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On several classes of monographs

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Abstract

Let $G = (V, E)$ be a finite (non-empty) graph. A monograph is a graph in which all vertices are assigned distinct real number labels so that the positive difference of the end-vertices of every edge is also a vertex label. In this paper we study the properties of monographs and construct signatures for several classes of graph, such as cycles, cycles with chord, fan graphs F_n, kite graphs, chains of monographs and necklaces of monographs.

1 Introduction

In this paper, all graphs are finite, simple and undirected. A graph G may be written as $G(V, E)$, where V and E are sets of the vertices and the edges of G. An *autograph labeling* is a map α from V to a set of real numbers $S \subset R$, with the property that there is an edge $xy \in E$, if and only if there is $z \in V$ such that $|\alpha(x) - \alpha(y)| = \alpha(z)$. The set $S = \{s \in R | s = \alpha(v), \text{for all } v \in V\}$ is called a *signature* of G. A graph that has an autograph labeling is called an *autograph.* An autograph is called a *proper autograph* if α is a mapping from V to a set of positive integers. An autograph that does not have duplicate elements in its signature is called a *monograph.* In this paper we consider only proper monographs.

Bloom, Hell and Taylor [2] introduced the notion of a monograph in 1979. According to Gallian's dynamic survey [5], there are several results on monographs. Bloom *et al.* [1, 2] proved that trees, cycles C_n, complete graphs K_n, complete bipartite graphs $K_{n,n}$ and $K_{n,n-1}$, pyramids and n-prisms are monographs. Wheels W_n are monographs only for $n = 3, 4$ or 6 [4]. Some researchers also studied directed monographs, see [3, 6] for details.

In this paper we study properties of (proper) monographs such as multiples of a monograph labeling and union of two monographs. Moreover, we give a signature

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construction for cycles C_n, fan graphs F_n, a class of graphs that we call *kites*, chains of monographs and necklaces of monographs.

2 Properties of monographs

We begin with the following observation.

Observation 1 *Let α be a monograph labeling for a graph G and k be a positive integer. Then $k\alpha$ is also a monograph labeling for G.*

Consequently, if G is a monograph then a signature of G is not unique. If S is a signature of G then $kS = \{ks|s \in S\}$ is also a signature of G. Note, however, that some graphs can have more than one signature that are not multiples of each other. Since one monograph can have more than one signature, then by choosing an appropriate k and using Observation 1, we obtain the following observation concerning the disjoint union of multiple copies of G.

Observation 2 *Let G be a monograph. Then mG, for some positive integer m, is also a monograph.*

Next, we consider the disjoint union of some non-isomorphic monographs. Let G_1 and G_2 be monographs. Bloom *et al.* proved the following theorem.

Theorem 2.1 [2] *If each of the components of a graph G is a proper autograph, then G is a proper autograph: G is not a proper autograph if any of its components are not.*

We can restate Theorem 2.1 for monographs as follows.

Theorem 2.2 *A graph is a monograph if and only if each of its components is a monograph.*

Figure 1 gives an example of monograph $F_5 \cup K_6$.

3 Signature for several classes of graphs

3.1 From cycles C_n to fan graph F_n

We start this subsection by giving a signature for the cycle C_n. We then add, one by one, chords to the cycle such that all chords have a common endpoint, culminating with a signature for the fan F_n.

The signature for cycles C_n can be found in [2]. However, we rewrite the result here for completeness sake.

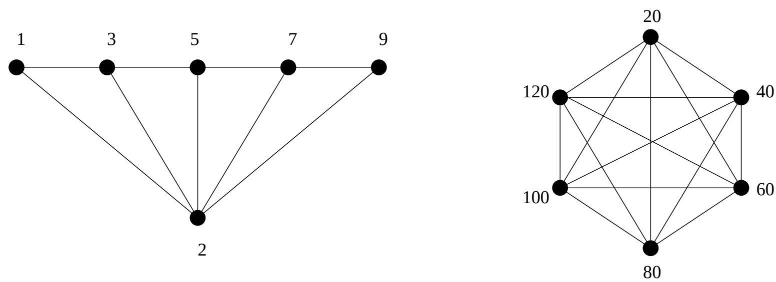

Figure 1: Union of F_5 and K_6.

Theorem 3.1 [2] *Every cycle C_n, $n > 2$ has an integer signature.*

Proof. Let C_n be a cycle with $n > 2$. Let v_i, $i = 1, 2, \ldots, n$, be the vertices of C_n. Define a labeling α as follows.

$$\alpha(v_i) = \begin{cases} 2^{i-1} & \text{if } i = 1, \ldots, n-1, \\ 2^{n-2} + 1 & \text{if } i = n. \end{cases}$$

Since the vertex labels constitute a signature $1, 2, 4, \ldots, 2^{n-2}, 2^{n-2}+1$, it follows that there is no additional edge. Thus α is a monograph labeling for C_n. $\square$.

We define an l-chord on a cycle as a chord that subtends a path on the cycle of length l. Note that an l-chord may equally be referred to as an $(n-l)$-chord. The end points of a 2-chord have a common neighbour on the cycle.

Next, we show that all cycles with one 2-chord have a positive integer signature.

Theorem 3.2 *Every cycle C_n, $n > 3$ with one 2-chord is a monograph.*

Proof. Let C_n be a cycle with $n > 3$. Denote the consecutive vertices of the cycle C_n by $v_1, v_2, \ldots, v_n$ with a chord joining v_2 and v_n. Define the labeling α as follows.

$$\alpha(v_i) = \begin{cases} i & \text{if } i = 1, 2, \\ 3 & \text{if } i = n, \\ 3.2^{n-i} & \text{if } i = 4,\ 5, \ldots, n-1, \\ 3.2^{n-4} + 2 & \text{if } i = 3. \end{cases}$$

The vertex labels form a signature $1, 2, 3.2^{n-4}+2, 3.2^{n-4}, 3.2^{n-5}, \ldots, 3.2^{n-i}$ By checking all elements of this signature, we obtain one additional edge (that is the 2-chord) between v_2 and v_n, i.e., $|\alpha(v_2) - \alpha(v_n)| = 1 = \alpha(v_1)$. Thus a cycle with one 2-chord has a positive integer signature. $\square$.

In the following theorem, we generalise the result for a cycle which has more than one chord, to the case of several chords, where all the chords have the same initial point v_2.

Theorem 3.3 *All cycles C_n with r chords, $2 \leq r \leq n-3$, in which all chords share the same vertex, at least one is a 2-chord and the end vertices (not including the common vertex) form a continuous path, are monographs.*

Proof. Let C_n be a cycle with $n > 3$. Suppose that the cycle C_n has $r, r = 2, \ldots, n-3$ chords. Define the labeling α as follows.

$$\alpha(v_i) = \begin{cases} i & \text{if } \; i = 1, 2, \\ 3 & \text{if } \; i = n, \\ 3 + 2(n-i) & \text{if } \; i = n-1, n-2, .., n-r, \\ 2\alpha(v_{i+1}) & \text{if } i = n-r+1, \ldots, 4, \\ \alpha(v_4) + 2 & \text{if } \; i = 3. \end{cases}$$

The vertex labels form the signature $1, 2, 3, 5, 7, \ldots, 3+2(r-1), 2(3+2(r-1)), \ldots,$ $2^{n-r-4}(3+2(r-1))$ By observing all elements in the signature, we obtain r additional edges (that is, chords) between v_2 and $v_n, v_{n-1}, \ldots, v_{n-r}$. Thus such a cycle with r chords is a monograph. $\square$.

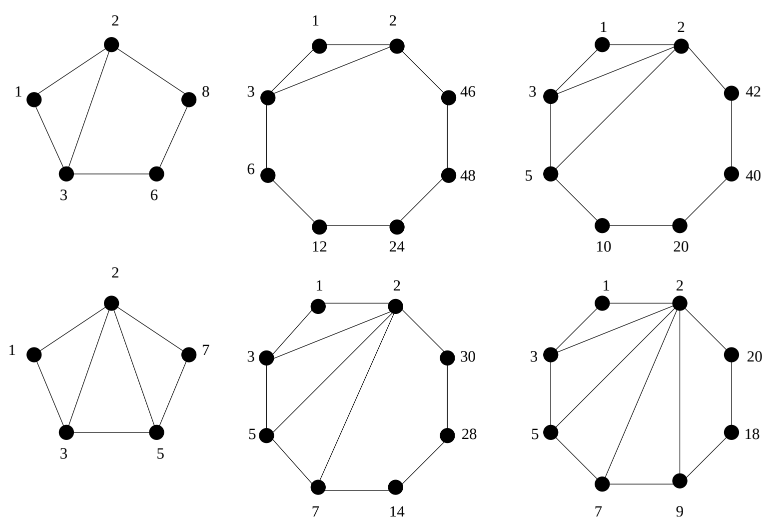

Figure 2: Examples of cycles with chords monographs.

A *fan* is a graph that can be generated by connecting all the vertices of a path P_n to one isolated vertex x; x is called the *centre* of the fan. Thus, a fan can be represented as $F_n = P_n + K_1$. Alternatively, a fan can be generated by adding $n-3$ chords to a cycle, where all chords have the same initial point. Using the previous theorem, we show that all fans are monographs. Figure 3 shows a signature for F_8.

Corollary 1 *Every fan F_n, $n \geq 2$, is a monograph.*

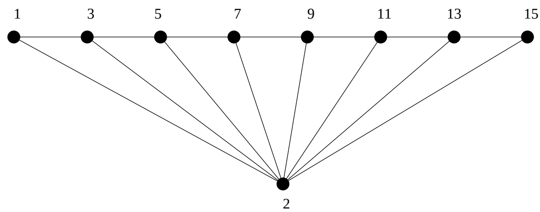

Figure 3: Monograph F_8.

3.2 Kite graphs

A *kite graph* K is a graph $G * P_m$, where G is a connected graph, and P_m is a path with m vertices, such that one of the end vertices of P_m coincides with one vertex of G. Thus, if $v_1, \dots, v_n$ are the vertices of G and $w_1, \dots, w_m$ are the consecutive vertices of P_m then we can assign $v_n = w_1$.

Theorem 3.4 *Every kite $G = K_n * P_m$ is a monograph.*

Proof. Label the vertices of G as follows.

$$\alpha(v) = \begin{cases} i & \text{if } v = v_i \in K_n,\ i = 1, 2, \dots, n; \\ 2^{j-1}n & \text{if } v = w_j \in P_m,\ j = 2, \dots, m. \end{cases}$$

From the definition, we can see that the positive monograph labels of vertices from K_n are labels of some vertices in K_n and the positive difference labels of vertices from P_m are labels from some vertices in P_m.

Next, we show that there is no additional edge between vertices in P_m or between a vertex in K_n and a vertex in P_m.

There is no additional edge among the tail vertices $V(P_m)$, since all vertices in the tail are of the form $n2^a$ for some a. If u, v are vertices in the tail, then $|\alpha(u) - \alpha(v)| = |n2^a - n2^b| = |n2^c|$, for some positive integers a and b. This can only happen when $a = 2b$ or $b = 2a$ which account for precisely the edges in the tail.

Suppose that there is an additional edge between vertex in the tail and vertex in the (original) complete graph. Let $u \in K_n$, $v \in P_m$ and $u \neq v$. Then $|\alpha(v) - \alpha(u)| = |2^{k-1}n - j|$ for some positive integer k and j. However, there is no vertex in G that has label $|2^{k-1}n - j|$. Thus, G is a monograph. $\square$

Using a similar argument as in the proof of Theorem 3.2, we can generalise the result as follow.

Theorem 3.5 *Let G be a monograph. Every kite graph $G * P_m$ where the tail begins on the largest labeled vertex of G is a monograph.*

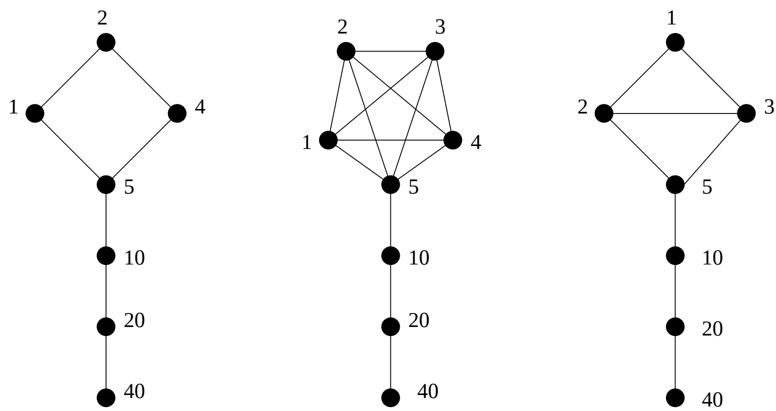

Figure 4: Examples of kite monographs.

3.2.1 Chain of monographs and necklace of monographs

Using a similar idea as in the kites monograph construction, we can construct a chain or a necklace of monographs. We define a *chain of graphs* $G_i,\ i = 1, \ldots, n$ as a graph that is constructed by gluing one vertex of G_i, say y, to one vertex of G_{i-1}, say x, and gluing a different vertex of G_i, say z, to one vertex of G_{i+1}, say w. Thus, $x = y$ and $z = w$. If G_n is connected to G_1, we call the chain a *necklace*. We denote a chain of graphs of $G_1, G_2, \ldots, G_n$ by $\mathbf{C}(G_i)_i = 1, \ldots, n$, and a necklace of the same graphs by $\mathbf{N}(G_i)_i = 1, \ldots, n$.

We first consider a chain of isomorphic monographs. Let G be a monograph. Choose $G_i = G$, for all $i = 1, \ldots, n$, for the chain. Glue the largest vertex label of G_i to the smallest vertex label of G_{i+1}, for $i = 1, \ldots, n-1$. To construct a signature of a chain of monographs, we have the following theorem.

Theorem 3.6 *Let $G_i = G,\ i = 1, \ldots, n$ be monographs. Then the chain of monographs $\mathbf{C}(G_i)$, $i = 1, \ldots, n$, is also a monograph.*

Proof. Let $G_i = G,\ i = 1, \ldots, n$, be monographs. Glue the largest vertex label of G_i to the smallest vertex label of G_{i+1}, for $i = 1, \ldots, n-2$ (or $n-1$). Multiply the signature of G_{i+1} by m_i, where m_i is the largest element of signature of G_i. In light of Observation 1, the new labeling of G_i will also be a signature of G_i. Continue the process until $i = n-1$.

Since every G_i is the same monograph G then we can guarantee that there is no additional edge inside each of the graphs, and since every signature of G_{i+1} is a multiple of signature of G_i then it will guarantee that there is no additional edge between one graph and another. Thus $\mathbf{C}(G_i)_i = 1, \ldots, n$, are monographs. $\square$.

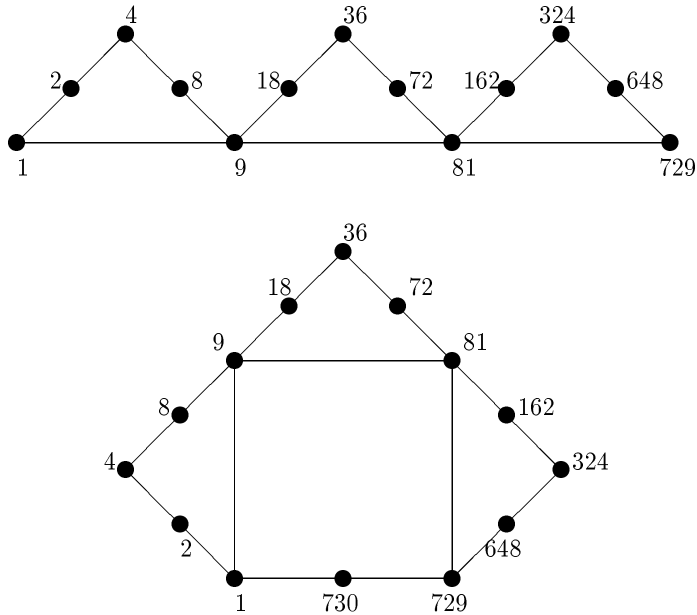

Figure 5: Examples of a chain of monographs and a necklace of monographs.

To construct a signature of a necklace of monographs, we need to choose G_n to be a path, as in the following result.

Theorem 3.7 *Let G_i, $i = 1, \ldots, n$, with G_n a path, be monographs. Then the necklace of monographs $\mathbf{N}(G_i)_i = 1, \ldots, n$, is a monograph.*

Proof. Using a similar technique as in the previous Theorem's proof, we can generate the signature construction α_i for $\mathbf{C}(G_i)_i = 1, \ldots, n-1$. To generate signature of a necklace $\mathbf{N}(G_i)_i = 1, \ldots, n$, define $G_n = P_m$, for some integer m. Let $x_1, \ldots, x_m$ be consecutive vertices of path P_m. Let x_1 be a vertex that is glued to G_{n-1}, and x_m be a vertex that is glued to G_1. Let α_1 and α_{n-1} be a monograph labeling for G_1 and G_{n-1}, respectively. Define a monograph labeling of P_m as follows. $\alpha(x_1) = max\{\alpha_{n-1}(v_i)|v_i \in G_{n-1}\}$, $\alpha(x_m) = min\{\alpha_1(w_i)|w_i \in G_1\}$ and $\alpha(x_j) = 2^{(j-1)}\alpha(x_1)$, for $j = 2, \ldots, x_{m-2}$, $\alpha(x_{m-1}) = \alpha(x_m) + \alpha(x_{m-2})$. Then $\mathbf{N}(G_i)_i = 1, \ldots, n$ is a monograph. $\square$

Figure 5 shows examples of a chain of monographs and a necklace of monographs. The chain example is generated by three C_5 and the example of the necklace is generated by three C_5 and P_3.

4 Conclusion

We proved and gave a monograph construction of cycles, cycles with chords, fan graphs F_n, kite graphs, chains of monographs and necklaces of monographs. To characterise classes of graphs that can be or cannot be a monograph is an interesting open problem for further research.

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