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Ternary codes through ternary designs

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Abstract

It is known that under certain conditions the incidence matrix of a balanced incomplete block design (v, b, r, k, λ) gives a binary code of length b and size $2(v+1)$. Here we investigate the conditions where a balanced ternary design gives a similar ternary code.

1 Introduction

Definition 2.1. A **balanced incomplete block design** (BIBD), denoted by (v, b, r, k, λ) is a finite collection B of b subsets (called blocks) of some finite set S with v elements, such that each block has k elements, each element of S appears in r blocks, and each pair of distinct elements from S appears in λ blocks.

A **balanced ternary design** or BTD is defined the same way as a BIBD; however elements may appear twice within a given block (which are therefore multisets). More formally, we say that a BTD has parameters $(V, B; \rho_1, \rho_2, R; K, \Lambda)$ if S has V elements such that the following properties hold: (1) each element appears only once in ρ_1 blocks and twice in exactly ρ_2 blocks; (2) There are B blocks containing K elements; (3) Each element appears R times altogether; (4) Each pair of elements appears Λ times.

In Fujitake, Kageyama, and Shimata [1], it was shown under which conditions a binary error correcting code results from the following structure:

$$\begin{bmatrix} \vec{0}_b \\ N \\ J_{v\times b} - N \\ \vec{1}_b \end{bmatrix}$$

where N is the incidence matrix of a BIBD, and $J_{v\times b} = (\vec{1}_v)^T\ \vec{1}_b$. In this paper we discuss some sufficient conditions on ternary designs such that the incidence matrix of a ternary design gives a ternary error-correcting code under the same structure. Adopting the notation from Fujitake, Kageyama, and Shimata [1], we call the above structure $(*)$.

2 Results

The first thing to consider is the possibility of two identical code words resulting from the structure $(*)$. We assume that in the process of computing $J - N$, subtraction is done modulo 3, so that -1 will be replaced by 2.

Theorem 2.1. *If two rows $\vec{m}, \vec{n}$ from the incidence matrix N of a balanced ternary design satisfy $\vec{m} = \vec{1} - \vec{n}$, then the design's parameters are*

$$(\frac{\rho_2 + 2\rho_1}{\rho_1 - 4\rho_2}, 2\rho_1 + \rho_2; \rho_1, \rho_2, \rho_1 + 2\rho_2; \frac{\rho_1 + 2\rho_2}{\rho_1 - 4\rho_2}, 4\rho_2).$$

Proof. Since $\vec{m} = \vec{1} - \vec{n}$ we have, without loss of generality,

$$\vec{m} = 22\ldots211\ldots100\ldots0$$
$$\vec{n} = 22\ldots200\ldots011\ldots1$$

We conclude that $\Lambda = 4\rho_2$ since all the twos line up and none of the ones line up. We also have $B = \rho_2 + 2\rho_1$ by counting the length of $\vec{m}$. The rest of the parameters follow from the design conditions $VR = BK$ and $\Lambda(V-1) = \rho_1(K-1) + 2\rho_2(K-2)$. □

Since we want to find out when $(*)$ gives a ternary error-correcting code, we need to investigate the minimum distance between code words given by $(*)$. We will have to find a lower bound on the minimum distance of any two code words given by $(*)$. Thus, we will sometimes need to assume the worst thing possible in relation to distance between codewords (the lowest possible distance). This way our result will encompass all BTDs, and their behavior in $(*)$.

Lemma 2.2. *The distance between the two rows $\vec{0}$ and $\vec{r}$, where $\vec{r}$ is a row from the incidence matrix of a BTD, is $\rho_1 + \rho_2$.*

Lemma 2.3. *The distance between two rows $\vec{0}$ and $\vec{r}'$ where $\vec{r}'$ is a row from $J_{vxb} - N$, where N is the incidence matrix of a BTD, is $B - \rho_1$.*

Lemma 2.4. *The distance between two rows $\vec{1}$ and $\vec{r}$, where $\vec{r}$ is from N, the incidence matrix of a BTD, is $B - \rho_1$.*

Lemma 2.5. *The distance between two rows $\vec{1}$ and $\vec{r}'$, where $\vec{r}'$ is a row from $J_{vxb} - N$, where N is the incidence matrix of a BTD, is $\rho_1 + \rho_2$.*

Lemma 2.6. *The distance between two rows $\vec{r}, \vec{r}'$, where $\vec{r}$ is from the incidence matrix of a BTD and $\vec{r}' = \vec{1} - \vec{r}$ is $B - \rho_2$.*

Lemma 2.7. *The distance between two distinct rows $\vec{m}, \vec{n}$ from N, the incidence matrix of a BTD, is greater than $2(\rho_1 + \rho_2 - \Lambda)$ if $\Lambda \leq \rho_1$ or $2(\rho_2 - \frac{\Lambda - \rho_1}{4})$ if $\Lambda > \rho_1$.*

Proof. In any BTD the least possible distance between two rows happens when the number of positions where both rows have a '1' is maximized. In the following discussion we consider only this case, assuming least possible distance. Suppose that

$\Lambda \le \rho_1$. Then, Λ ones line up in corresponding positions between $\vec{m}$ and $\vec{n}$ and these are the only places where nonzero entries line up. However there are still $\rho_1 - \Lambda$ ones and ρ_2 twos in both $\vec{m}$ and $\vec{n}$. Thus the distance is $2(\rho_1 + \rho_2 - \Lambda)$.

Now, suppose that $\Lambda > \rho_1$. Now we have four cases to consider:

Case #1: $(\Lambda - \rho_1) \equiv 0 \pmod 4$.
In this case, all the ones line up in $\vec{m}$ and $\vec{n}$ and some twos line up. The number of twos that line up is $\frac{\Lambda-\rho_1}{4}$. This is the case because the inner product of $\vec{m}$ and $\vec{n}$ must be Λ. Without loss of generality, the structure is:

$$\begin{array}{c} 11\ldots122\ldots222\ldots200\ldots000\ldots0 \\ 11\ldots122\ldots200\ldots022\ldots220\ldots0 \end{array}$$

So we have the distance between the two is

$$2(\rho_2 - \frac{\Lambda-\rho_1}{4}). \tag{1}$$

Case #2: $(\Lambda - \rho_1) \equiv 1 \pmod 4$.
In this case $\rho_1 - 1$ ones line up between $\vec{m}$ and $\vec{n}$, in one place a 1 lines up with a 2, and some 2s line up. This happens as a natural consequence of maximizing the number of 1s that line up. To see this, notice that $(\Lambda - \rho_1) \pmod 4 = 1$. If all the ones lined up, then the remaining part $\Lambda - \rho_1$ would not be divisible by 4. So even if as many twos lined up as possible, the inner product between $\vec{m}$ and $\vec{n}$ will still only be $\Lambda - 1$. But we need the inner product to be Λ so if one 1 lines up with a 2 and all the other ones line up then we can line up some twos to get Λ. Without loss of generality, the structure is:

$$\begin{array}{c} 11\ldots11022\ldots222\ldots200\ldots000\ldots0 \\ 11\ldots12122\ldots200\ldots022\ldots220\ldots0 \end{array}$$

The distance is $2 + \rho_2 - \frac{\Lambda-\rho_1-1}{4} + \rho_2 - \frac{\Lambda-\rho_1-1}{4} - 1 = 1 + 2(\rho_2 - \frac{\Lambda-\rho_1-1}{4}) =$

$$\frac{3}{2} + 2(\rho_2 - \frac{\Lambda-\rho_1}{4}) \tag{2}$$

Case #3: $(\Lambda - \rho_1) \equiv 2 \pmod 4$.
In this case $\rho_1 - 2$ ones line up between $\vec{m}$ and $\vec{n}$, and some twos line up. It may also be the case that 2 ones line up with 2 twos. It can be shown that this would give the same distance. We consider the former case where no one lines up with a two. Here we have that $\rho_1 - 2$ ones line up so 4 ones line up with zeros. Also $\frac{\Lambda-\rho_1+2}{4}$ twos must line up so $2(\rho_2 - \frac{\Lambda-\rho_1-2}{4} - 1)$ twos line up with zeros. Without loss of generality, the structure is:

$$\begin{array}{c} 11\ldots1110022\ldots222\ldots200\ldots000\ldots0 \\ 11\ldots1001122\ldots200\ldots022\ldots220\ldots0 \end{array}$$

We have that the distance between $\vec{m}$ and $\vec{n}$ is $4 + 2(\rho_2 - \frac{\Lambda-\rho_1-2}{4} - 1) = 2 + 2(\rho_2 - \frac{\Lambda-\rho_1-2}{4}) =$

$$3 + 2(\rho_2 - \frac{\Lambda-\rho_1}{4}) \tag{3}$$

Case #4: $(\Lambda - \rho_1) \equiv 3 \pmod 4$.
In this case $\rho_1 - 1$ ones line up between m and n, and some twos line up. Hence we have that 2 ones line up with zeros. Also $\frac{\Lambda-\rho_1+1}{4}$ twos line up, so $2(\rho_2 - \frac{\Lambda-\rho_1+1}{4})$ twos line up with zeros. Without loss of generality, the structure is:

$$\begin{array}{c} 11\ldots11022\ldots222\ldots200\ldots000\ldots0 \\ 11\ldots10122\ldots200\ldots022\ldots220\ldots0 \end{array}$$

We have that the distance between $\vec{m}$ and $\vec{n}$ is $2 + 2(\rho_2 - \frac{\Lambda-\rho_1+1}{4}) =$

$$\frac{3}{2} + 2(\rho_2 - \frac{\Lambda - \rho_1}{4}) \tag{4}$$

It is clear that the minimum of (1), (2), (3), and (4) is $2(\rho_2 - \frac{\Lambda-\rho_1}{4})$. □

Corollary 2.8. *The distance between two distinct rows $\vec{m}\,', \vec{n}\,'$ from $J_{vxb} - N$, where N is the incidence matrix of a BTD, is greater than $2(\rho_1 + \rho_2 - \Lambda)$ if $\Lambda \leq \rho_1$ or $2(\rho_2 - \frac{\Lambda-\rho_1}{4})$ if $\Lambda > \rho_1$.*
Proof. This is due to the fact that the distance between $\vec{m}\,'$ and $\vec{n}\,'$ is equal to the distance between $\vec{m}$ and $\vec{n}$ where $\vec{n} = \vec{1} - \vec{n}\,'$ and $\vec{m} = \vec{1} - \vec{m}\,'$. However, $\vec{m}$ and $\vec{n}$ are distinct rows from N, hence we have the desired result. □

Theorem 2.9. *The distance between two distinct rows $\vec{m}, \vec{n}\,'$, where $\vec{m}$ is from N, the incidence matrix of a BTD, and $\vec{n}\,'$ is from $J_{vxb} - N$, is greater than $B - 9\lfloor\frac{\Lambda}{4}\rfloor - 2\rho_1 + 2\Lambda - 3\lfloor\frac{\Lambda \bmod 4}{2}\rfloor$ if $\Lambda \leq 4\rho_2$ or $B + 2\Lambda - 9\rho_2 - 2\rho_1$ if $\Lambda \geq 4\rho_2$.*
Proof. Since $\vec{n}\,'$ is from $J_{vxb} - N$, there is a corresponding row $\vec{n}$ from N such that $\vec{n}\,' = \vec{1} - \vec{n}$. Intuititively, $\vec{n}\,'$ is $\vec{n}$ with the ones and zeros swapped. Let us examine the general structure of $\vec{m}$ and $\vec{n}$ lined up one on top of the other, with similar patterns grouped into blocks:

$$\begin{array}{c} 22\ldots2|22\ldots2|22\ldots2|11\ldots1|11\ldots1|11\ldots1|00\ldots0|00\ldots0|00\ldots0 \\ 22\ldots2|11\ldots1|00\ldots0|22\ldots2|11\ldots1|00\ldots0|22\ldots2|11\ldots1|00\ldots0 \end{array}$$

Now we examine the corresponding general structure of $\vec{m}$ and $\vec{n}\,'$:

$$\begin{array}{c} 22\ldots2|22\ldots2|22\ldots2|11\ldots1|11\ldots1|11\ldots1|00\ldots0|00\ldots0|00\ldots0 \\ 22\ldots2|00\ldots0|11\ldots1|22\ldots2|00\ldots0|11\ldots1|22\ldots2|00\ldots0|11\ldots1 \end{array}$$

Clearly the only blocks that do not add to the distance between the codewords $\vec{m}$ and $\vec{n}\,'$ are 1, 6, and 8. If we are looking for a lower bound on distance, we would want to maximize the size of these blocks, that is, look for the worst case (when the size of these blocks are biggest). This is equivalent to maximizing the size of blocks 1, 6, and 8 among the former structure comparison between $\vec{m}$ and $\vec{n}$. Clearly to do this, we want to line up as many corresponding twos as possible. This directly maximizes block 1; however it eliminates the need to line up a lot of ones, so lots of ones will line up with zeros, indirectly maximizing blocks 6 and 8. We must think about the case where lining up twos does not suffice to produce Λ. This could occur if all the twos are lined up but Λ is not achieved, or if Λ is not divisible by 4 so something other than twos need to be lined up. The latter case is very common. We will always line up as many twos as possible. If we use all the twos up doing so (former case), then we simply line up as many ones as are left. However if we have enough twos but the problem is that Λ is not divisible by 4, then we still line up as many twos as possible but we still may have 1, 2, or 3 left over to reach Λ. If it is 1 we have no choice but to line up one pair of ones. Otherwise we may line up 1 one with a two, or line up several ones. It is best to line up 1 one with a two, because this indirectly increases the size of blocks 6 and 8 in comparison to only lining up ones which decreases the size of blocks 6 and 8.

Case #1: $\Lambda \leq 4\rho_2$: As discussed, we want to line up as many twos as possible. We can see that we want to line up $\lfloor \frac{\Lambda}{4} \rfloor$ twos. Now if $\Lambda \pmod 4$ is 2 or 3 we want to line up a two with a one, so in general we want to line up $\lfloor \frac{\Lambda \pmod 4}{2} \rfloor$ twos from m with ones from n. This number will be either 0 or 1, and corresponds to block 2 from the general structure above. So there will be $\rho_2 - \lfloor \frac{\Lambda}{4} \rfloor - \lfloor \frac{\Lambda \bmod 4}{2} \rfloor$ twos left in $\vec{m}$ that will need to be lined up with zeros from $\vec{n}$ (block 3) and $\rho_2 - \lfloor \frac{\Lambda}{4} \rfloor$ twos from $\vec{n}$ that will need to be lined up with zeros from $\vec{m}$ (block 7). However Λ still may not be achieved. Hence we need to line up $\Lambda - 4\lfloor \frac{\Lambda}{4} \rfloor - 2\lfloor \frac{\Lambda \bmod 4}{2} \rfloor$ ones between $\vec{m}$ and $\vec{n}$. This number will again be either zero or one and corresponds to block 5 from the general structure above. We do not line up any ones from $\vec{m}$ with twos from $\vec{n}$; thus block 4 from above is nonexistent. We can see from the general structure of $\vec{m}$ and $\vec{n}\,'$ that the only blocks adding distance to these codewords are 2,3,4,5,7 and 9. We have found the size of blocks 2,3,4,5 and 7, but what about 9? The size of block 9 is just the block size B minus the size of all the other blocks. To find this we need to consider the size of blocks 6 and 8. The size of block 6 equals the number of ones from $\vec{m}$ lined up with zeros from $\vec{n}$. But ones from $\vec{m}$ are either lined up with ones from $\vec{n}$ or zeros from $\vec{n}$ and we know how many are lined up with ones from $\vec{n}$; hence the size of block 6 is $\rho_1 - \Lambda + 4\lfloor \frac{\Lambda}{4} \rfloor + 2\lfloor \frac{\Lambda \bmod 4}{2} \rfloor$. The size of block 8 equals the number of ones from $\vec{n}$ lined up with zeros from $\vec{m}$. Ones from $\vec{m}$ may be lined up with either zeros, ones, or twos from $\vec{n}$ but we already know how many are lined up with ones and twos. Hence the size of block 8 is $\rho_1 - \Lambda + 4\lfloor \frac{\Lambda}{4} \rfloor + \lfloor \frac{\Lambda \bmod 4}{2} \rfloor$. Finding the distance is now simplified to adding the size of blocks 2,3,4,5,7 and 9, which gives the distance

$$B - 9\left\lfloor \frac{\Lambda}{4} \right\rfloor - 2\rho_1 + 2\Lambda - 3\left\lfloor \frac{\Lambda \pmod 4}{2} \right\rfloor$$

Case #2: $\Lambda \geq 4\rho_2$: This case is simpler because we just line up all the twos from $\vec{m}$ and $\vec{n}$ and line up as many ones from $\vec{m}$ and $\vec{n}$ that are needed. We need to line up ρ_2 twos (block 1), $\Lambda - 4\rho_2$ ones (block 5), $2(\rho_1 - \Lambda + 4\rho_2)$ ones with zeros (blocks 6 and 8 combined), and $B - 5\rho_2 - 2\rho_1 + \Lambda$ zeros (block 9). All other blocks are nonexistent in this case. Adding the size of blocks 2,3,4,5,7, and 9 gives the distance

$$B + 2\Lambda - 9\rho_2 - 2\rho_1$$

This concludes case #2. □

Summary 2.10. *If N is the incidence matrix of a BTD satisfying $\Lambda \leq \rho_1$ and $\Lambda \leq 4\rho_2$ then N under the structure $*$ gives a one-error correcting ternary code if the* *every element of the set* $\{\rho_1+\rho_2,\ B-\rho_1,\ B-\rho_2,\ 2(\rho_1+\rho_2-\Lambda),\ B-9\lfloor\frac{\Lambda}{4}\rfloor-2\rho_1+2\Lambda-3\lfloor\frac{\Lambda \bmod 4}{2}\rfloor\}$ *is greater than or equal to* 3.

Summary 2.11. *If* N *is the incidence matrix of a BTD satisfying* $\Lambda>\rho_1$ *and* $\Lambda\le 4\rho_2$ *then* N *under the structure* $*$ *gives a one-error correcting ternary code if the every element of the set* $\{\rho_1+\rho_2,\ B-\rho_1,\ B-\rho_2,\ 2(\rho_2-\frac{\Lambda-\rho_1+1}{4}),\ B-9\lfloor\frac{\Lambda}{4}\rfloor-2\rho_1+2\Lambda-3\lfloor\frac{\Lambda \bmod 4}{2}\rfloor\}$ *is greater than or equal to* 3.

Summary 2.12. *If* N *is the incidence matrix of a BTD satisfying* $\Lambda\le\rho_1$ *and* $\Lambda\ge 4\rho_2$ *then* N *under the structure* $*$ *gives a one-error correcting ternary code if the every element of the set* $\{\rho_1+\rho_2,\ B-\rho_1,\ B-\rho_2,\ 2(\rho_1+\rho_2-\Lambda),\ B+2\Lambda-9\rho_2-2\rho_1\}$ *is greater than or equal to* 3.

Summary 2.13. *If* N *is the incidence matrix of a BTD satisfying* $\Lambda>\rho_1$ *and* $\Lambda\ge 4\rho_2$ *then* N *under the structure* $*$ *gives a one-error correcting ternary code if the every element of the set* $\{\rho_1+\rho_2,\ B-\rho_1,\ B-\rho_2,\ 2(\rho_2-\frac{\Lambda-\rho_1+1}{4}),\ B+2\Lambda-9\rho_2-2\rho_1\}$ *is greater than or equal to* 3.

3 Examples

Example 2.14. Suppose we are given the following $BTD(19,19;6,1,8;8,3)$:

$$\begin{bmatrix}
2&0&0&0&0&0&0&0&1&0&1&0&1&1&0&1&0&0&1\\
1&2&0&0&0&0&0&0&0&1&0&1&0&1&1&0&1&0&0\\
0&1&2&0&0&0&0&0&0&0&1&0&1&0&1&1&0&1&0\\
0&0&1&2&0&0&0&0&0&0&0&1&0&1&0&1&1&0&1\\
1&0&0&1&2&0&0&0&0&0&0&0&1&0&1&0&1&1&0\\
0&1&0&0&1&2&0&0&0&0&0&0&0&1&0&1&0&1&1\\
1&0&1&0&0&1&2&0&0&0&0&0&0&0&1&0&1&0&1\\
1&1&0&1&0&0&1&2&0&0&0&0&0&0&0&1&0&1&0\\
0&1&1&0&1&0&0&1&2&0&0&0&0&0&0&0&1&0&1\\
1&0&1&1&0&1&0&0&1&2&0&0&0&0&0&0&0&1&0\\
0&1&0&1&1&0&1&0&0&1&2&0&0&0&0&0&0&0&1\\
1&0&1&0&1&1&0&1&0&0&1&2&0&0&0&0&0&0&0\\
0&1&0&1&0&1&1&0&1&0&0&1&2&0&0&0&0&0&0\\
0&0&1&0&1&0&1&1&0&1&0&0&1&2&0&0&0&0&0\\
0&0&0&1&0&1&0&1&1&0&1&0&0&1&2&0&0&0&0\\
0&0&0&0&1&0&1&0&1&1&0&1&0&0&1&2&0&0&0\\
0&0&0&0&0&1&0&1&0&1&1&0&1&0&0&1&2&0&0\\
0&0&0&0&0&0&1&0&1&0&1&1&0&1&0&0&1&2&0\\
0&0&0&0&0&0&0&1&0&1&0&1&1&0&1&0&0&1&2
\end{bmatrix}.$$

Summary 2.10. tells us that this produces a one error-correction code (corrects at least 1 error, this one corrects more). The code that is constructed is:

{ 0000000000000000000, 2000000010101101001, 1200000001010110100,
0120000000101011010, 0012000000010101101, 1001200000001010110,
0100120000000101011, 1010012000000010101, 1101001200000001010,
0110100120000000101, 1011010012000000010, 0101101001200000001,
1010110100120000000, 0101011010012000000, 0010101101001200000,
0001010110100120000, 0000101011010012000, 0000010101101001200,
0000001010110100120, 0000000101011010012, 2111111101010010110,
0211111110101001011, 1021111111010100101, 1102111111101010010,
0110211111110101001, 1011021111111010100, 0101102111111101010,
0010110211111110101, 1001011021111111010, 0100101102111111101,
1010010110211111110, 0101001011021111111, 1010100101102111111,
1101010010110211111, 1110101001011021111, 1111010100101102111,
1111101010010110211, 1111110101001011021, 1111111010100101102,
1111111111111111111 }.

If we look closely at the results used to justify Summary 2.10 we see that the minimum distance of this code is guaranteed to be greater than the minimum of $\{\rho_1+\rho_2,\, B-\rho_1,\, B-\rho_2,\, 2(\rho_1+\rho_2-\Lambda),\, B-9\lfloor\frac{\Lambda}{4}\rfloor-2\rho_1+2\Lambda-3\lfloor\frac{\Lambda \bmod 4}{2}\rfloor\} = \{7, 13, 18, 8, 10\}$. We know from Lemmas 2.2, 2.3, 2.4, 2.5, and 2.6 that there will be codewords with minimum distance 7, 13, and 18 (so this code is really a three error-correcting code); however Theorem 2.7 states that the distance between two codewords both from the incidence matrix N is greater than or equal to 8. Also, Theorem 2.9 ensures the distance between codewords $\vec{m}$ and $\vec{n}\,'$ where $\vec{m}$ is from N and $\vec{n}\,'$ is from $J_{vxb}-N$ is greater than or equal to 10. Since these are lower bounds it is natural to ask how good are these bounds. Is equality every achieved? This example shows that in both of these bounds equality can be achieved. The two codewords

2000000010101101001   0012000000010101101

have distance 8 and equality is satisfied in the first bound and

0012000000010101101   0100101102111111101

are codewords of distance 10 that prove equality can occur in the second bound. In both cases, just picking two codewords from the specified place will not guarantee equality because, for example

2000000010101101001   1010012000000010101

are two codewords from N with distance $11 > 8$, while

2000000010101101001   1111110101001011021

are two codewords with distance $13 > 10$ where one is from N and the other from $J_{vxb}-N$.

Example 2.15. Using Billington and Robinson [2], we can construct a $BTD(4,8;\,2,3,8;4,6)$. Summary 2.11 tells us that this produces a one error-correcting code. It satisfies the conditions: $\rho_1+\rho_2 = 2+3 = 5 \geq 3$, $B-\rho_1 = 8-2 = 6 \geq$

3, $B - \rho_2 = 8 - 3 = 5 \geq 3$, $2(\rho_2 - \frac{\Lambda - \rho_1 + 1}{4}) = 2(3 - \frac{5}{4}) = 2(\frac{7}{4}) = \frac{7}{2} \geq 3$, $B - 9\lfloor \frac{\Lambda}{4} \rfloor - 2\rho_1 + 2\Lambda - 3\lfloor \frac{\Lambda \bmod 4}{2} \rfloor = 8 - 9(1) - (2)(2) + (2)(6) - (3)(1) = 4 \geq 3$, thus it is one error-correcting. The code that is constructed is:

$$\begin{array}{llllll} \{ & 00000000, & 11222000, & 11200220, & 11020202, & 11002022, \\ & 00222111, & 00211221, & 00121212, & 00112122, & 11111111 \quad \}. \end{array}$$

Example 2.16. Using Billington and Robinson [2], we can construct a BTD with the parameters $(9, 18; 10, 2, 14, 7, 10)$ Summary 2.12 does not help because the lower bound on the distance is less than zero. However, the code is at least a one error-correcting code:

$$\begin{array}{llll} \{ & 000000000000000000, & 210110011201101011, & 121011001120110101, \\ & 112101100112011010, & 011210110011201101, & 001121011101120110, \\ & 100112101010112011, & 110011210101011201, & 011001121110101120, \\ & 101100112011010112, & 201001100210010100, & 020100110021001010, \\ & 002010011002100101, & 100201001100210010, & 110020100010021001, \\ & 011002010101002100, & 001100201010100210, & 100110020001010021, \\ & 010011002100101002, & 111111111111111111 & \}. \end{array}$$

The lower bound given by Summary 2.12 is not ideal in this example because the assumption that as many 2s as possible will line up. In this example no pairs of 2s line up so we actually get good distances.

Example 2.17. Using Billington and Robinson [2], we can construct a BTD with the parameters $(3, 11; 7, 2, 11; 3, 9)$. Summary 2.13 does not help because the lower bound on the distance is less than zero. However when we construct the code we do get a one error-correcting code:

$$\begin{array}{lllll} \{ & 00000000000, & 20120111111, & 12012011111, & 01201211111 \\ & 21021000000, & 02102100000, & 10210200000, & 11111111111 \quad \}. \end{array}$$

The minimum distance of the code is 4 so it is one error-correcting. Why does Summary 2.13 have a lower bound less than zero? It has to do with the case of the distance between a row $\vec{m}$ of the incidence matrix N and a row $\vec{n}'$ from $J_{vxb} - N$. Then it assumes that all the twos line up between $\vec{m}$ and $\vec{n} = \vec{1} - \vec{n}'$. If this happened in this design then two pairs of 2s would line up and one pair of 1s would line up. Then the distance between $\vec{m}$ and $\vec{n}'$ would be negative. However, this is clearly impossible since more than one pair of 1s must line up. Hence two pairs of 2s cannot possibly line up. This suggests that some additional results might be obtained under the conditons of Summary 2.13 where the additional condition of $\rho_1 - \Lambda + 4\rho_2 > B - \rho_1 - \rho_2$ is satisfied. However, under these conditions it becomes very complicated to describe the distances of codewords for general BTDs. Since the minimum distance of the code is 4, no more than 1 error can be corrected.

The four preceding examples illustrate each of the four summaries. It is clear that while Summary 2.10 and Summary 2.11 give very good lower bounds, Summary 2.12 and Summary 2.13 give lower bounds that are sometimes not very helpful. These summaries (along with their associated theorems) are useful from a theoretical viewpoint since they provide at the very least a starting point for the theory under those conditions and they do provide lower bounds. However, for the practitioner only interested in creating Ternary codes, the lower bounds of Summaries 2.12 and 2.13 associated with Theorem 2.9 (and in some cases Theorem 2.7) should not discourage the use of certain designs.

References

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