that $A(S)$ consists of all arcs of D whose endvertices are in $V(S)$. We also say that S is induced by $V(S)$ and use $D\langle V(S)\rangle$ to denote it. Let H and S be two subdigraphs of D or two subsets of $V(D)$. If every vertex of H dominates every vertex of S, then we say that H *completely dominates* S or S is *completely dominated by* H and denote this by $H \Rightarrow S$. When H consists of single vertex x or S consists of a single vertex y, we shall write $x \Rightarrow S$ or $H \Rightarrow y$

Two digraphs D and D' are *isomorphic*, denoted by $D \cong D'$, if there is a bijection $f : V(D) \to V(D')$ such that $xy \in A(D)$ if and only if $f(x)f(y) \in A(D')$.

Let D be a digraph and S be a subdigraph of D or a subset of $V(D)$. We say that S is *coned* by a vertex $x \in V(D)$ or x *cones* S if either $x \Rightarrow S$ or $S \Rightarrow x$. If S is coned by each vertex of $D - S$, then we say that S is *coned in* D.

Let D and H be two digraphs and x be a vertex of D. To *substitute* H for x means to form a new digraph from D by replacing x with H such that in the new digraph every vertex of H dominates each out-neighbour of x and is dominated by each in-neighbour of x.

A (directed) cycle of length three is referred to as a (directed) *triangle*. We shall use Δ_{xyz} to denote the triangle $xyzx$. If a digraph contains no cycles, then it is called *acyclic*. A digraph is called *nearly-acyclic* if it is obtained from an acyclic digraph by substituting a triangle or a single vertex for each vertex of the acyclic digraph. Clearly, every acyclic digraph is nearly-acyclic and every subdigraph of an acyclic (resp. nearly-acyclic) digraph is acyclic (resp. nearly-acyclic).

If a digraph can be made acyclic (resp. nearly-acyclic) using the push operation, then it is called *acyclically pushable* (resp. *nearly-acyclically pushable*).

A *tournament* is an oriented complete graph. It is well-known that if a tournament is not acyclic then it contains a triangle. A *nearly-acyclic* tournament is a tournament obtained from an acyclic tournament by substituting a triangle or a single vertex for each vertex of the acyclic tournament. Clearly, all nearly-acyclic tournaments form a larger class than the class of acyclic tournaments.

Proposition 2.1 *A digraph D is nearly-acyclically pushable if and only if every subdigraph of D is nearly-acyclically pushable.*

Proof: If every subdigraph of D can be made nearly-acyclic using the push operation, then D can be made nearly-acyclic using the push operation as D is a subdigraph of itself. Conversely, suppose that D^X is nearly-acyclic for some $X \subseteq V(D)$. Then, for any subdigraph D' of D, $(D')^{X \cap V(D')}$ is a subdigraph of D^X and hence nearly-acyclic. ◇

Observe that the relation $\equiv$ on all digraphs, defined by $D \equiv D'$ if and only if $D^X \cong D'$ for some $X \subseteq V(D)$, is an equivalence relation. The equivalence class that contains D shall be denoted by $[D]$. One can easily verify that the two tournaments in Fig. 1 form an equivalence class.

Proposition 2.2 *Let D be a digraph and $D' \in [D]$. Then D is nearly-acyclically pushable if and only if D' is nearly-acyclically pushable.* $\diamond$

The following is a reformulation of the main result in [9].

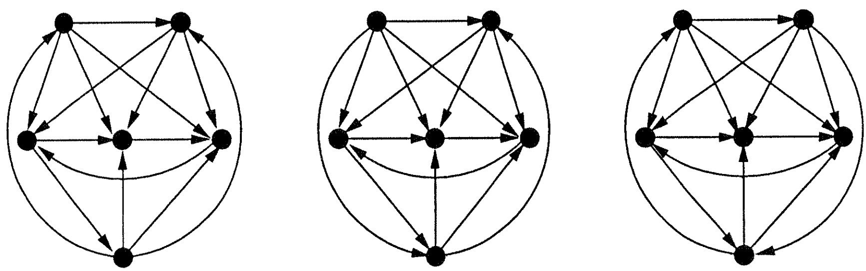
Proposition 2.3 *[9] A tournament is acyclically pushable if and only if no triangle is coned by any vertex.* $\diamond$

3 Nearly-acyclically pushable tournaments

Every triangle of a nearly-acyclic digraph D is coned in D. But the converse is not necessarily true in general. For instance, every cycle satisfies the property that every triangle is coned, but any cycle of length greater than three is not nearly-acyclic. However, the converse is true for tournaments.

Lemma 3.1 *Let T be a tournament. If every triangle is coned in T, then T is nearly-acyclic.*

Proof: We first show that no two triangles share a vertex. Suppose that Δ_{xyz} and Δ_{abc} are two triangles of T with $x = a$. Without loss of generality, assume that $c \notin \{x, y, z\}$. Since $c \to a = x$ and Δ_{xyz} is coned in T, $c \to z$. But then we have $c \to z \to x = a$ and that Δ_{abc} is not coned by z, contradicting the assumption. Let T' be the tournament obtained from T by shrinking each triangle of T to a single vertex and deleting all multiple arcs. Clearly, T' is acyclic and T can be seen as the tournament obtained from T' by substituting a triangle or a vertex for each vertex of T'. Hence T is nearly-acyclic. $\diamond$



R₁ **R₂** **R₃**

Figure 2: Funddamental forbidden subtournaments for nearly-acyclically pushable tournaments.

One should have little difficulty verifying that none of the four tournaments R_1, R_2, R_3 in Fig. 2 are nearly-acyclically pushable. Thus by Propositions 2.1 and 2.2 we have the following:

Lemma 3.2 *If a tournament contains any tournament in* $[R_1] \cup [R_2] \cup [R_3]$ *as a subtournament, then it is not nearly-acyclically pushable.* ◇

Theorem 3.3 *Let T be a tournament. Then T is nearly-acyclically pushable if and only if T contains no tournament in* $[R_1] \cup [R_2] \cup [R_3]$ *as a subtournament.*

Proof: By Lemma 3.2, we only need to show sufficiency. Suppose T contains no tournament in $[R_1] \cup [R_2] \cup [R_3]$ as a subtournament. By Proposition 2.3, we may assume that T contains a triangle which is coned by at least one vertex.

We claim that either T is nearly-acyclically pushable or some tournament in $[T]$ contains a triangle which is both coned by at least one vertex and not coned by at least one vertex. Let Δ_{xyz} be a triangle of T which is coned by at least one vertex. Partition $V(T) - \{x, y, z\}$ into three sets A, B, C such that $A \Rightarrow \Delta_{xyz}$, $\Delta_{xyz} \Rightarrow B$, and C consists of those vertices that do not cone Δ_{xyz}. Let $T' = T^B$. Then $T' \in [T]$ and Δ_{xyz} is also a triangle in T' completely dominated by $A \cup B$. Since Δ_{xyz} is coned by at least one vertex in T, $A \cup B \neq \emptyset$. If $C \neq \emptyset$, then Δ_{xyz} is a desired triangle in T'. So assume that $C = \emptyset$. If every triangle of T' is coned in T', then, by Lemma 3.1, T' is nearly-acyclic. Hence, by Proposition 2.2, T is nearly-acyclically pushable and we are done. Thus assume that T' contains a triangle which is not coned by at least one vertex. Since $(A \cup B) \Rightarrow \Delta_{xyz}$ in T', any triangle of T' which is not coned by at least one vertex must completely lie in $T'\langle A \cup B \rangle$. Moreover, such a triangle is coned by each vertex of Δ_{xyz}. Therefore T' contains a triangle that is coned by at least one vertex and is not coned by at least one vertex.

For sake of simplicity, we may just assume that $B = \emptyset$ and $C \neq \emptyset$. That is, in T, both A and C are nonempty and Δ_{xyz} is coned (dominated) by every vertex in A and is not coned by every vertex in C. Furthermore, we assume that each vertex in C dominates exactly two vertices of Δ_{xyz}, as otherwise we push those vertices in C which dominate exactly one vertex of Δ_{xyz} and consider the resulting tournament.

Suppose that $|A| \geq 2$. Let w, w' be any two vertices in A and v be any vertex in C. Then it is easy to see that the subtournament of T induced by $\{w, w', x, y, z, v\}$ is isomorphic either to R_i for some $i = 1, 2, 3$ or a tournament in $[R_1]$, contradicting the hypothesis. Suppose that $|A| = |C| = 1$. Let $A = \{w\}$ and $C = \{v\}$. Without loss of generality, assume that v dominates y and z and is dominated by x. If $vw \in A(T)$, then pushing $\{x\}$ results in a nearly-acyclic tournament. On the other hand, if $wv \in A(T)$, then pushing $\{z\}$ results in a nearly-acyclic tournament. Hence we assume that $|A| = |\{w\}| = 1$ and $|C| \geq 2$. Denote

$$X = \{v \in C \mid xv,\ vy,\ vz \in A(T)\},$$

$$Y = \{v \in C \mid yv,\ vy,\ vz \in A(T)\},$$

$$Z = \{v \in C \mid zv,\ vx,\ vy \in A(T)\}.$$

We consider the following two cases:

Case 1. Two of X, Y, and Z are empty. Without loss of generality, assume that $Y = Z = \emptyset$, i.e., that $C = X$ and every vertex in C dominates y and z and is dominated by x.

Suppose that there are two vertices, say $u, v \in X$, such that $wu, vw \in A(T)$. If $vu \in A(T)$, then pushing $\{x, y, z\}$ results in a tournament in which the subtournament induced by $\{u, v, w, x, y, z\}$ is isomorphic to R_1. This implies that T contains a tournament in $[R_1]$ as a subtournament, a contradiction. If $uv \in A(T)$, then pushing $\{w, x\}$ results in a tournament in which the subtournament induced by $\{u, v, w, x, y, z\}$ is isomorphic to R_2. It implies that T contains a tournament in $[R_2]$ as a subtournament, a contradiction.

So we must have $w \Rightarrow X$ or $X \Rightarrow w$. We only consider the case when $w \Rightarrow X$ as the other case can be treated analogously. Let $T_1 = T^{\{y,z\}}$. Then X is coned in T_1. If $T_1\langle X \rangle$ contains a triangle, say Δ_{abc}, which is not coned by a vertex, say d, then $\{a, b, c, d, x, y\}$ induces in T_1 a subtournament isomorphic to a tournament in $[R_1]$. This implies that T contains a tournament in $[R_1]$ as a subtournament, a contradiction. Thus every triangle of $T_1\langle X \rangle$ is coned in T_1. Hence, every triangle of T_1 is coned in T_1, as Δ_{wxz} is the only triangle which is not contained in $T_1\langle X \rangle$ and Δ_{wxz} is coned in T_1. Thus T_1 is nearly-acyclic by Lemma 3.1 and hence T is nearly-acyclically pushable.

Case 2. At least two of X, Y, and Z are nonempty. Without loss of generality, assume that $X \neq \emptyset$ and $Y \neq \emptyset$.

From the discussion in Case 1, we know that either $w \Rightarrow X$ or $X \Rightarrow w$ and similarly, either $w \Rightarrow Y$ or $Y \Rightarrow w$. Suppose that $w \Rightarrow X$ and $Y \Rightarrow w$. Then, in $T^{X \cup \{y,z\}}$, the subtournament induced by $\{u, v, w, x, y, z\}$ with $u \in X$ and $v \in Y$ is isomorphic to R_1 or R_2. This means that T contains a tournament in $[R_1]$ or in $[R_2]$ as a subtournament, a contradiction. Suppose that $w \Rightarrow X$ and $w \Rightarrow Y$. Then, in $T^{X \cup Y \cup \{y,z\}}$, the subtournament induced by $\{u, v, w, x, y, z\}$ with $u \in X$ and $v \in Y$ is isomorphic to R_3. This means that T contains a tournament in $[R_3]$ as a subtournament, a contradiction. Hence we must have $X \Rightarrow w$. If $Y \Rightarrow x$, then pushing $Y \cup \{x, z\}$ results in a tournament in which the subtournament induced by $\{u, v, w, x, y, z\}$ with $u \in X$ and $v \in Y$ is isomorphic to R_1 or R_3. This means that T contains a tournament in $[R_1]$ or in $[R_3]$ as a subtournament, a contradiction. Hence we must have $w \Rightarrow Y$. From this analysis, we see that $Z = \emptyset$, as otherwise by considering Y and Z instead of X and Y and applying the above argument, we see that $Y \Rightarrow w$, a contradiction.

So we know that $X \Rightarrow w$, $w \Rightarrow Y$ and $Z = \emptyset$. Suppose that $X \Rightarrow Y$. Let $T_2 = T^{X \cup \{x\}}$. Then both X and Y are coned in T_2. If either $T_2\langle X \rangle$ or $T_2\langle Y \rangle$ contains a triangle, say Δ_{abc}, which is not coned by some vertex, say d, then $\{a, b, c, d, x, y\}$ induces a subtournament of T_2 isomorphic to a tournament in $[R_1]$. This implies that T contains a tournament in $[R_1]$ as a subtournament, a contradiction. Thus every triangle of $T_2\langle X \rangle$ and $T_2\langle Y \rangle$ is coned. Hence every triangle of T_2 is coned and so T_2 is nearly-acyclic. This means that T is nearly-acyclically pushable.

Suppose now that T contains at least one arc from Y to X. We will show that in fact T contains exactly one arc from Y to X. Assume that this is not the case. Let vu and $v'u'$ be two arcs from Y. There are three possibilities: $u = u'$ and $v \neq v'$, $u \neq u'$ and $v = v'$, or $u \neq u'$ and $v \neq v'$. When $u = u'$, pushing $\{w\}$ results in a tournament whose subtournament induced by $\{u, v, v', w, y, z\}$ is isomorphic to R_1. This implies that T contains a tournament in $[R_1]$ as a subtournament, a contradiction. When $v = v'$, pushing $\{x, u, u'\}$ results in a tournament whose subtournament induced by $\{x, y, u, u', v, z\}$ is isomorphic to R_1, a contradiction. For the case when $u \neq u'$ and $v \neq v'$, we may assume that $u \rightarrow v'$ and $u' \rightarrow v$. Then the subtournament of T induced by $\{w, x, u, u', v, v'\}$ is isomorphic to a tournament in $[R_1]$. This can be seen by pushing $\{x, u, u'\}$. Thus T contains a tournament in $[R_1]$ as a subtournament, a contradiction. Therefore T contains exactly one arc from Y to X.

Denote the only arc from Y to X by vu and let $X' = X - \{u\}$ and $Y' = Y - \{v\}$. Then we have that $u \Rightarrow Y'$, $X' \Rightarrow v$ and $X' \Rightarrow Y'$. We claim that $u \Rightarrow X'$ and $Y' \Rightarrow v$. Assume that v dominates a vertex $y' \in Y'$. Then, pushing $\{u\}$ results in a tournament whose subtournament induced by $\{u, v, w, y, y', z\}$ is isomorphic to a tournament in $[R_1]$, a contradiction. Assume that there is a vertex $x' \in X'$ which dominates u. Then, pushing $\{y, z\}$ results in a tournament whose subtournament induced by $\{x', z, u, v, w, y\}$ is isomorphic to R_1, a contradiction. Therefore $u \Rightarrow X'$ and $Y' \Rightarrow v$.

Let $T_3 = T^{X \cup \{w,y\}}$. Then we have in T_3 that $Y' \Rightarrow \{u, v, z\} \Rightarrow X' \Rightarrow \{x, w, y\}$ and $(Y' \cup \{u, v, z\}) \Rightarrow (X' \cup \{x, w, y\})$. If either of $T_3\langle X' \rangle$ and $T_3\langle Y' \rangle$ contains a triangle which is not coned in T_3, then T_3 and hence T contain a tournament in $[R_1]$ as an induced subtournament, a contradiction. Thus every triangle is coned in T_3 and hence, by Lemma 3.1, T_3 is nearly-cyclic. Therefore T is nearly-acyclically pushable. $\diamond$

According to the remark at the beginning of this section, for a digraph D whose underlying graph contains no cycles of length less than four, D is is nearly-acyclic (nearly-acyclically pushable) if and only if it is acyclic (acyclically pushable). Since it is NP-complete to decide if a given digraph whose underlying graph is bipartite is acyclically pushable [4], it is NP-complete to decide if a given digraph is nearly-acyclically pushable. However, Theorem 3.3 implies immediately the following:

Corollary 3.4 *The problem of deciding if a given tournament is nearly-acyclically pushable is solvable in polynomial time.* $\diamond$

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