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On generalized weights for codes over $\mathbb{Z}_k$

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Abstract

We generalize the definition of higher weights for codes over $\mathbb{Z}_k$ and define weight enumerators corresponding to these weights. We provide MacWilliams relations for the weight enumerators. We define generalized Lee weights for a linear code over $\mathbb{Z}_4$ and give bounds for these weights. Moreover, we determine these weights for some codes over $\mathbb{Z}_4$.

1 Introduction

For a linear code over a finite field, Helleseth, Kløve and Mykkeltveit ([14]) introduced generalized Hamming weights while studying the weight distribution of irreducible cyclic codes and later Wei ([26]) rediscovered the idea of generalized Hamming weights. Following these, numerous papers dealing with these weights have been published (cf. [12, 25], etc.). Recently, generalized Hamming weights for codes over $\mathbb{Z}_4$ have been defined and studied (see [1, 4, 15, 27, 28], for example). In this work, we generalize the definition of higher weight enumerators for a linear code over $\mathbb{Z}_k$ and prove MacWilliams relations for this weight enumerator.

The Lee weight of a codeword plays an important role in studying a code over $\mathbb{Z}_4$. The Lee weight of a codeword over $\mathbb{Z}_4$ corresponds to the Hamming weight of its binary Gray map image (cf. [11]). Additionally, we give an alternate definition for the higher weight of a linear code over $\mathbb{Z}_4$ to the one that has been given in [1, 4, 15, 27, 28]. In [18], Hove studied the concept of generalized Lee weights for codes over $\mathbb{Z}_4$ with respect to the order of a code. Our definition of generalized Lee weights is another natural extension of generalized Hamming weights.

2 Definitions and Notation

2.1 Generalized Hamming Weights

Let $\mathbb{Z}_k$ be the ring of integers modulo k. A *code* of length n over $\mathbb{Z}_k$ is a subset of the free module $\mathbb{Z}_k^n$ and the code is *linear* if it is a $\mathbb{Z}_k$-submodule of $\mathbb{Z}_k^n$.

For $\boldsymbol{v}, \boldsymbol{x} \in \mathbb{Z}_k^n$, we define the *inner product* by

$$[\boldsymbol{v}, \boldsymbol{x}] = \sum v_i x_i.$$

For a linear code C of length n over $\mathbb{Z}_k$, we define the *rank* of C, denoted by $\mathrm{rank}(C)$, to be the minimum number of generators of C and define the *free rank* of C, denoted by f-rank(C), to be the maximum of the ranks of $\mathbb{Z}_k$-free submodules of C (cf. [9, 23]). We shall say that a linear code is *free* if the free rank is equal to the rank, that is, a code is a free $\mathbb{Z}_k$-submodule.

Define the following norm for a vector $\boldsymbol{v} \in \mathbb{Z}_k^n$:

$$||\boldsymbol{v}|| = |\mathrm{supp}(\boldsymbol{v})|$$

where

$$\mathrm{supp}(\boldsymbol{v}) = \{i : v_i \neq 0\}.$$

We extend this norm to subcodes, specifically let C be a linear code of length n and let D be any subset of C. Define

$$||D|| = |\mathrm{supp}(D)|,$$

where

$$\begin{aligned}\mathrm{supp}(D) &= \{i : \text{ there exists } \boldsymbol{v} \in D \text{ with } \ v_i \neq 0\} \\ &= \bigcup_{\boldsymbol{v} \in D} \mathrm{supp}(\boldsymbol{v}).\end{aligned}$$

For a linear code C over a ring $\mathbb{Z}_k$ and any g, $1 \leq g \leq \mathrm{rank}(C)$, we define the g-th *generalized Hamming weight with respect to rank* (GHWR) as follows:

$$d_g^H(C) = \min\{||D|| : \ D \text{ is a } \mathbb{Z}_k\text{-submodule of } C \text{ with } \mathrm{rank}(D) = r\}.$$

We note that the minimum Hamming weight of a linear code C is $d_1^H(C)$. In [17], they introduced the GHWR of a linear code C over a finite chain ring and studied some properties of the GHWR.

For any g, $1 \leq g \leq \mathrm{rank}(C)$, we define the higher weight spectrum as

$$A_i^g = |\{D : D \text{ is a } \mathbb{Z}_k\text{-submodule of } C \text{ with } \mathrm{rank}(D) = g \text{ and } ||D|| = i\}|$$

which naturally gives higher weight enumerators

$$W_C^g(x,y) = \sum A_i^g x^{n-i} y^i.$$

These definitions are of course, the natural extensions of the definitions used for codes over finite fields. The next two extensions are a broader generalization of these ideas.

Let $a_1, a_2, \ldots, a_s$ be the divisors of k, with $a_i < a_j$. This forces $a_1 = 1$. Any linear code over $\mathbb{Z}_k$ has a generator matrix which can be put in the following form (cf. [2]):

$$\begin{pmatrix} a_1 I_{k_1} & A_{1,2} & A_{1,3} & A_{1,4} & \cdots & \cdots & A_{1,s+1} \\ 0 & a_2 I_{k_2} & a_2 A_{2,3} & a_2 A_{2,4} & \cdots & \cdots & a_2 A_{2,s+1} \\ 0 & 0 & a_3 I_{k_3} & a_3 A_{3,4} & \cdots & \cdots & a_3 A_{3,s+1} \\ \vdots & \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_s I_{k_s} & a_s A_{s,s+1} \end{pmatrix},$$

where $A_{i,j}$ are binary matrices for $i > 1$. A linear code of this form is said to be of *type* $\{k_1, k_2, k_3, \ldots, k_s\}$ and has $\prod_{i=1}^{s} \frac{k}{a_i}^{k_i}$ elements.

Moreover, define

$$\delta_{k_1,\ldots,k_s}(C) = \min\{||D|| : D \text{ is a } \mathbb{Z}_k\text{-submodule of } C \text{ with } \mathrm{type}(D) = \{k_1, \ldots, k_s\}\}.$$

We extend the definition of the higher weight spectrum as

$$A_i^{k_1,k_2,\ldots,k_s} = |\{D : D \text{ is a } \mathbb{Z}_k\text{-submodule of } C \text{ with } \mathrm{type}(D) = \{k_1, \ldots, k_s\} \text{ and } ||D|| = i\}|$$

which naturally extends higher weight enumerators as follows:

$$W_C^{k_1,\ldots,k_s}(x,y) = \sum A_i^{k_1,\ldots,k_s} x^{n-i} y^i.$$

Hence for each type we have a weight enumerator.

If C is a linear code over $\mathbb{F}_2 + u\mathbb{F}_2$ or $\mathbb{Z}_4$ then the image under the corresponding Gray map of a linear subcode D has support $2||D||$, since any non-zero coordinate is mapped to two non-zero coordinates. Of course, it is necessary for the subcode to be linear for this to be true. If the ring is $\mathbb{F}_2 + u\mathbb{F}_2$ then the image is linear, but in neither case would it account for all binary subcodes. For example, the image of the ambient space of length 1 over $\mathbb{F}_2 + u\mathbb{F}_2$ is $\mathbb{F}_2^2$, but the subcode $\{00, 10\}$ is a binary subcode but corresponds to a non-linear subcode of $\mathbb{F}_2 + u\mathbb{F}_2$.

2.2 Generalized Lee Weights

It is known that a linear code C of length n over $\mathbb{Z}_4$ is permutation-equivalent to a linear code with generator matrix of the form

$$(1) \qquad \begin{pmatrix} I_{k_1} & X & Y \\ 0 & 2I_{k_2} & 2Z \end{pmatrix},$$

where X and Z are binary matrices and Y is a matrix over $\mathbb{Z}_4$. In this case, it gives that $|C| = 4^{k_1}2^{k_2}$ and $\mathrm{rank}(C) = k_1 + k_2$. We shall define a code with a generator matrix of the form given in matrix (1) as being of type $\{k_1, k_2\}$ and then say C is an $[n; k_1, k_2]$ code. Sometimes we also write (1) as

$$G = \begin{bmatrix} G_1 \\ 2G_2 \end{bmatrix},$$

where G_1 and G_2 are $k_1 \times n$ and $k_2 \times n$ matrices over $\mathbb{Z}_4$. Let $\hat{C}$ denote the subcode $[n; 0, k_1]$ of C generated by the matrix $[2G_1]$ and let $\tilde{C}$ denote the subcode $[n; 0, k_1+k_2]$ of C with generator matrix $\begin{bmatrix} 2G_1 \\ 2G_2 \end{bmatrix}$ (see [1]).

A vector $\boldsymbol{v}$ is a 2-*linear combination* of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ if $\boldsymbol{v} = \lambda_1\boldsymbol{v}_1 + \ldots + \lambda_k\boldsymbol{v}_k$ with $\lambda_i \in \mathbb{Z}_2$ for $1 \leq i \leq k$. A subset $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_k\}$ of C is called a 2-*basis* for C if for each $i = 1, 2, ..., k-1$, $2\boldsymbol{v}_i$ is a 2−linear combination of $\boldsymbol{v}_{i+1}, ..., \boldsymbol{v}_k$, $2\boldsymbol{v}_k = 0$, C is the 2-linear span of S and S is 2-linearly independent ([4]). The number of elements in a 2-basis for C is called the 2-*dimension* of C and is denoted by 2-dim(C). It is easy to verify that the rows of the matrix

$$(2) \qquad \begin{pmatrix} I_{k_1} & X & Y \\ 2I_{k_1} & 2X & 2Y \\ 0 & 2I_{k_2} & 2Z \end{pmatrix}$$

form a 2-basis for the code C generated by matrix (1). Thus the 2-dimension of C is $2k_1 + k_2$.

For a vector $\boldsymbol{x} \in \mathbb{Z}_4^n$, we denote the Hamming weight and Lee weight by $\mathrm{wt}(\boldsymbol{x})$ and L-wt$(\boldsymbol{x})$, respectively.

Let C be a linear code of length n over $\mathbb{Z}_4$. Let $A(C)$ be the $|C| \times n$ array of all codewords in C. It is well-known that each column of $A(C)$ corresponds to the following three cases: (i) the column contains only 0 (ii) the column contains 0 and 2 equally often (iii) the column contains all elements of $\mathbb{Z}_4$ equally often (cf. [28]). For the three columns (i), (ii) and (iii), we define the *Lee support weights* of these columns by 0, 2 and 1 respectively. Thus we define the *Lee support weight* $\mathrm{wt}_L(C)$ of C by the sum of the Lee support weights of all columns of $A(C)$. For example, if

$$C = \{(0,0,0), (1,0,1), (2,0,2), (3,0,3), (0,2,2), (1,2,3), (2,2,0), (3,2,1)\},$$

then $\mathrm{wt}_L(C) = 1+2+1 = 4$. We remark that if C is generated by only one vector $\boldsymbol{x}$, then the Lee support weight $\mathrm{wt}_L(C)$ corresponds to the original Lee weight L-wt$(\boldsymbol{x})$ of $\boldsymbol{x}$. Then we have the following theorem.

Theorem 2.1 *Let C be an $[n; k_1, k_2]$ code over $\mathbb{Z}_4$. Then we have*

$$\begin{aligned} \mathrm{wt}_L(C) &= \frac{1}{4^{k_1-1}2^{k_2}} \sum_{\boldsymbol{x}\in C} (\text{L-wt}(\boldsymbol{x}) - \mathrm{wt}(\boldsymbol{x})) \\ &= \frac{1}{4^{k_1-1}2^{k_2}} \sum_{\boldsymbol{x}\in C} |\{i : \ x_i = 2\}|. \end{aligned}$$

Proof. In the array $A(C)$, let n_0 be the number of columns in which 0 and 2 are balanced and let n_1 be the number of columns in which 0,1,2 and 3 occurs equally often. So we have $2n_0 + n_1 = \mathrm{wt}_L(C)$. Hence we have

$$\begin{aligned} \sum_{\boldsymbol{x}\in C} (\text{L-wt}(\boldsymbol{x}) - \mathrm{wt}(\boldsymbol{x})) &= (n_0(|C|/2 \cdot 2) + n_1(|C|/4 \cdot 1 + |C|/4 \cdot 2 + |C|/4 \cdot 1)) \\ &\quad -(n_0(|C|/2 \cdot 1) + n_1(|C|/4 \cdot 1 + |C|/4 \cdot 1 + |C|/4 \cdot 1)) \\ &= |C|/4((4n_0 + 4n_1) - (2n_0 + 3n_1)) \\ &= |C|/4 \cdot \mathrm{wt}_L(C). \end{aligned}$$

□

Now, for $1 \leq r \leq \mathrm{rank}(C)$, we define the r-th *generalized Lee weight with respect to rank* (GLWR) $d_r^L(C)$ of C as follows:

$$d_r^L(C) = \min\{\mathrm{wt}_L(D) : \ D \text{ is a } \mathbb{Z}_4\text{-submodule of } C \text{ with } \mathrm{rank}(D) = r\}.$$

We note that $d_1^L(C)$ corresponds to the minimum Lee weight of C. As a connection between the GHWR and the GLWR for a linear code C over $\mathbb{Z}_4$, we remark that

$$d_r^L(C) \leq 2d_r^H(C). \tag{3}$$

Additionally, we define the (k_1, k_2)-*generalized Lee weight with respect to type* as follows:

$$d_{k_1,k_2}^L = \min\{\mathrm{wt}_L(D) : \ D \text{ is a } \mathbb{Z}_4\text{-submodule of } C \text{ with type } \{k_1, k_2\} \ \}.$$

Also, for $1 \leq r \leq 2k_1 + k_2$, we define the r-th *generalized Lee weight with respect to 2-dimension* (GLWT) of C as follows:

$$2\text{-}d_r^L(C) = \min\{\mathrm{wt}_L(D) : \ D \text{ is a } \mathbb{Z}_4\text{-submodule of } C \text{ with } 2\text{-}\dim(D) = r\}.$$

Note that with respect to 2-dimension $2\text{-}d_1^L(C)$ does not always corresponds to the minimum Lee weight of C. In each case, the set $\{d_r^L(C)\}$ or $\{d_{k_1,k_2}^L(C)\}$ or $\{2\text{-}d_r^L(C)\}$ is called the *Lee weight hierarchy* of C.

In this paper, we shall derive several basic properties of these weights.

3 MacWilliams Relations

We define the following weight enumerator which is a natural generalization of the joint weight enumerator for codes over $\mathbb{Z}_k$. Let $C_1, C_2, .., C_g$ be codes such that C_i is a code over $\mathbb{Z}_k$. The complete joint weight enumerator of genus g for codes $C_1, \ldots, C_g$ of length n is defined as

$$\mathfrak{J}_{C_1,C_2,\ldots,C_g}(X_{\boldsymbol{a}} \text{ with } \boldsymbol{a} \in \mathbb{Z}_k^g) = \sum_{(\boldsymbol{c}_1,\boldsymbol{c}_2,\ldots,\boldsymbol{c}_g)\in C_1\times C_2\times\ldots\times C_g} \prod_{\boldsymbol{a}\in\mathbb{Z}_k^g} X_{\boldsymbol{a}}^{n_{\boldsymbol{a}}(\boldsymbol{c}_1,\boldsymbol{c}_2,\ldots,\boldsymbol{c}_g)}$$

where $n_{\boldsymbol{a}}(\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_g) = |\{j : ((\boldsymbol{c}_1)_j, (\boldsymbol{c}_2)_j, \ldots, (\boldsymbol{c}_g)_j) = \boldsymbol{a}\}|$, and $\boldsymbol{c}_i = ((\boldsymbol{c}_i)_1, \ldots, (\boldsymbol{c}_i)_n)$.

We shall describe the matrix we need to produce the MacWilliams relations for codes over $\mathbb{Z}_k$.

We want the orthogonality given by the character group associated to the additive group G of the ring $\mathbb{Z}_k$ to match the given inner product, where the orthogonality given by the character group is:

$$\chi(C) = \{v : \chi_v(\omega) = 1, \forall \omega \in C\}$$

where $\chi_v \in \widehat{G}$, the character group.

Let σ be a character in $\widehat{G}$ associated with the element 1. For $a, b \in \mathbb{Z}_k$ set $\chi_b(a) = \sigma(ab)$. We see that χ_b is a character associated with the element b.

This gives

$$\prod \chi_{w_i}(v_i) = \prod \sigma(v_i w_i) = \sigma(\sum v_i w_i).$$

If $(\sum v_i w_i) = 0$ then $\sigma(\sum v_i w_i) = 1$.

It is shown in [6] that the matrix produced by these characters gives the MacWilliams relation for the complete weight enumerator, where the complete weight enumerator for a code C is

$$W_C(x_0, \ldots, x_{r-1}) = \sum_{c\in C} A_{a_0,\ldots,a_{r-1}} x_0^{a_0} x_1^{a_1} \ldots x_{r-1}^{a_{r-1}}$$

where the number of coordinates in the vector c with an i in them is a_i.

To produce the MacWilliams relations we define the matrix T by

$$T_{\alpha_i,\alpha_j} = \chi_{\alpha_j}(\alpha_i). \tag{4}$$

Let η be a complex k-th root of unity. Noting that $\sigma(\alpha) = \eta^\alpha$, then indexing the matrix T with the elements of $\mathbb{Z}_k$ we have that $T_{i,j} = \eta^{ij}$.

Then the MacWilliams relations for the complete weight enumerator are given by:

$$W_{C^\perp}(x_0, \ldots, x_{k-1}) = \frac{1}{|C|} W_C(T(x_0, \ldots, x_{k-1})).$$

For a complete description, see [6].

The MacWilliams relations for the joint weight enumerator over $\mathbb{Z}_k$ were corrected in [5]. They can be generalized to the following lemma.

Lemma 3.1 *Let $C_1, C_2, \ldots, C_g$ be linear codes in $\mathbb{Z}_k$ and let $\tilde{C}$ denote either C or $C^\perp$. Then*

$$\mathfrak{J}_{\tilde{C}_1,\tilde{C}_2,\ldots,\tilde{C}_g}(X_{\boldsymbol{a}}) = \frac{1}{\prod_{i=1}^{g} |C_i|^{\delta_{\tilde{C}_i}}} \cdot (\otimes_{i=1}^{g} T^{\delta_{\tilde{C}}}) \mathfrak{J}_{C_1,\ldots,C_g}(X_{\boldsymbol{a}}), \tag{5}$$

where

$$\delta_{\tilde{C}} = \begin{cases} 0 & \text{if } \tilde{C} = C, \\ 1 & \text{if } \tilde{C} = C^\perp. \end{cases}$$

Note that the matrix $\otimes_{i=1}^{g} T^{\delta_{\tilde{C}}}$ is an k^g by k^g matrix and that $\mathfrak{J}_{\tilde{C}_1,\tilde{C}_2,\ldots,\tilde{C}_g}(X_{\boldsymbol{a}})$ is a polynomial in k^g variables. The proof of this lemma is given in the preprint [6]. Denote by $\mathfrak{J}(C,g)(X_{\boldsymbol{a}}) = \mathfrak{J}_{C_1,C_2,\ldots,C_g}(X_{\boldsymbol{a}})$ with $C_i = C$ for $i = 1, \ldots, g$.

Let $\mathfrak{A}_{g,\mathbf{h}} = \{\mathbf{j}$ such that a subcode of type $\mathbf{j}$ can be generated from a type $\mathbf{h}$ code using g (not necessarily independent) vectors$\}$, where $\mathbf{h} = \{h_1, h_2, \ldots, h_s\}$ and $\mathbf{j} = \{j_1, j_2, \ldots, j_s\}$.

Lemma 3.2 *Let C be a linear code over $\mathbb{Z}_k$; then*

$$\mathfrak{J}(C,g)(X_{\boldsymbol{a}}) = \sum_{\mathbf{j} \in \mathfrak{A}_{g,\mathbf{h}}} \Psi(g,\mathbf{h},\mathbf{j}) W_C^{\mathbf{j}}(X_{\mathbf{0}} = x, X_{\boldsymbol{a}} = y, (\boldsymbol{a} \neq 0)) \tag{6}$$

where $\Psi(g,\mathbf{h},\mathbf{j})$ denotes the number of ways a subcode of type $\mathbf{j}$ can be generated from a subspace of type $\mathbf{h}$ using g vectors.

Proof. Given a set of g vectors represented by $X_{\boldsymbol{a}}$, then the number of $X_{\boldsymbol{a}}$ that are not 0 is equal to the support of the space generated by the vectors. Moreover, each subspace is generated $\Psi(g,\mathbf{h},\mathbf{j})$ different times. □

Note that a similar thing cannot be done by simply considering ranks because from knowing only the rank of a code it is not possible to determine how many subcodes of a given rank exist. For example, a rank 1 code over $\mathbb{Z}_6$ may have a subcode of rank 1 or it may not, depending on whether the code is $\mathbb{Z}_6$ or $\{0,3\}$.

This lemma allows us to generate MacWilliams relations for the higher weight enumerators.

Theorem 3.3 *Let C be a linear code over $\mathbb{Z}_k$; then*

$$\sum_{\mathbf{j} \in \mathfrak{A}_{g,\mathbf{h}}} \Psi(g,\mathbf{h},\mathbf{j}) W_{C^\perp}^{\mathbf{j}}(x,y) = \frac{1}{|C|^g} \sum_{\mathbf{j} \in \mathfrak{A}_{g,\mathbf{h}}} \Psi(g,\mathbf{h},\mathbf{j}) W_{C^\perp}^{\mathbf{j}}(x + (k^g - 1)y, x - y). \tag{7}$$

Proof. Specializing the variables collapses the matrix $\otimes_{i=1}^{g} T$, the first row of which is all 1 and hence collapses to $k^g - 1$.

Every other row has a 1 in the first column and then noticing that $\sum_{a \in \mathbb{Z}_k} \chi_b(a) = 0$, so summing all but the first row gives -1. Hence the matrix becomes

$$\begin{pmatrix} 1 & k^g - 1 \\ 1 & -1 \end{pmatrix} \tag{8}$$

□

A similar technique was used for codes over fields in [7].

Example: Let C be the linear code of length 2 over $\mathbb{Z}_4$ generated by $(1,0)$ and $(0,2)$. The code has type $\{1,1\}$. We have

$$\begin{aligned} \mathfrak{J}(C,2)(x,y,\ldots,y) &= W^{0,0}(x,y) + 12W^{1,0}(x,y) \\ &+ 3W^{0,1}(x,y) + 6W^{0,2}(x,y) + 24W^{1,1}(x,y), \end{aligned}$$

where $W^{0,0}(x,y) = x^2$, $W^{1,0}(x,y) = xy + y^2$, $W^{0,1}(x,y) = 2xy + y^2$, $W^{0,2}(x,y) = y^2$, $W^{1,1}(x,y) = y^2$. Then

$$\begin{aligned} \frac{1}{64}(W^{0,0}(x+15y, x-y) &+ 12W^{1,0}(x+15y,x-y) + 3W^{0,1}(x+15y,x-y) \\ &+ 6W^{0,2}(x+15y,x-y) + 24W^{1,1}(x+15y,x-y) \\ &= x^2 + 3xy. \end{aligned}$$

Now, $C^\perp = \{(0,0),(0,2)\}$ and is of type $\{0,1\}$, with $W^{0,0}(x,y) = x^2$, $W^{0,1}(x,y) = xy$ and $\mathfrak{J}(C,2)(x,y,\ldots,y) = W^{0,0}(x,y) + 3W^{0,1}(x,y)$.

Notice also that

$$\begin{aligned} W_C(x,y) = \mathfrak{J}(C,1)(x,y) &= x^2 + 4xy + 3y^2 \\ &= W^{0,0}(x,y) + 2W^{1,0}(x,y) + W^{0,1}(x,y) \\ &= x^2 + 2(xy+y^2) + (2xy+y^2). \end{aligned}$$

4 Bounds

4.1 A Singleton Bound

A *chain ring* R is a finite ring with Jacobson radical $J(R) \neq 0$ whose principal left ideals form a chain (see [21]). It follows easily that $\mathbb{Z}_{p^m}$ is a kind of chain ring, where p is a prime. In [17], Horimoto and Shiromoto proved the following Singleton type bound for GHWR of linear codes over finite chain rings:

Proposition 4.1 *Let C be a linear code of length n over a finite chain ring R. For any r, $1 \le r \le \operatorname{rank}(C)$, we have*

$$d_r^H(C) \le n - \operatorname{rank}(C) + r.$$

In this subsection we shall find the corresponding Singleton bound for the higher weights over a kind of non-chain rings $\mathbb{Z}_k$.

The Chinese Remainder Theorem was used in [9] to form MDR codes over $\mathbb{Z}_k$. Here we recall the basic definitions and a few facts. Let k and q be integers with q dividing k, and define the map

$$\Psi_q : (\mathbb{Z}/k\mathbb{Z})^n \to (\mathbb{Z}/q\mathbb{Z})^n$$

by

$$\Psi_q(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\alpha_1 \pmod q, \alpha_2 \pmod q, \cdots, \alpha_n \pmod q)$$

where $v = (\alpha_1, \alpha_2, \cdots, \alpha_n)$.

If k is a positive integer with $k = \prod_{i=1}^{s} q_i$ and $\gcd(q_i, q_j) = 1$ then define the map

$$\Psi : (\mathbb{Z}/k\mathbb{Z})^n \to (\mathbb{Z}/q_1\mathbb{Z})^n \times (\mathbb{Z}/q_2\mathbb{Z})^n \times \cdots \times (\mathbb{Z}/q_s\mathbb{Z})^n$$

by

$$\Psi(\boldsymbol{v}) = (\Psi_{q_1}(\boldsymbol{v}), \Psi_{q_2}(\boldsymbol{v}), \cdots, \Psi_{q_s}(\boldsymbol{v})).$$

If $C^{(q_1)}, C^{(q_2)} \cdots C^{(q_s)}$ are codes of length n, with $C^{(q_i)}$ a code over $\mathbb{Z}_{q_i}$, define the Chinese product by

$$\mathrm{CRT}(C^{(q_1)}, C^{(q_2)}, \cdots, C^{(q_s)}) = \{\Psi^{-1}(\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_s) \mid \boldsymbol{v}_i \in C^{(q_i)}\},$$

where $\Psi^{-1}(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_s)$ is the unique vector in $(\mathbb{Z}/\mathbb{Z}_k)^n$ that is congruent component wise to $\boldsymbol{v}_i \pmod{q_i}$.

The generalized Chinese Remainder Theorem implies that CRT is the inverse image of the map Ψ.

We have the following fact. Let $C^{(q_1)}$,$C^{(q_2)}, \cdots$,$C^{(q_s)}$ be codes over $\mathbb{Z}_{q_1}, \mathbb{Z}_{q_2}, \cdots, \mathbb{Z}_{q_s}$ respectively. Then

$$\mathrm{rank}(\mathrm{CRT}(C^{(q_1)}, C^{(q_2)}, \cdots, C^{(q_s)})) = \mathrm{Max}\{\mathrm{rank}(C^{(q_i)})\}.$$

Additionally, we can see that if $C = (\mathrm{CRT}(C^{(q_1)}, C^{(q_2)}, \cdots, C^{(q_s)}))$ and D is a subcode of rank h of C then

$$D = \mathrm{CRT}(D^{(q_1)}, D^{(q_2)}, \cdots, D^{(q_s)})$$

where $D^{(q_i)} \subseteq C^{(q_i)}$ and $\mathrm{Max}\{\mathrm{rank}(D^{(q_i)})\}$ is h.

Lemma 4.2 *Let $C = (\mathrm{CRT}(C^{(q_1)}, C^{(q_2)}, \cdots, C^{(q_s)}))$; then $d_g(C) = \mathrm{Min}\{d_g^H(C^{(q_i)})\}$.*

Proof. This follows from the fact that $D = \mathrm{CRT}(\mathbf{0}, \ldots, D^{(q_i)}, \ldots, \mathbf{0}, \ldots)$ is an R-submodule of C of rank g for all i if $D^{(q_i)}$ has rank g. □

Theorem 4.3 *Let C be a linear code of length n over $\mathbb{Z}_k$ of rank r. Then*

$$d_g(C) \leq n - r + g,$$

for any h, $1 \leq g \leq r$.

Proof. Follows directly from Proposition 4.1 and Lemma 4.2. □

We shall call codes meeting this bound *g-th Maximum Hamming Distance Separable with respect to Rank* (g-th MHDR) codes.

The following theorem and proof is similar to that for MDR codes given in [9].

Theorem 4.4 *Let $C^{(k_1)}, C^{(k_2)}, \cdots, C^{(k_s)}$ be codes over $\mathbb{Z}_{k_1}, \mathbb{Z}_{k_2}, \ldots, \mathbb{Z}_{k_s}$ respectively. If $C^{(k_i)}$ is an g-th MHDR code for all i (not necessary the same rank), then $\mathrm{CRT}(C^{(k_1)}, C^{(k_2)}, \cdots, C^{(k_s)})$ is a g-th MHDR code.*

Proof. Let $C = \mathrm{CRT}(C^{(k_1)}, C^{(k_2)}, \cdots, C^{(k_s)})$. We have $\mathrm{rank}(C) = \mathrm{Max}\{\mathrm{rank}(C^{(k_i)})\}$. So

$$\begin{aligned} d_g^H(C) &= \min\{d_g^H(C^{(k_i)})\} = \min\{n - \mathrm{rank}(C^{(k_i)}) + g\} \\ &= n - \mathrm{Max}\{\mathrm{rank}(C^{(k_i)})\} + g = n - \mathrm{rank}(C) + g. \end{aligned}$$

□

4.2 Bounds for GLWR

In this section, we give some bounds for GLWR of linear codes over $\mathbb{Z}_4$.

Lemma 4.5 *If C is a linear code of length n over $\mathbb{Z}_4$ with $\mathrm{rank}(C) = 2$, then there exists a codeword $\mathbf{0} \neq \boldsymbol{v} \in C$ such that $\text{L-wt}(\boldsymbol{v}) \leq \mathrm{wt}_L(C)$.*

Proof. We assume that C is generated by $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$, where both x_i and y_i are not 0 for any i. If either x_i or y_i is 1 or 3 and the other is 0 or 2, then the Lee weight of $\alpha x_i + \beta y_i$ are at most 1 for any units α, β in $\mathbb{Z}_4$. If $2x_i = 2y_i = 0$, then the Lee weights of $\alpha x_i + \beta y_i$ are at most 2 for any units α, β in $\mathbb{Z}_4$. So if $|\{i : x_i = y_i = 1 \text{ or } 3\}| \leq |\{i : \{x_i, y_i\} = \{1,3\} \text{ or } \{3,1\}\}|$ (Resp., $|\{i : x_i = y_i = 1 \text{ or } 3\}| \geq |\{i : \{x_i, y_i\} = \{1,3\} \text{ or } \{3,1\}\}|$), then $\text{L-wt}(\boldsymbol{x} + \boldsymbol{y}) \leq \mathrm{wt}_L(C)$ (Resp., $\text{L-wt}(\boldsymbol{x} + 3\boldsymbol{y}) \leq \mathrm{wt}_L(C)$). The lemma follows. □

Theorem 4.6 *Let C be a linear code of length n over $\mathbb{Z}_4$ with $\mathrm{rank}(C) \geq 2$. Then we have $1 \leq d_1^L(C) \leq d_2^L(C)$.*

Proof. Let D be a submodule of C with $\mathrm{wt}_L(D) = d_2^L(C)$ and $\mathrm{rank}(D) = 2$. From Lemma 4.5, there exists a codeword $\mathbf{0} \neq \boldsymbol{v} \in D$ such that $\text{L-wt}(\boldsymbol{v}) \leq \mathrm{wt}_L(D)$. Since $d_1^L(C) \leq \text{L-wt}(\boldsymbol{v})$, the theorem follows. □

The following monotonicity is well-known for a linear code C of rank k over a chain ring ([17, 26]):

$$1 \leq d_1^H(C) < d_2^H(C) < \cdots < d_k^H(C) \leq n.$$

Based on the above inequality, with respect to the GLWR, we had conjectured as follows for a linear code C of length n over $\mathbb{Z}_4$ with $\mathrm{rank}(C) = k > 0$:

$$1 \le d_1^L(C) \le d_2^L(C) \le \cdots \le d_k^L(C) \le 2n.$$

However, Hashimoto ([13]) recently found a counter-example to the conjecture.

Example 4.7 ([13]) Let C be a linear code of length 21 over $\mathbb{Z}_4$ having a generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 2 & 1 & 2 & 0 & 1 & 0 & 1 & 1 & 0 & 3 & 2 & 1 & 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 3 & 1 & 0 & 1 & 2 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 & 2 & 0 & 1 & 0 & 1 & 1 & 0 & 3 & 1 & 1 & 1 & 2 & 1 & 1 & 3 \end{pmatrix}.$$

Then it follows that $d_2^L(C) = 22$ and $d_3^L(C) = 21$. Therefore it shows that the conjecture is false and this is a counter-example of a code whose lengths are a minimum.

Now, we give a Singleton type bound on the GLWR.

Theorem 4.8 *For a linear code C of length n over $\mathbb{Z}_4$ and any r, $1 \le r \le \mathrm{rank}(C)$,*

$$\left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor \le n - \mathrm{rank}(C).$$

Proof. We set $d_r^L = d_r^L(C)$ and $k = \mathrm{rank}(C)$. Now, we assume that

$$\left\lfloor \frac{d_r^L - 2r + 1}{2} \right\rfloor > n - k. \tag{9}$$

Note that

$$\left\lfloor \frac{d_r^L - 2r + 1}{2} \right\rfloor = \begin{cases} (d_r^L - 2r)/2 & d_r^L : \text{even} \\ (d_r^L - 2r + 1)/2 & d_r^L : \text{odd}. \end{cases}$$

If d_r^L is even, then the bound (9) is $d_r^L > 2n - 2k + 2r$. On the other hand, from (3) and Proposition 4.1, we have

$$d_r^L \le 2n - 2k + 2r. \tag{10}$$

A contradiction.

If d_r^L is odd, then the bound (9) is $d_r^L > 2n - 2k + 2r - 1$. Thus we have $d_r^L = 2n - 2k + 2r$ from (10). This contradicts that d_r^L is odd. Therefore the theorem follows. □

Remark 4.9 In [8, 23], it is shown that for a linear code C of length n over $\mathbb{Z}_4$ with minimum Lee weight d_L,

$$\left\lfloor \frac{d_L - 1}{2} \right\rfloor \le n - \mathrm{rank}(C).$$

Since $d_L = d_1^L(C)$, the bound in Theorem 4.8 is a generalization of the above bound.

If a linear code C of length n over $\mathbb{Z}_4$ meets the bound in Theorem 4.8 for r, that is, $\left\lfloor (d_r^L(C) - 2r + 1)/2 \right\rfloor = n - \mathrm{rank}(C)$, then we shall call the code C a *r-th Maximum Lee Distance Separable with respect to Rank* (r-th MLDR) code. Now we shall give a connection between r-th MLDR codes and r-th MHDR codes.

Lemma 4.10 *If C is an r-th MLDR code, then $d_r^L(C) = 2d_r^H(C) - 1$ or $2d_r^H(C)$.*

Proof. Since C is an r-th MLDR code, we have

$$\left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor = n - \mathrm{rank}(C). \tag{11}$$

We assume that $d_r^L(C) < 2d_r^H(C) - 1$. If $d_r^L(C)$ is odd, then we have the following equation from (11):

$$d_r^L(C) = 2n - 2\mathrm{rank}(C) + 2r - 1.$$

Since $d_r^L(C) < 2d_r^H(C) - 1$, we have

$$2n - 2\mathrm{rank}(C) + 2r - 1 < 2d_r^H(C) - 1 \Longleftrightarrow n - \mathrm{rank}(C) + r < d_r^H(C).$$

A contradiction from the bound in Proposition 4.1. In the case that $d_r^L(C)$ is even, the proof follows. □

Theorem 4.11 *Let C be a linear code C of length n over $\mathbb{Z}_4$. If C is an r-th MLDR code, then C is an r-th MHDR code.*

Proof. From the above lemma, we have $d_r^L(C) = 2d_r^H(C) - 1$ or $2d_r^H(C)$. In both case,

$$n - \mathrm{rank}(C) = \left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor = d_r^H(C) - r.$$

□

Theorem 4.12 *Let C be an r-th MHDR code of length n over $\mathbb{Z}_4$. C is an r-th MLDR code if and only if $d_r^L(C) = 2d_r^H(C) - 1$ or $2d_r^H(C)$.*

Proof. Since C is an r-th MLDR code if and only if

$$\left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor = d_r^H(C) - r.$$

If $d_r^L(C)$ is odd, then

$$\left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor = \frac{d_r^L(C) - 2r + 1}{2} = d_r^H(C) - r,$$

and if $d_r^L(C)$ is even, then

$$\left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor = \frac{d_r^L(C) - 2r}{2} = d_r^H(C) - r.$$

The theorem follows. □

It is known that if C is a linear code of length n over $\mathbb{Z}_4$ with minimum Hamming weight d_H and minimum Lee weight d_L, then

$$d_H \geq \left\lceil \frac{d_L}{2} \right\rceil \tag{12}$$

(cf. [22]). In [24], they proved the following Griesmer type bound for linear codes over finite quasi-Frobenius rings.

Lemma 4.13 *Let C be a linear code of length n over $\mathbb{Z}_4$ with* $\operatorname{rank}(C) = k$ *and minimum Hamming weight d_H. Then*

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d_H}{2^i} \right\rceil.$$

Using (12) and Lemma 4.13, we have the following Griesmer type bound for minimum Lee weights of linear codes over $\mathbb{Z}_4$.

Proposition 4.14 *Let C be a linear code of length n over $\mathbb{Z}_4$ with* $\operatorname{rank}(C) = k$ *and minimum Lee weight d_L. Then*

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{\lceil d_L/2 \rceil}{2^i} \right\rceil.$$

Now we have a generalized Griesmer type bound for GLWR.

Theorem 4.15 *For a linear code C of length n over $\mathbb{Z}_4$ and any r, $1 \leq r \leq \operatorname{rank}(C)$, we have*

$$d_r^L(C) \geq \sum_{i=0}^{r-1} \left\lceil \frac{\lceil d_1^L(C)/2 \rceil}{2^i} \right\rceil.$$

Proof. For a $\mathbb{Z}_4$-submodule D of C with $\mathrm{wt}_L(D) = d_r^L(C)$ and $\operatorname{rank}(D) = r$, let D' be the code having a generator matrix obtained from a generator matrix of D by deleting the zero columns. Since the length of D' is less than or equal to $\mathrm{wt}_L(D)$ and the minimum Lee weight of D' is greater than or equal to $d_1^L(C)$, the theorem follows from Proposition 4.14. □

Let C be a linear code C of length n over $\mathbb{Z}_4$. From the definitions of GLWR and GHWR, we have

$$d_r^H \geq \left\lceil \frac{d_r^L}{2} \right\rceil \tag{13}$$

for any r. It is known that if C is a linear code C of length n over $\mathbb{Z}_4$ with $\text{rank}(C) = k$ and minimum Hamming weight d_H, then $\text{Soc}(C)$ is isomorphic to a binary $[n, k, d]$ code (cf. [17]).

Lemma 4.16 ([17]) *For any r, $1 \le r \le \text{rank}(C)$, we have*

$$d_r^H(C) = d_r^H(\text{Soc}(C)).$$

Using the above lemma and Theorem 3.19 (p. 35 in [12]), the lemma follows:

Lemma 4.17 *Let C be a linear code C of length n over $\mathbb{Z}_4$ with* $\text{rank}(C) = k$. *Then*

$$n \ge d_r^H(C) + \sum_{i=1}^{k-r} \left\lceil \frac{d_r^H(C)}{2^i(2^i - 1)} \right\rceil,$$

for any r, $1 \le r \le k$.

Now we have a generalized Griesmer type bound for GLWR.

Theorem 4.18 *Let C be a linear code C of length n over $\mathbb{Z}_4$ with* $\text{rank}(C) = k$. *Then*

$$n \ge \left\lceil \frac{d_r^L(C)}{2} \right\rceil + \sum_{i=1}^{k-r} \left\lceil \frac{\left\lceil d_r^L(C)/2 \right\rceil}{2^i(2^i - 1)} \right\rceil,$$

for any r, $1 \le r \le k$.

Proof. The theorem follows from the above lemma and inequality (13). □

Let C be a free linear code of length n over $\mathbb{Z}_4$ with $\text{rank}(C) = r$ and minimum Lee weight d_L then the following Griesmer type bound is known [1].

Lemma 4.19

$$n \ge \sum_{i=0}^{r-1} \left\lceil \frac{3 \cdot 2^{i(i-1)}/2}{4 \cdot \Pi_{j=0}^{i-1}(2^{i+1-j} + 1)} d_L \right\rceil.$$

Thus we have the following bound for the free codes. This is better than the bound given by the Theorem 4.15 for free codes. Its proof is similar.

Theorem 4.20

$$d_r^L(C) \ge \sum_{i=0}^{r-1} \left\lceil \frac{3 \cdot 2^{i(i-1)}/2}{4 \cdot \Pi_{j=0}^{i-1}(2^{i+1-j} + 1)} d_L \right\rceil.$$

It is known that the octacode meets the bound of the Lemma 4.19. It will be interesting to construct codes over $\mathbb{Z}_4$ that meets the above bound of Theorem 4.20. However, except for $r = 1$ the octacode meets the above bound for GLWR (see Theorem 5.7).

5 Determination of Generalized Weight

In this section we look at the Generalized weights for some well known classes of codes. Let C be a linear code over $\mathbb{Z}_4$ of length n and 2-dimension k. For $\boldsymbol{x} \in C$ let $\omega_2(\boldsymbol{x}) = |\{i : x_i = 2\}|$. The following remark follows from Theorem 2.1.

Remark 5.1 *For* $1 \leq r \leq k$,

$$d_r^L(C) = \frac{1}{2^{r-2}} \min \left\{ \sum_{\boldsymbol{x} \in D} \omega_2(\boldsymbol{x}) \mid D : [n, r] \text{ subcode of } C \right\}.$$

It is clear from remark 5.1 that it is difficult to find the generalized Lee weight since $\omega_2(\boldsymbol{x})$ is not a metric. Now we find the generalized Lee weight for the several known classes of codes.

The following lemma follows from definition.

Lemma 5.2 *Let C be a linear code over $\mathbb{Z}_4$ with generator matrix $G = [2g_1, 2g_2, \ldots, 2g_k]$; then for $1 \leq r \leq k$ we have*

$$d_r^L(C) = 2d_r^H(C)$$

where $d_r^H(C)$ is the Hamming weight hierarchy of C.

5.1 First-Order Reed Muller Code

The first order Reed Muller code $R^{1,m}$ over $\mathbb{Z}_4$ is a code of length $n = 2^{m-1}$, rank m, 2-dimension $m+1$ with minimum Hamming weight 2^{m-2} and minimum Lee weight 2^{m-1}.

Theorem 5.3 *The Lee weight hierarchy of $R^{1,m}$ with respect to 2-dimension is given by 2-$d_r^L = 2^{m-r}(2^r - 1), 1 \leq r \leq m-1$, 2-$d_m^L = 2^m$ and 2-$d_{m+1}^L = 2^{m-1}$.*

Proof. This follows from Lemma 5.2 (see [10]).

□

Remark 5.4 Note that the monotonicity fails for GLWT as in Theorem 5.3, 2-$d_m^L >$ 2-d_{m+1}^L.

5.2 Simplex Codes

The Hamming weight hierarchy of quaternary simplex codes of type α and β with respect to 2-dimension were studied in [4]. The next theorem finds the Hamming weight hierarchy with respect to rank. Note that the rank of both the simplex codes is k.

Theorem 5.5 *The Hamming weight hierarchy of S_k^α and S_k^β with respect to rank is given by*

$$d_r^H\left(S_k^\alpha\right) = 2d_r^H\left(S_k^\beta\right) = 2^{2k-r}(2^r - 1), 1 \leq r \leq k.$$

Proof. We will prove it only for S_k^β, since the other case is similar. By Lemma 4.16 and Lemma 5 of [4] the result follows.

□

5.3 Quaternary Golay Code

The quaternary lifted Golay code has length 24, rank 12, 2-dimension 24, minimum Hamming weight 8 and minimum Lee weight 12.

Theorem 5.6 *The quaternary Golay code QR_{24} has Lee weight hierarchy (with respect to rank)* $\{12, 14, 16, 16, 17, 18, 19, 20, 21, 22, 23, 24\}$.

Proof. It is a straightforward computation.

□

5.4 Octacode

The octacode QR_8 is a code over $\mathbb{Z}_4$ of length 8, 2-dimension 8, minimum Hamming weight 4 and minimum Lee weight 6.

Theorem 5.7 *The quaternary octacode QR_8 has Lee weight hierarchy (with respect to rank)* $\{6, 6, 7, 8\}$.

Proof. Straightforward.

□

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