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Decomposing complete 3-uniform hypergraphs into Hamiltonian cycles

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Abstract

Using the Katona-Kierstead definition of a Hamiltonian cycle in a uniform hypergraph, we continue the investigation of the existence of a decomposition of the complete 3-uniform hypergraph into Hamiltonian cycles began by Bailey and Stevens. We also discuss two extensions of the problem: to the complete 3-uniform hypergraph from which a parallel class of triples has been removed, and to the complete 3-uniform (multi)hypergraph of higher index. We also briefly consider decompositions of 3-uniform hypergraphs into (not necessarily Hamiltonian) cycles and comment on a possible analogue of Alspach's conjecture for cycle decompositions of the ordinary complete graph.

1 Introduction

In a recent paper [2], the authors introduce, as an obvious generalization of the problem of decomposing the complete graph into Hamiltonian cycles, the problem of decomposing the complete k-uniform hypergraph into Hamiltonian cycles. The definition of a Hamiltonian cycle that they use is that of [9]: A Hamiltonian cycle in a complete k-uniform hypergraph ${K_n}^{(k)}$ is a cyclic ordering of its vertices such that each consecutive k-tuple of vertices is a hyperedge. A Hamiltonian decomposition of the complete k-uniform hypergraph is a partition of the set of the hyperedges of ${K_n}^{(k)}$ into disjoint Hamiltonian cycles.

[Let us note that a different notion of a Hamiltonian cycle in a hypergraph that predates that of [9] is due to Berge [3]; a decomposition of the complete 3-uniform hypergraph into Hamiltonian cycles in the sense of Berge has been considered in [4] and the proof of the existence of decompositions of the complete 3-uniform hypergraphs into this kind of Hamiltonian cycles was completed in [12].]

It is observed in [2] that a necessary condition for the existence of a decomposition of the complete k-uniform hypergraph on n vertices into Hamiltonian cycles is that n must divide $\binom{n}{k}$, and that due to taking complements, and also taking into account the well known existence of Hamiltonian decompositions of the complete graph, it suffices to consider, for given n, only values of k for which $3 \le k \le \frac{n}{2}$.

The authors of [2] make the conjecture that the obvious necessary condition for the existence of a Hamiltonian decomposition of ${K_n}^{(k)}$ is also sufficient.

Bailey-Stevens (BS) conjecture (cf. [2]). Let $n \ge 5$ and $2 \le k \le n-2$. Then there exists a Hamiltonian decomposition of ${K_n}^{(k)}$ if and only if n divides $\binom{n}{k}$.

In [2], two methods are described for finding Hamiltonian decompositions: the clique finding method which allows the authors to find Hamiltonian decompositions of ${K_7}^{(3)}$, ${K_8}^{(3)}$ and ${K_9}^{(4)}$, and a difference method which results in their finding Hamiltonian decompositions of ${K_{10}}^{(3)}$, ${K_{11}}^{(3)}$ and ${K_{16}}^{(3)}$.

The evidence presented in [2] in support of the BS conjecture is somewhat scarce. It is the purpose of this article to provide some more substantial evidence in support of the BS conjecture, at least for $k = 3$. We describe an extension of the difference method which enables us to obtain Hamiltonian decompositions of ${K_n}^{(3)}$ for all admissible $n \le 32$, as well as of ${K_{13}}^{(4)}$.

We also consider the more general problem of decomposing the complete 3-uniform hypergraph into cycles (not necessarily Hamiltonian cycles). And finally, for $k = 3$, we discuss two extensions of the Hamiltonian decomposition problem for the complete uniform hypergraph, namely to the case of ${K_n}^{(3)} - T$, the complete uniform hypergraph from which a parallel class T of triples was removed, and to the case of higher index λ.

2 A method for Hamiltonian decompositions of ${K_n}^{(3)}$

A necessary condition for the existence of a decomposition of ${K_n}^{(3)}$ into Hamiltonian cycles is $n \equiv 1$ or 2 (mod 3), or, equivalently, $n \equiv 1, 2, 4$ or 5 (mod 6).

a) Consider first the case $n \equiv 1 \pmod 6$, say, $n = 6t+1$. In this case the number of Hamiltonian cycles in any decomposition of ${K_n}^{(3)}$ into Hamiltonian cycles is $t(6t-1)$. Take as the set of vertices $V = Z_{6t-1} \cup \{\infty_1, \infty_2\}$, and consider α given by $\alpha(i) = i+1 \pmod{6t-1}$ for $i \in Z_{6t-1}$, $\alpha(\infty_j) = \infty_j$, $j = 1, 2$. The number of orbits of triples under α equals $t(6t+1)$. Thus if one succeeds in producing t Hamiltonian cycles $H_1, \dots, H_t$ such that each of the $t(6t+1)$ orbits of triples is represented among the $t(6t+1)$ triples in $\bigcup_{i=i}^{t} H_i$ exactly once then applying α will produce $t(6t-1)$ Hamiltonian cycles and thus a Hamiltonian decomposition. This works already for $t = 1$, i.e. when $n = 7$, and e.g. the Hamiltonian cycle $(\infty_1, 0, \infty_2, 1, 4, 3, 2)$ has the property required. We also found, with the aid of a computer, Hamiltonian decompositions of ${K_n}^{(3)}$ for $n \in \{13, 19, 25, 31\}$. There exists apparently a large number of solutions: already for $n = 13$, he search was interrupted after over ten thousand solutions of this kind were obtained. Sample solutions are provided in the

Appendix.

b) When $n \equiv 2 \pmod 6$, say, $n = 6t + 2$, a slight modification is needed. In this case, the number of Hamiltonian cycles in any decomposition of ${K_n}^{(3)}$ into Hamiltonian cycles is $t(6t + 1)$ which suggests to take as the set of vertices $V = Z_{6t+1} \cup \{\infty\}$, and consider β given by $\beta(i) = i + 1 \pmod{6t + 1}$ for $i \in Z_{6t+1}$, $\beta(\infty) = \infty$. Now the number of orbits of triples under β equals $t(6t + 2)$, so again we need t Hamiltonian cycles such that each of the orbits of triples is represented among the triples in the t Hamiltonian cycles exactly once; applying β then again produces Hamiltonian decomposition of ${K_{6t+2}}^{(3)}$. Here the method does *not* work when $t = 1$, e.g. when $n = 8$ but it works for $n \in \{14, 20, 26, 32\}$. (A Hamiltonian decomposition of ${K_8}^{(3)}$ was provided in [2]).

c) Yet another different approach is needed when $n \equiv 4$ or $5 \pmod 6$. Basically, this approach was already utilized in [2]; in that paper, the authors have, in effect, looked for a set S of disjoint Hamiltonian cycles with the property that every consecutive triple of any cycle in S, and also of the union of the cycles of S, belongs to a different orbit of triples under Z_n. Having found such a set S, they subsequently looked at those remaining orbits of triples which have no representatives in S, and checked whether one can form a Hamiltonian cycle from the triples of one orbit, or possibly two Hamiltonian cycles from triples of two "conjugate" orbits. This is how Hamiltonian decompositions of ${K_n}^{(3)}$ for $n = 10, 11$ and 16 were found in [2].

The disadvantage of this approach is in that for some sets S found in the first stage, no Hamiltonian cycles can be formed from the unused orbits of triples. We have instead reversed the two steps, and selected in the first stage (a proper number of) those orbits whose triples may be arranged in one (or two, as the case may be) Hamiltonian cycles, and in the second stage tried to select one representative each from the remaining orbits to form the set of Hamiltonian cycles S. This approach proved more efficient, and by using it we were able to find Hamiltonian decompositions of ${K_n}^{(3)}$ for $n = 10, 11, 16, 17, 22, 23, 28, 29$. Sample solutions can be found in the Appendix.

3 An extension of the problem

Let $n \geq 4$. When $n \equiv 0 \pmod 3$, a Hamiltonian decomposition of ${K_n}^{(3)}$ cannot exist. By analogy with the case of ordinary complete graphs of even order where one must delete a 1-factor from the complete graph for a Hamiltonian decomposition of the remaining graph (i.e. the cocktail-party graph) to become possible, we delete from ${K_n}^{(3)}$ a parallel class of triples T. Since $n | [\binom{n}{3} - \frac{n}{3}]$, the necessary condition for the existence of a Hamiltonian decomposition of ${K_n}^{(3)} - T$ become satisfied.

For example, when $n = 6$, the three Hamiltonian cycles $(0, 1, 2, 3, 4, 5), (0, 2, 5, 4, 1, 3), (0, 3, 5, 1, 2, 4)$ contain all edges of ${K_6}^{(3)}$, except for the edges of $T = \{\{0, 1, 4\}, \{2, 3, 5\}\}$. For $n = 9$, if we set $V = Z_9$ and $T = \{\{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 8\}\}$, then, for example, $(0, 1, 2, 7, 6, 4, 8, 5, 3) \pmod 9$ is a Hamiltonian decomposition of ${K_9}^{(3)} - T$.

(There exist altogether 12 nonisomorphic solutions of this kind.)

A Hamiltonian decomposition of ${K_{12}}^{(3)} - T$ is obtained by taking $V = Z_{12}$, developing $(0, 1, 2, 4, 5, 8, 11, 6, 10, 9, 3, 7)$ modulo 12, and developing $(0, 2, 4, 8, 3, 10, 1, 6, 11, 7, 5, 9)$ modulo 12 with step 2 (that is, adding $2j, j = 1, 2, 3, 4, 5$ to each element). This yields 18 Hamiltonian cycles; here $T = \{\{0, 4, 8\}, \{1, 5, 9\}, \{2, 6, 10\}, \{3, 7, 11\}\}$.

It seems reasonable to conjecture that for all $n \equiv 0 \pmod 3$ there exists a Hamiltonian decomposition of ${K_n}^{(3)} - T$ where T is some parallel class of triples in ${K_n}^{(3)}$.

4 The case of higher index and its connection to the Dudeney's problem

Another extension of the problem, again by analogy with the case of ordinary graphs, is as follows. When n is even, the complete multigraph $2K_n$ can be decomposed into Hamiltonian cycles. In the case of the complete 3-uniform hypergraph, can the complete 3-uniform (multi)-hypergraph $3{K_n}^{(3)}$ be decomposed into Hamiltonian cycles? Clearly, if $n \equiv 1$ or $2 \pmod 3$ and there exists a Hamiltonian decomposition of ${K_n}^{(3)}$, the answer is yes. But what if $n \equiv 0 \pmod 3$ or if a Hamiltonian decomposition of ${K_n}^{(3)}$ is not known? At least a partial answer can be obtained by considering the relationship of this problem to a problem of Dudeney.

In 1908, Dudeney [4] formulated the following problem.

Can one seat n people at a round table on $\binom{n-1}{2}$ occasions so that no person has the same two neighbours more than once?

The arithmetic conditions imply that if such a seating is possible, any person must have any pair of other persons as neighbours exactly once. Any solution to the above problem is called a *Dudeney configuration* of order n. There exists an extensive literature on Dudeney's problem (see, e.g., [5], [7], [8], [10], [11]). A Dudeney configuration of order n is known to exist for all even n, for small $n \leq 15$, and for several infinite classes of odd orders (see, e.g., [10]), but for odd n, the problem remains largely open. (An equivalent formulation asks for an exact covering of all 2-paths by Hamiltonian cycles.)

For example, a Dudeney configuration of order 4 consists of the three Hamiltonian cycles $(1, 2, 3, 4), (1, 2, 4, 3), (1, 3, 2, 4)$. An example of the ten Hamiltonian cycles of a Dudeney configuration of order 6 on $V = Z_5 \cup \{\infty\}$ is provided by $(\infty, 0, 1, 4, 2, 3), (\infty, 0, 2, 3, 4, 1)$ mod 5.

The connection of Dudeney configurations to Hamiltonian decompositions of $3{K_n}^{(3)}$ is given by the following.

Theorem 4.1 *Let $n \geq 4$. If there exists a Dudeney configuration of order n then there exists a Hamiltonian decomposition of $3{K_n}^{(3)}$.*

Proof. Consider three distinct elements a, b, c. Each of the 2-paths abc, bac, bca occurs in exactly one Hamiltonian cycle of a Dudeney configuration of order n. Thus the triple $\{a, b, c\}$ occurs in three Hamiltonian cycles of a decomposition into Hamiltonian cycles. □

Corollary 4.1 *A Hamiltonian decomposition of* $3{K_n}^{(3)}$ *exists for all even* n.

The following conjecture is an extension of the BS conjecture for $k = 3$.
Conjecture. Let $n \geq 4$. A Hamiltonian decomposition of the complete 3-uniform n-multihypergraph $\lambda {K_n}^{(3)}$ exists if and only if
(i) $n \equiv 1$ or $2 \pmod 3$ when $\lambda \equiv 1$ or $2 \pmod 3$
(ii) $n \geq 4$ when $\lambda \equiv 0 \pmod 3$.

5 Decompositions into cycles

Finally, one may consider decompositions of ${K_n}^{(3)}$ (or ${K_n}^{(3)} - T, T$ a parallel class) into cycles — not necessarily Hamiltonian cycles. The smallest nontrivial case occurs when $n = 6$. The admissible partitions of $\binom{6}{3} - 2$ into parts π_i, $4 \leq \pi_i \leq 6$ are $6+6+6, 6+4+4+4$, and $5+5+4+4$, and in each case, there is a decomposition of ${K_6}^{(3)} - T$ into cycles of prescribed lengths:
$6+6+6 : (0,1,2,3,4,5), (0,2,5,4,1,3), (0,3,5,1,2,4)$,
$T = \{\{0,1,4\},\{2,3,5\}\}$
$6+4+4+4 : (0,1,2,3,4,5), (0,1,3,4), (0,2,3,5), (1,2,4,5)$,
$T = \{\{0,2,4\},\{1,3,5\}\}$
$5+5+4+4 : (0,1,2,3,4), (0,4,2,1,5), (0,2,3,5), (1,3,4,5)$,
$T = \{\{0,1,3\},\{2,4,5\}\}$

At this point, one is encouraged to consider an analogue of the well-known Alspach's conjecture [1] for cycle decompositions of (ordinary) complete graphs: Given a partition $\pi = (\pi_1, \ldots, \pi_s)$ of the number of edges $\binom{n}{2}$ (or $\binom{n}{2} - \frac{n}{2}$) of the complete graph K_n (or of the cocktail-party graph $K_n - F$ where F is a 1-factor) into parts $3 \leq \pi_i \leq n$, there exists a decomposition of K_n or $K_n - F$ into cycles whose lengths equal the parts of the partition π.

Any hope of such an analogue for decompositions of ${K_n}^{(3)}$ into cycles of prescribed lengths is quickly quashed when one examines the case of $n = 7$: of the 19 possible partitions of 35 into parts between 4 and 7, there exists a decomposition into cycles with lengths equal to the parts of the partition only for 10 partitions while for the remaining 9 there exists none. For $n = 8$, there are 159 partitions of 56 into parts between 4 and 8; for 156 of these, there exists a decomposition into cycles but for the remaining three, there is none. The smaller proportion of those partitions for which there is no decomposition is somewhat encouraging, unfortunately, the next case of $n = 9$ (which would require to examine decompositions into cycles according to partitions of the number 81 into parts between 4 and 9) is too large to be handled at present. So whether the nonexistence of cycle decompositions according to some partitions for $n = 7$ and $n = 8$ is an anomaly specific to small orders remains a good open question.

One might therefore turn the focus to the existence of decompositions of ${K_n}^3$ into cycles of uniform length. Here we are aided by the following theorem and its corollary.

Theorem 5.1 *If there exists a Steiner system $S(3, s, v)$ and there exists a decomposition of ${K_s}^{(3)}$ into k-cycles then there exists a decomposition of ${K_v}^{(3)}$ into k-cycles.*

Proof is obvious: replace each block (of size s) with the k-cycles of any decomposition of ${K_s}^{(3)}$ into k-cycles.

In particular, we get the following corollary.

Corollary 5.1 *If there exists a Steiner system $S(3, k, v)$, $k \equiv 1$ or $2 \pmod 3$, and there exists a Hamiltonian decomposition of ${K_k}^{(3)}$ then there exists a decomposition of ${K_v}^{(3)}$ into k-cycles.*

Let us remark that one can formulate a much more general version of this theorem by using 3-wise balanced designs in place of Steiner systems $\mathrm{S}(3, k, v)$ in the theorem, however, due to lack of general results on the existence of 3-wise balanced designs, we refrain from formulating this (necessarily more complicated) version.

Corollary 5.2 *A decomposition of ${K_n}^{(3)}$ into 4-cycles exists if and only if $n \equiv 2$ or $4 \pmod 6$.*

This follows from the classical result of Hanani [6].

However, already the problem of decomposing the complete uniform 3-hypergraph ${K_n}^{(3)}$ into 5-cycles is open. A necessary condition for the existence of such a decomposition is that $n \equiv 1, 2, 5, 7, 10$ or $11 \pmod{15}$. An application of the Theorem above yields a decomposition of ${K_{17}}^{(3)}$, and more generally of ${K_{4^n+1}}^{(3)}$ into 5-cycles. A decomposition into 5-cycles trivially exists for $n = 5$, and it also exists for $n = 7, 10, 11$, and 16. There are exactly two nonisomorphic solutions for $n = 7$:
1. $(0, 1, 2, 3, 4), (0, 1, 3, 4, 5), (0, 2, 3, 5, 6), (0, 3, 6, 2, 5), (0, 4, 2, 1, 6)$,
$(1, 3, 6, 4, 5), (1, 5, 2, 4, 6)$,
2. $(0, 1, 2, 3, 4), (0, 1, 3, 4, 5), (0, 2, 3, 6, 4), (0, 2, 5, 3, 6), (0, 5, 3, 1, 6)$,
$(1, 2, 5, 4, 6), (1, 4, 2, 6, 5)$.

For $n = 10$, a solution is obtained, with $V = Z_{10}$, by taking the two base 5-cycles $(0, 1, 2, 5, 6), (0, 2, 5, 4, 7)$ modulo 10, and adjoining the four individual cycles $(0, 2, 4, 6, 8), (1, 3, 5, 7, 9), (0, 4, 8, 2, 6), (1, 5, 9, 3, 7)$, to obtain a total of 24 5-cycles, as needed.

An example for $n = 11$ is given by taking three base 5-cycles
$(0, 2, 1, 7, 4), (0, 5, 3, 6, 7), (0, 6, 8, 5, 4) \pmod{11}$.

And an example for $n = 16$ is obtained by developing the seven base 5-cycles
$(0, 1, 2, 4, 5), (0, 2, 4, 7, 8), (0, 3, 5, 9, 6), (0, 4, 6, 15, 9), (0, 5, 11, 4, 8)$,
$(0, 6, 1, 14, 5), (0, 7, 14, 6, 12) \pmod{16}$.

Thus a decomposition of ${K_n}^{(3)}$ into 5-cycles exists for all admissible $n \leq 17$, and for all $n = 4^m + 1, m$ a positive integer. However, in general, the existence of a decomposition into 5-cycles remains open, as does the existence of decompositions into k-cycles for $k > 5$.

On the other hand, we note that another application of the Theorem above yields the existence of decompositions of ${K_{q^n+1}}^{(3)}$ into $(q + 1)$-cycles for all prime powers $q \leq 31$.

6 The case of larger k

In [2], the existence of a Hamiltonian decomposition of ${K_9}^{(4)}$ is reported. A necessary condition for the existence of a Hamiltonian decomposition of ${K_n}^{(4)}$ is $n \not\equiv 0, 4, 6 \pmod 8$. We have managed to construct, in two essentially different ways, a Hamiltonian decomposition of ${K_{13}}^{(4)}$. In one construction, one takes for the element set $V = Z_{11} \cup \{\infty_1, \infty_2\}$, similarly to the method described for $n \equiv 1 \pmod 6$ when $k = 3$. The second construction has $V = Z_{13}$, and is similar to the method for $n \equiv 4 \pmod 6$ when $k = 3$. Below are two examples of these constructions:

1) $(\infty_1, 0, 1, \infty_2, 2, 4, 5, 3, 7, 9, 6, 10, 8)$
$(\infty_1, 0, 2, \infty_2, 4, 7, 1, 6, 5, 3, 10, 9, 8)$
$(\infty_1, 0, 4, \infty_2, 6, 1, 7, 10, 8, 5, 2, 3, 9)$
$(\infty_1, 0, 6, \infty_2, 2, 9, 8, 5, 10, 3, 7, 1, 4)$
$(\infty_1, 0, 8, \infty_2, 7, 3, 6, 1, 2, 5, 9, 10, 4) \pmod{11}$.

2) $(0, 1, 2, 3, 5, 6, 7, 10, 11, 12, 4, 8, 9)$
$(0, 1, 2, 6, 4, 10, 7, 9, 3, 11, 12, 5, 8)$
$(0, 1, 3, 6, 9, 2, 4, 11, 7, 5, 12, 10, 8)$
$(0, 1, 8, 6, 9, 11, 5, 2, 3, 12, 7, 10, 4) \pmod{13}$;
three further individual cycles are
$(0, 2, 4, 6, 8, 10, 12, 1, 3, 5, 7, 9, 11)$, $(0, 5, 10, 2, 7, 12, 4, 9, 1, 6, 11, 3, 8)$, $(0, 6, 12, 5, 11, 4, 10, 3, 9, 2, 8, 1, 7)$.

7 Miscellanea

We established that there are exactly three nonisomorphic decompositions of ${K_7}^{(3)}$ into Hamiltonian cycles. The representatives of the three isomorphism classes are as follows:

1. $(0, 1, 2, 3, 4, 5, 6), (0, 1, 3, 4, 6, 2, 5), (0, 2, 3, 5, 1, 4, 6), (0, 3, 6, 1, 5, 2, 4), (0, 4, 1, 2, 6, 3, 5)$,
2. $(0, 1, 2, 3, 4, 5, 6), (0, 1, 3, 4, 6, 2, 5), (0, 2, 4, 5, 1, 3, 6), (0, 3, 2, 5, 1, 6, 4), (0, 4, 1, 2, 6, 3, 5)$,
3. $(0, 1, 2, 3, 4, 5, 6), (0, 1, 3, 6, 2, 5, 4), (0, 2, 4, 1, 5, 6, 3), (0, 2, 5, 3, 1, 4, 6), (0, 3, 4, 6, 2, 1, 5)$.

There are exactly 312 nonisomorphic decompositions of ${K_8}^{(3)}$ into Hamiltonian cycles. The list of the 312 solutions is available from the authors at
`http://home.agh.edu.pl/~meszka/hh8_3.html`.

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Appendix

$n = 10$

$(0, 1, 3, 5, 6, 9, 4, 8, 2, 7)$ mod 10 and two individual cycles
$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$, $(0, 3, 6, 9, 2, 5, 8, 1, 4, 7)$.

$n = 11$

$(0, 1, 3, 6, 2, 8, 4, 7, 9, 10, 5)$ mod 11
and four individual cycles
$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$, $(0, 2, 4, 6, 8, 10, 1, 3, 5, 7, 9)$
$(0, 3, 6, 9, 1, 4, 7, 10, 2, 5, 8)$, $(0, 4, 8, 1, 5, 9, 2, 6, 10, 3, 7)$.

$n = 13$

$(\infty_1, 0, \infty_2, 1, 3, 2, 5, 6, 9, 8, 4, 10, 7)$
$(\infty_1, 0, 5, \infty_2, 9, 6, 4, 8, 2, 3, 7, 10, 1)$ mod 11

$n = 14$

$(\infty, 0, 1, 2, 5, 6, 8, 9, 4, 10, 11, 3, 12, 7)$
$(\infty, 0, 2, 7, 4, 1, 10, 8, 12, 5, 3, 11, 6, 9)$ mod 13

$n = 16$

$(0, 1, 3, 4, 7, 2, 5, 9, 11, 13, 6, 14, 10, 8, 15, 12)$
$(0, 2, 9, 3, 11, 14, 4, 5, 10, 6, 15, 12, 7, 13, 1, 8)$ mod 16
and three individual cycles
$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15)$,
$(0, 3, 6, 9, 12, 15, 2, 5, 8, 11, 14, 1, 4, 7, 10, 13)$,
$(0, 5, 10, 15, 4, 9, 14, 3, 8, 13, 2, 7, 12, 1, 6, 11)$.

$n = 17$

$(0, 1, 3, 4, 7, 2, 5, 6, 12, 14, 8, 13, 16, 9, 11, 15, 10)$
$(0, 2, 10, 5, 4, 12, 6, 3, 13, 7, 16, 11, 15, 8, 14, 1, 9)$ mod 17
and six individual cycles
$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16)$,
$(0, 2, 4, 6, 8, 10, 12, 14, 16, 1, 3, 5, 7, 9, 11, 13, 15)$,
$(0, 3, 6, 9, 12, 15, 1, 4, 7, 10, 13, 16, 2, 5, 8, 11, 14)$,
$(0, 4, 8, 12, 16, 3, 7, 11, 15, 2, 6, 10, 14, 1, 5, 9, 13)$,
$(0, 5, 10, 15, 3, 8, 13, 1, 6, 11, 16, 4, 9, 14, 2, 7, 12)$,
$(0, 6, 12, 1, 7, 13, 2, 8, 14, 3, 9, 15, 4, 10, 16, 5, 11)$.

$n = 19$

$(\infty_1, 0, \infty_2, 1, 3, 2, 5, 4, 8, 6, 11, 7, 12, 10, 16, 13, 9, 15, 14)$
$(\infty_1, 0, 4, \infty_2, 1, 6, 8, 13, 2, 3, 11, 14, 5, 12, 16, 7, 9, 15, 10)$
$(\infty_1, 0, 11, \infty_2, 2, 9, 16, 8, 3, 13, 4, 1, 5, 12, 6, 14, 10, 7, 15)$ mod 17

$n = 20$

$(\infty, 0, 1, 2, 4, 5, 14, 13, 8, 7, 11, 12, 16, 15, 3, 9, 10, 18, 17, 6)$
$(\infty, 0, 2, 4, 9, 11, 14, 16, 3, 1, 12, 6, 8, 18, 13, 7, 15, 5, 17, 10)$
$(\infty, 0, 3, 6, 10, 17, 7, 11, 16, 1, 4, 15, 12, 2, 9, 13, 5, 8, 18, 14)$ mod 19

$n = 22$

$(0, 1, 3, 4, 7, 2, 5, 6, 10, 8, 14, 9, 15, 17, 11, 20, 19, 12, 16, 21, 13, 18)$,
$(0, 2, 7, 9, 15, 1, 3, 10, 4, 11, 13, 21, 14, 5, 19, 18, 6, 17, 20, 8, 16, 12)$,
$(0, 2, 13, 6, 16, 1, 19, 10, 20, 3, 12, 18, 7, 14, 4, 15, 21, 9, 5, 17, 8, 11)$ mod 22
and four individual cycles
$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)$,
$(0, 3, 6, 9, 12, 15, 18, 21, 2, 5, 8, 11, 14, 17, 20, 1, 4, 7, 10, 13, 16, 19)$,
$(0, 5, 10, 15, 20, 3, 8, 13, 18, 1, 6, 11, 16, 21, 4, 9, 14, 19, 2, 7, 12, 17)$,
$(0, 7, 14, 21, 6, 13, 20, 5, 12, 19, 4, 11, 18, 3, 10, 17, 2, 9, 16, 1, 8, 15)$.

$n = 23$

$(0, 1, 3, 4, 7, 2, 5, 6, 10, 11, 16, 8, 9, 15, 17, 21, 14, 18, 20, 13, 19, 22, 12)$,
$(0, 2, 7, 8, 16, 3, 1, 9, 6, 15, 4, 11, 17, 22, 13, 18, 5, 19, 10, 20, 12, 21, 14)$,
$(0, 2, 12, 8, 20, 17, 4, 7, 16, 1, 5, 11, 19, 18, 6, 3, 14, 21, 10, 15, 22, 9, 13)$
mod 23
and eight individual cycles
$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22)$,
$(0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21)$,
$(0, 3, 6, 9, 12, 15, 18, 21, 1, 4, 7, 10, 13, 16, 19, 22, 2, 5, 8, 11, 14, 17, 20)$,
$(0, 4, 8, 12, 16, 20, 1, 5, 9, 13, 17, 21, 2, 6, 10, 14, 18, 22, 3, 7, 11, 15, 19)$,
$(0, 5, 10, 15, 20, 2, 7, 12, 17, 22, 4, 9, 14, 19, 1, 6, 11, 16, 21, 3, 8, 13, 18)$,
$(0, 6, 12, 18, 1, 7, 13, 19, 2, 8, 14, 20, 3, 9, 15, 21, 4, 10, 16, 22, 5, 11, 17)$,
$(0, 7, 14, 21, 5, 12, 19, 3, 10, 17, 1, 8, 15, 22, 6, 13, 20, 4, 11, 18, 2, 9, 16)$,
$(0, 8, 16, 1, 9, 17, 2, 10, 18, 3, 11, 19, 4, 12, 20, 5, 13, 21, 6, 14, 22, 7, 15)$.

$n = 25$

$(\infty_1, 0, \infty_2, 1, 3, 2, 5, 4, 8, 6, 11, 7, 10, 12, 16, 9, 13, 17, 22, 14, 20, 19, 15, 21, 18)$
$(\infty_1, 0, 4, \infty_2, 1, 6, 8, 13, 2, 3, 9, 10, 15, 5, 11, 21, 20, 12, 19, 22, 14, 16, 7, 18, 17)$
$(\infty_1, 0, 7, \infty_2, 1, 9, 2, 13, 4, 3, 12, 5, 19, 15, 10, 22, 14, 11, 20, 17, 6, 18, 16, 8, 21)$
$(\infty_1, 0, 9, \infty_2, 22, 11, 19, 2, 13, 21, 7, 10, 17, 6, 1, 15, 5, 18, 16, 3, 8, 14, 4, 20, 12)$
mod 23

$n = 26$

$(\infty, 0, 1, 2, 4, 5, 8, 9, 13, 14, 23, 24, 15, 16, 10, 11, 22, 21, 3, 20, 12, 19, 18, 6, 17, 7)$
$(\infty, 0, 2, 4, 7, 9, 1, 11, 24, 22, 13, 6, 20, 18, 3, 10, 8, 16, 23, 14, 12, 19, 21, 15, 17, 5)$
$(\infty, 0, 3, 6, 10, 13, 20, 23, 4, 17, 7, 1, 14, 11, 19, 8, 22, 5, 16, 24, 21, 12, 18, 2, 15, 9)$
$(\infty, 0, 4, 8, 13, 1, 6, 20, 24, 7, 11, 21, 17, 2, 12, 16, 5, 10, 19, 15, 3, 23, 18, 9, 22, 14)$
mod 25

$n = 28$

(0, 1, 3, 4, 7, 2, 5, 6, 10, 8, 14, 9, 13, 15, 20, 11, 12, 18, 21, 25, 16, 24, 26, 19, 27, 23, 17, 22),
(0, 2, 8, 10, 1, 3, 11, 4, 14, 5, 15, 9, 6, 16, 19, 27, 22, 7, 18, 23, 26, 12, 20, 25, 13, 21, 24, 17),
(0, 2, 11, 13, 1, 3, 14, 4, 15, 8, 21, 5, 6, 17, 23, 7, 20, 24, 10, 25, 18, 9, 26, 19, 12, 27, 22, 16),
(0, 2, 14, 26, 19, 13, 1, 20, 6, 9, 25, 8, 16, 24, 10, 23, 4, 17, 5, 22, 12, 7, 21, 11, 3, 27, 18, 15) mod 28
and five individual cycles

(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27),
(0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 2, 5, 8, 11, 14, 17, 20, 23, 26, 1, 4, 7, 10, 13, 16, 19, 22, 25),
(0, 5, 10, 15, 20, 25, 2, 7, 12, 17, 22, 27, 4, 9, 14, 19, 24, 1, 6, 11, 16, 21, 26, 3, 8, 13, 18, 23),
(0, 9, 18, 27, 8, 17, 26, 7, 16, 25, 6, 15, 24, 5, 14, 23, 4, 13, 22, 3, 12, 21, 2, 11, 20, 1, 10, 19),
(0, 11, 22, 5, 16, 27, 10, 21, 4, 15, 26, 9, 20, 3, 14, 25, 8, 19, 2, 13, 24, 7, 18, 1, 12, 23, 6, 17).

$n = 29$

(0, 1, 3, 4, 7, 2, 5, 6, 10, 11, 16, 8, 9, 15, 13, 19, 18, 12, 20, 23, 27, 17, 26, 21, 28, 25, 14, 24, 22),
(0, 2, 8, 11, 1, 3, 10, 12, 21, 4, 5, 13, 14, 23, 7, 9, 19, 24, 6, 16, 25, 28, 20, 17, 26, 22, 15, 27, 18),
(0, 2, 13, 1, 5, 11, 16, 3, 4, 14, 7, 20, 6, 10, 19, 26, 15, 27, 23, 9, 24, 21, 8, 25, 18, 12, 28, 22, 17),
(0, 2, 14, 18, 3, 11, 23, 28, 7, 13, 21, 6, 12, 26, 17, 1, 15, 10, 25, 9, 20, 27, 8, 24, 5, 16, 22, 4, 19) mod 29
and ten individual cycles
(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28),
(0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27),
(0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 2, 5, 8, 11, 14, 17, 20, 23, 26),
(0, 4, 8, 12, 16, 20, 24, 28, 3, 7, 11, 15, 19, 23, 27, 2, 6, 10, 14, 18, 22, 26, 1, 5, 9, 13, 17, 21, 25),
(0, 5, 10, 15, 20, 25, 1, 6, 11, 16, 21, 26, 2, 7, 12, 17, 22, 27, 3, 8, 13, 18, 23, 28, 4, 9, 14, 19, 24),
(0, 6, 12, 18, 24, 1, 7, 13, 19, 25, 2, 8, 14, 20, 26, 3, 9, 15, 21, 27, 4, 10, 16, 22, 28, 5, 11, 17, 23),

$(0, 7, 14, 21, 28, 6, 13, 20, 27, 5, 12, 19, 26, 4, 11, 18, 25, 3, 10, 17, 24, 2, 9, 16, 23, 1, 8, 15, 22)$,
$(0, 8, 16, 24, 3, 11, 19, 27, 6, 14, 22, 1, 9, 17, 25, 4, 12, 20, 28, 7, 15, 23, 2, 10, 18, 26, 5, 13, 21)$,
$(0, 9, 18, 27, 7, 16, 25, 5, 14, 23, 3, 12, 21, 1, 10, 19, 28, 8, 17, 26, 6, 15, 24, 4, 3, 22, 2, 11, 20)$,
$(0, 10, 20, 1, 11, 21, 2, 12, 22, 3, 13, 23, 4, 14, 24, 5, 15, 25, 6, 16, 26, 7, 17, 27, 8, 18, 28, 9, 19)$.

$n = 31$
$(\infty_1, 0, \infty_2, 1, 3, 2, 5, 4, 8, 6, 11, 7, 10, 12, 16, 9, 13, 14, 19, 20, 26, 15, 18, 23, 28, 21, 27, 24, 17, 22, 25)$
$(\infty_1, 0, 5, \infty_2, 1, 4, 10, 2, 3, 11, 7, 13, 15, 6, 8, 16, 17, 26, 9, 20, 23, 12, 25, 21, 14, 22, 27, 18, 24, 28, 19)$
$(\infty_1, 0, 6, \infty_2, 13, 1, 2, 12, 5, 17, 3, 4, 16, 7, 18, 9, 19, 20, 8, 10, 23, 25, 11, 26, 21, 15, 27, 28, 14, 24, 22)$
$(\infty_1, 0, 8, \infty_2, 17, 1, 4, 14, 16, 27, 2, 9, 15, 5, 18, 10, 3, 23, 19, 6, 26, 20, 12, 28, 22, 13, 25, 21, 7, 24, 11)$
$(\infty_1, 0, 14, \infty_2, 25, 15, 5, 27, 12, 4, 19, 23, 2, 13, 24, 7, 10, 21, 26, 8, 17, 22, 3, 16, 9, 28, 20, 11, 6, 18, 1) \bmod 29$

$n = 32$
$(\infty, 0, 1, 2, 4, 5, 8, 9, 3, 10, 11, 23, 12, 24, 25, 15, 16, 30, 29, 21, 20, 26, 19, 18, 14, 13, 27, 28, 17, 7, 6, 22)$
$(\infty, 0, 2, 4, 7, 9, 1, 3, 11, 13, 26, 15, 6, 25, 23, 18, 16, 22, 20, 29, 27, 17, 5, 19, 21, 30, 8, 10, 28, 12, 14, 24)$
$(\infty, 0, 3, 6, 10, 13, 1, 4, 11, 14, 29, 26, 8, 18, 21, 27, 30, 12, 20, 23, 15, 7, 24, 16, 5, 19, 22, 2, 25, 28, 9, 17)$
$(\infty, 0, 4, 8, 13, 17, 1, 5, 16, 9, 24, 20, 10, 3, 26, 22, 15, 29, 2, 23, 27, 14, 18, 6, 30, 19, 12, 28, 21, 11, 7, 25)$
$(\infty, 0, 5, 10, 16, 1, 7, 21, 15, 2, 8, 27, 22, 12, 6, 20, 26, 13, 18, 25, 30, 17, 4, 29, 9, 24, 3, 19, 14, 28, 23, 11) \bmod 31$

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