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Lower bound on the weakly connected domination number of a cycle-disjoint graph*

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Abstract

For a connected graph G and any non-empty $S \subseteq V(G)$, S is called a *weakly connected dominating set* of G if the subgraph obtained from G by removing all edges each joining any two vertices in $V(G) \setminus S$ is connected. The *weakly connected domination number* $\gamma_w(G)$ is defined to be the minimum integer k with $|S| = k$ for some weakly connected dominating set S of G. In this note, we extend a result on the lower bound for the weakly connected domination number $\gamma_w(G)$ on trees to cycle-e-disjoint graphs, i.e., graphs in which no cycles share a common edge. More specifically, we show that if G is a connected cycle-e-disjoint graph, then $\gamma_w(G) \geq (|V(G)| - v_1(G) - n_c(G) - n_{oc}(G) + 1)/2$, where $n_c(G)$ is the number of cycles in G, $n_{oc}(G)$ is the number of odd cycles in G and $v_1(G)$ is the number of vertices of degree 1 in G. The graphs for which equality holds are also characterised.

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1 Introduction

Let $G = (V, E)$ be a (simple) graph. For any vertex $v \in V$, the *open neighbourhood* $N(v)$ *of* v is the set $\{u \in V \mid uv \in E\}$, while the *closed neighbourhood* $N[v]$ is $N(v) \cup \{v\}$. For $S \subseteq V$, the *closed neighbourhood* $N[S]$ is $\cup_{v \in S} N[v]$. We call S a *dominating set* if $N[S] = V$.

Let $S \subseteq V$. The *subgraph* $\langle S \rangle_w$ *of* G *weakly induced by* S is the graph $(N[S], E \cap (S \times N[S]))$. We call S a *weakly connected dominating set* (WCDS) of G if S is a dominating set of G and $\langle S \rangle_w$ is connected, i.e., the subgraph obtained from G by removing all edges joining any two vertices in $V(G) \setminus S$ is connected. The *weakly connected domination number* $\gamma_w(G)$ *of* G is the minimum cardinality among all weakly connected dominating sets in G. For any WCDS S of G, if $|S| = \gamma_w(G)$, then we call it a MWCDS. The parameter $\gamma_w(G)$ was first introduced in [2]. For some existing results on $\gamma_w(G)$, see [1, 2, 3].

A vertex in a graph is called an *end-vertex* if it is of degree 1. Let $\mathcal{R}$ be the family of trees defined recursively as follows:

(a) $K_{1,p} \in \mathcal{R}$ for $p \geq 2$;

(b) for any $T \in \mathcal{R}$ and any $p \geq 2$, the graph obtained from T and $K_{1,p}$ by identifying any end-vertex in T with any end-vertex in $K_{1,p}$ belongs to $\mathcal{R}$.

Let G be a connected graph. Denote by $v(G)$ the number of vertices of G and $v_1(G)$ the number of vertices of degree 1 (i.e. leaves) of G. Lemanska [3] proved the following result:

Theorem 1.1. *If T is a tree with $v(T) \geq 2$, then $\gamma_w(T) \geq \frac{v(T)-v_1(T)+1}{2}$; and equality holds if and only if T belongs to the family $\mathcal{R}$.*

A connected graph G is said to be *cycle-e-disjoint* if no two cycles in G have an edge in common. In this paper, we shall establish a lower bound of $\gamma_w(G)$ for a cycle-e-disjoint graph in terms of $v(G)$, $v_1(G)$ and the number of cycles in G. The structure of cycle-e-disjoint graphs attaining the lower bound is also characterised.

2 Preliminary results

To begin with, we introduce two operations to combine two connected graphs G_1 and G_2 to form a graph G, and obtain relations among $\gamma_w(G)$, $\gamma_w(G_1)$ and $\gamma_w(G_2)$.

Operation 1: Edge linking

Let G_1 and G_2 be two connected graphs with $V(G_1) \cap V(G_2) = \emptyset$. For $x \in V(G_1)$ and $y \in V(G_2)$, let $G_1(x) - G_2(y)$ denote the graph obtained from G_1 and G_2 by adding an edge joining x to y.

Lemma 2.1. *Let $G = G_1(x) - G_2(y)$ be the graph defined above, $S \subseteq V(G)$ and $S_i = S \cap V(G_i)$ for $i = 1, 2$. Assume that $v(G_i) \geq 2$ for $i = 1, 2$. Then*

(a) *S is a WCDS of G if and only if S_i is a WCDS of G_i for $i = 1, 2$ and $\{x, y\} \cap (S_1 \cup S_2) \neq \emptyset$;*

(b) *$\gamma_w(G) \geq \gamma_w(G_1) + \gamma_w(G_2)$, where the equality holds if and only if S_i is a MWCDS of G_i for $i = 1, 2$ and $\{x, y\} \cap (S_1 \cup S_2) \neq \emptyset$.*

Proof. (a) Let S be a WCDS of G. Since $v(G_1) \geq 2$ and $\langle S \rangle_w$ is connected, we have $N[x] \cap S_1 \neq \emptyset$. So S_1 is a WCDS of G_1, and similarly S_2 is a WCDS of G_2.

It is obvious that $S = S_1 \cup S_2$ is a WCDS of G if S_i is a WCDS of G_i for $i = 1, 2$ and $\{x, y\} \cap (S_1 \cup S_2) \neq \emptyset$.

(b) If S is a MWCDS of G, then by (a), S_i is a WCDS of G_i and so

$$\gamma_w(G) \;=\; |S| \;=\; |S_1| + |S_2| \;\geq\; \gamma_w(G_1) + \gamma_w(G_2).$$

Assume that $\gamma_w(G) = \gamma_w(G_1) + \gamma_w(G_2)$. Then $|S_i| = \gamma_w(G_i)$ for $i = 1, 2$. Since $\langle S \rangle_w$ is a connected spanning subgraph of G, we have either $x \in S_1$ or $y \in S_2$.

On the other hand, if S_i is a MWCDS of G_i for $i = 1, 2$ and either $x \in S_1$ or $y \in S_2$, then by (a), $S_1 \cup S_2$ is a WCDS of G and

$$\gamma_w(G) \;\leq\; |S_1 \cup S_2| \;=\; |S_1| + |S_2| \;=\; \gamma_w(G_1) + \gamma_w(G_2).$$

Hence the result holds. □

Operation 2: Vertex gluing

Let G_1 and G_2 be two graphs with $V(G_1) \cap V(G_2) = \emptyset$. For any $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$, let $G_1(u_1) \cdot G_2(u_2)$ denote the graph obtained from G_1 and G_2 by gluing(identifying) u_1 with u_2.

Lemma 2.2. *Let G be the graph $G_1(u_1) \cdot G_2(u_2)$, $S \subseteq V(G)$ and $S_i = S \cap V(G_i)$ for $i = 1, 2$.*

(a) *S is a WCDS of G if and only if S_i is a WCDS of G_i for $i = 1, 2$.*

(b) *$\gamma_w(G) \geq \gamma_w(G_1) + \gamma_w(G_2) - 1$, where the equality holds if and only if S_i is a MWCDS of G_i and each u_i is contained in a MWCDS of G_i, for $i = 1, 2$.*

Proof. (a) Assume that S is a WCDS of G. Consider H_1, the subgraph of $\langle S \rangle_w$ when restricted to G_1. Clearly, H_1 is a connected spanning subgraph of G_1. Now every edge of H_1 has an end in S_1 and every edge of $\langle S_1 \rangle_w$ is in H_1, so $\langle S_1 \rangle_w = H_1$ and S_1 is indeed a WCDS of G_1. Similarly, S_2 is a WCDS of G_2.

It is obvious that if S_i is a WCDS of G_i for $i = 1, 2$, then S is a WCDS of G.

(b) Assume that S is a MWCDS of G. Since $|S_1 \cap S_2| \leq 1$, we have

$$\gamma_w(G) = |S| = |S_1 \cup S_2| \;=\; |S_1| + |S_2| - |S_1 \cap S_2| \;\geq\; \gamma_w(G_1) + \gamma_w(G_2) - 1.$$

Note that the above equality holds if and only if $|S_i| = \gamma_w(G_i)$ for $i = 1, 2$ and $|S_1 \cap S_2| = 1$. Thus (b) holds. □

3 Cycle-e-disjoint graphs

In this section we shall find a lower bound for $\gamma_w(G)$ for a cycle-e-disjoint graph G. We first establish the following results.

Let us state the following result which will be applied later. It can be proven by induction.

Lemma 3.1. *Let $A_1, A_2, \ldots, A_m$ be any m finite sets, where $m \geq 1$. Then*

$$|A_1 \cup A_2 \cup \cdots \cup A_m| \;=\; \sum_{i=1}^{m} |A_i| - \sum_{i=2}^{m} |A_i \cap (A_1 \cup \cdots \cup A_{i-1})|. \tag{1}$$

By Lemma 3.1, we have:

Corollary 3.2. *Let $A_1, A_2, \ldots, A_m$ be any m finite sets, where $m \geq 1$. If $|A_i| \geq a_i$ for $i = 1, 2, \ldots, m$ and $|A_i \cap (A_1 \cup \cdots \cup A_{i-1})| \leq 1$ for all $2 \leq i \leq m$, then*

$$|A_1 \cup A_2 \cup \cdots \cup A_m| \geq a_1 + a_2 + \cdots + a_m - m + 1, \tag{2}$$

where the equality holds if $|A_i| = a_i$ for all $i = 1, 2, \ldots, m$ and $|A_i \cap (A_1 \cup \cdots \cup A_{i-1})| = 1$ for all $2 \leq i \leq m$. □

For any connected non-trivial graphs $G_1, G_2, \ldots, G_m$, let $\mathcal{G}(G_1, G_2, \ldots, G_m)$ be the family of graphs defined recursively as follows:

(i) $\mathcal{G}(G_1) = \{G_1\}$;

(ii) for $m \geq 2$, $H(x) \cdot G_m(y) \in \mathcal{G}(G_1, G_2, \ldots, G_m)$ for any $H \in \mathcal{G}(G_1, G_2, \ldots, G_{m-1})$, where $x \in V(H)$ and $y \in V(G_m)$.

Note that each G_i is an induced subgraph of any graph in $\mathcal{G}(G_1, G_2, \ldots, G_m)$.

For a graph H and a subgraph G of H, write

$$F_{G,H} = \{x \in V(G) : xy \in E(H) \text{ for some } y \in V(H) \setminus V(G)\}.$$

Lemma 3.3. *Let $G_1, G_2, \ldots, G_m$ be any m connected non-trivial graphs. Then, for any graph $H \in \mathcal{G}(G_1, G_2, \ldots, G_m)$,*

$$\gamma_w(H) \geq \sum_{i=1}^{m} \gamma_w(G_i) - m + 1, \tag{3}$$

where the equality holds if and only if $F_{G_i,H}$ is a subset of some MWCDS of G_i for all $i = 1, 2, \ldots, m$.

Proof. Let $H \in \mathcal{G}(G_1, G_2, \ldots, G_m)$ and S an MWCDS of H. By Lemma 2.2, S_i is a WCDS of G_i for $i = 1, 2, \ldots, m$, where $S_i = V(G_i) \cap S$. Note that

(i) $|S_i| \geq \gamma_w(G_i)$ for all $i = 1, 2, \ldots, m$;

(ii) $|S_i \cap (S_1 \cup \cdots \cup S_{i-1})| \leq 1$ for all $2 \leq i \leq m$.

By Corollary 3.2, we have

$$\gamma_w(H) = |S| = |S_1 \cup S_2 \cup \cdots \cup S_m| \geq \sum_{i=1}^{m} \gamma_w(G_i) - m + 1, \tag{4}$$

where the equality holds if $|S_i| = \gamma_w(G_i)$ for all $i = 1, 2, \ldots, m$ and $|S_i \cap (S_1 \cup \cdots \cup S_{i-1})| = 1$ for all $2 \leq i \leq m$. Observe that

(i) $|S_i| = \gamma_w(G_i)$ if and only if S_i is an MWCDS of G_i;

(ii) $|S_i \cap (S_1 \cup \cdots \cup S_{i-1})| = 1$ for all $2 \leq i \leq m$ if and only if $F_{G_i,H} \subseteq S_i$ for all $i = 1, 2, \ldots, m$.

Hence the result holds. □

Let G be a connected graph and x any vertex in G. If $G - x$ is disconnected, where $G - x$ is the graph obtained from G by deleting x and all edges incident with x, then x is called a *cut-vertex* of G.

Let G be any connected cycle-e-disjoint graph and $\mathcal{C}(G)$ the family of cycles in G.

A connected graph is said to be *separable* if it contains a cut-vertex, and *non-separable* otherwise. A *block* in a graph G is a maximal induced subgraph of G which is non-separable. Recall that if G is cycle-e-disjoint, then $E(C_1) \cap E(C_2) = \emptyset$ for for any two distinct $C_1, C_2 \in \mathcal{C}(G)$. Then we can get the following characterization on cycle-e-disjoint graphs.

Lemma 3.4. *Let G be a connected cycle-e-disjoint graph. Then every cycle of G is a block and hence every block of G is a cycle or a bridge of G.* □

Let $n_c(G)$ be the number of cycles in G and $n_{oc}(G)$ the number of odd cycles in G. Applying Lemmas 3.3 and 3.4, we find a lower bound for $\gamma_w(G)$, where G is a connected cycle-e-disjoint graph in which each bridge has an end-vertex as one end.

Lemma 3.5. *Let G be a connected cycle-e-disjoint graph with $v(G) \geq 3$. Assume that one end of each bridge in G is an end-vertex of G. Then*

$$\gamma_w(G) \geq \frac{v(G) - v_1(G) + 1 - n_c(G) - n_{oc}(G)}{2}, \qquad (5)$$

where the equality holds if and only if for each cycle C in G, $F_{C,G}$ is a subset of some MWCDS of C.

Proof. If G does not contain a cycle, then G is a star and equality (5) holds by Theorem 1.1.

Assume that G contains m cycles, where $m \geq 1$. Since G is cycle-e-disjoint by Lemma 3.4, every block of G is either a cycle or a bridge. As one end of each bridge in G is an end-vertex of G, every block of G is a cycle or a bridge uv with u on some cycle of G and v is an end-vertex of G. Then there is an ordering of blocks $G_1, G_2, \ldots, G_m, \ldots, G_k$, where $k \geq m$ such that

(a) each G_i is a cycle for $i = 1, 2, \ldots, m$ and each $G_j \cong K_2$ for $m + 1 \leq j \leq k$;

(b) for $i = 2, 3, \ldots, m$, $|V(G_i) \cap (V(G_1) \cup \cdots \cup V(G_{i-1})| = 1$, and for $i = m+1, \ldots, k$, $|V(G_i) \cap (V(G_1) \cup \cdots \cup V(G_m)| = 1$.

By Lemma 3.3, we have

$$\gamma_w(G) \geq \sum_{i=1}^{k} \gamma_w(G_i) - k + 1,$$

where the equality holds if and only if for each G_i, $F_{G_i,G}$ is a subset of some MWCDS of G_i. Note that for $i = m + 1, \ldots, k$, $F_{G_i,G}$ is indeed a subset of some MWCDS of G_i. Hence this condition is equivalent to that for each cycle C in G, $F_{C,G}$ is a subset of some MWCDS of C.

It remains to show that

$$\sum_{i=1}^{k} \gamma_w(G_i) - k + 1 = \frac{v(G) - v_1(G) + 1 - n_c(G) - n_{oc}(G)}{2}. \qquad (6)$$

Since $\gamma_w(G_i) = 1$ for $m < i \leq k$, we need only consider the case that $k = m$, i.e., G contains no bridges.

For $i = 1, 2, \ldots, m$, G_i is a cycle and so $\gamma_w(G_i) = \lfloor v(G_i) \rfloor$. Observe that $v_1(G) = 0$ and $|V(G_1)| + |V(G_2)| + \cdots + |V(G_m)| = v(G) + m - 1$ by Corollary 3.2. Thus

$$\begin{aligned}
\sum_{i=1}^{m} \lfloor |V(C_i)|/2 \rfloor - m + 1 &= \sum_{i=1}^{m} |V(C_i)|/2 - n_{oc}(G)/2 - m + 1 \\
&= \frac{v(G) + m - 1 - n_{oc}(G)}{2} - m + 1 \\
&= \frac{v(G) + 1 - n_c(G) - n_{oc}(G)}{2},
\end{aligned}$$

because $n_c(G) = m$. □

If $d(x) = 2$, let $G \circ x$ denote the graph obtained from $G - x$ by identifying the two neighbours of x.

Lemma 3.6. *Let G be a connected graph of order at least 4. If x is a cut-vertex of G with $d(x) = 2$, then $\gamma_w(G) = \gamma_w(G \circ x) + 1$.*

Proof. Assume that $N(x) = \{y, z\}$. Let w be the new vertex in $G \circ x$ after identifying y and z.

Assume that S is an MWCDS of $G \circ x$. If $w \in S$, then $(S \setminus \{w\}) \cup \{y, z\}$ is a WCDS of G; otherwise, $S \cup \{x\}$ is a WCDS of G. Thus $\gamma_w(G) \leq \gamma_w(G \circ x) + 1$.

Now assume that T is a MWCDS of G. As x is a cut-vertex of G, both xy and xz are bridges of G. If $x \notin T$, then $\{y, z\} \subseteq T$ and thus $(T \setminus \{y, z\}) \cup \{w\}$ is a WCDS of $G \circ x$. If $x \in T$ and $\{y, z\} \cap T = \emptyset$, then $T \setminus \{x\}$ is a WCDS of $G \circ x$, as G is connected and $v(G) \geq 4$. If $x \in T$ and $\{y, z\} \cap T \neq \emptyset$, then $(T \setminus \{x, y, z\}) \cup \{w\}$ is a WCDS of $G \circ x$. Hence $\gamma_w(G \circ x) \leq \gamma_w(G) - 1$.

Therefore the result holds. □

Let G be a graph. Let $\mathcal{P}_b(G)$ be the set of paths P of G such that every edge of P is a bridge of G. Let $\mathcal{P}_1(G)$ the set of paths $u_0u_1 \cdots u_k \in \mathcal{P}_b(G)$ such that $d_G(u_0) \geq 3, d_G(u_k) \geq 3$ but $d_G(u_i) = 2$ for all $1 \leq i \leq k-1$, and $\mathcal{P}_2(G)$ the set of paths $u_0u_1 \cdots u_k \in \mathcal{P}_b(G)$ such that $d_G(u_0) \geq 3, d_G(u_k) = 1$ but $d_G(u_i) = 2$ for all $1 \leq i \leq k-1$.

A path is said to be *odd* if it contains an odd number of edges and *even* otherwise.

Theorem 3.7. *Let G be a connected cycle-e-disjoint graph which is not a tree. Then*

$$\gamma_w(G) \geq \frac{v(G) - v_1(G) + 1 - n_c(G) - n_{oc}(G)}{2}, \tag{7}$$

where the equality holds if and only if the following conditions are satisfied:

(a) there are no odd paths in $\mathcal{P}_1(G)$;

(b) there are no even paths in $\mathcal{P}_2(G)$; and

(c) $F_{C,G}$ is a subset of some MWCDS of C for every cycle C in G.

Proof. By Lemma 3.5, the result holds if G contains no bridges uv such that $d(u) \geq 2$ and $d(v) \geq 2$.

Assume that the result holds if the order of G is less than m, where $m \geq 4$. Now let G be a connected cycle-e-disjoint graph of order m with $\delta(G) \geq 2$. By Lemma 3.5, we need only to consider the case that G contains some bridges uv with $d(u) \geq 2$ and $d(v) \geq 2$.

Then, one of the following situations occurs:

(1) G has a bridge with $d_G(u) \geq 3$ and $d_G(v) \geq 3$;

(2) G contains a cut-vertex x with $d(x) = 2$, $d(u) \geq 2$ and $d(v) \geq 2$, where u, v are the two neighbours of x;

(3) G has a cut-vertex x with $d(x) = 2$, $d(u) \geq 3$ and $d(v) = 1$, where u, v are the two neighbours of x.

Case 1: G contains a bridge uv with $d(u) \geq 3$ and $d(v) \geq 3$.

Note that in this case, there is an odd path in $\mathcal{P}_1(G)$ and we need to show that inequality (7) is strict.

Let G_1 and G_2 be the two components of $G - uv$. It is clear that each G_i is either a tree or a connected cycle-e-disjoint graph with at least one cycle. By Theorem 1.1 or by induction,

$$\gamma_w(G_i) \geq \frac{v(G_i) - v_1(G_i) + 1 - n_c(G_i) - n_{oc}(G_i)}{2},$$

for $i = 1, 2$. Notice that $v(G) = v(G_1) + v(G_2)$, $n_c(G) = n_c(G_1) + n_c(G_2)$, $v_1(G) = v_1(G_1) + v_1(G_2)$ and $n_{oc}(G) = n_{oc}(G_1) + n_{oc}(G_2)$. Thus, by Lemma 2.1,

$$\begin{aligned} \gamma_w(G) &\geq \gamma_w(G_1) + \gamma_w(G_2) \\ &\geq \frac{v(G) - v_1(G) + 2 - n_c(G) - n_{oc}(G)}{2} \\ &> \frac{v(G) - v_1(G) + 1 - n_c(G) - n_{oc}(G)}{2}. \end{aligned}$$

Case 2: G contains a cut-vertex x with $d(x) = 2$, $d(u) \geq 2$ and $d(v) \geq 2$, where u, v are the two neighbours of x.

It is clear that $G \circ x$ is a connected cycle-e-disjoint graph with at least one cycle. By Lemma 3.6, $\gamma_w(G) = \gamma_w(G \circ x) + 1$. Also notice that

- G and $G \circ x$ have the same cycle set;
- $v(G) = v(G \circ x) + 2$ and $v_1(G) = v_1(G \circ x)$;
- for each cycle C in G, $F_{C,G} = F_{C,G \circ x}$.

Hence the result also holds for G since the result holds for $G \circ x$ by induction.

Case 3: G has a cut-vertex x with $d(x) = 2$, $d(u) \geq 3$ and $d(v) = 1$, where u, v are the two neighbours of x.

Note that in this case, there is an even path in $\mathcal{P}_2(G)$ and we need to show that inequality (7) is strict.

It is clear that each $G \circ x$ is a connected cycle-e-disjoint graph with at least one cycle. By induction,

$$\gamma_w(G \circ x) \geq \frac{v(G \circ x) - v_1(G \circ x) + 1 - n_c(G \circ x) - n_{oc}(G \circ x)}{2}.$$

Notice that $v(G) = v(G \circ x) + 2$, $n_c(G) = n_c(G \circ x)$, $v_1(G) = v_1(G \circ x) + 1$ and $n_{oc}(G) = n_{oc}(G \circ x)$. Thus, by Lemma 3.6,

$$\begin{aligned}\gamma_w(G) &= \gamma_w(G \circ x) + 1 \\ &\geq \frac{v(G \circ x) - v_1(G \circ x) + 1 - n_c(G \circ x) - n_{oc}(G \circ x)}{2} + 1 \\ &> \frac{v(G) - v_1(G) + 1 - n_c(G) - n_{oc}(G)}{2}.\end{aligned}$$

□

Remarks: Inequality (7) holds for a path G and it is strict if and only if G is an odd path. If G is a tree but not a path, then the result of Theorem 3.7 holds. It can be proved by an idea similar to that used in the proof of Theorem 3.7 or by the definition of $\mathcal{R}$.

Lemma 3.8. *Let T be any tree. If T is a path, then $T \in \mathcal{R}$ if and only if T is an even path; otherwise, $T \in \mathcal{R}$ if and only if the following conditions are satisfied:*

(a) there are no odd paths in $\mathcal{P}_1(T)$; and

(b) there are no even paths in $\mathcal{P}_2(T)$. □

Let $V_t(G)$ be the set of vertices $x \in V(C)$ for some $C \in \mathcal{C}(G)$ such that x is incident to some bridges of G. Let $V_t(G) = \{x_1, \ldots, x_k\}$ and G' the graph obtained from G by adding k vertices $w_1, \ldots, w_k$ and k edges $w_i x_i$ for $i = 1, 2, \ldots, k$.

Lemma 3.9. *Let G be a connected cycle-e-disjoint graph. Then G satisfies condition (a) and (b) in Theorem 3.7 if and only if each component of the graph $G' - \cup_{C \in \mathcal{C}(G)} E(C)$ belongs to $\{K_1\} \cup \mathcal{R}$.*

Proof. Let $V'_t(G)$ be the set of vertices $x \in V_t(G)$ such that x is incident to only one bridge of G. Let $\mathcal{P}'_s(G)$ be the set of paths P in $\mathcal{P}_s(G)$ such that only one end of P belongs to $V'_t(G)$ for $s = 1, 2$. Let $\mathcal{P}''_1(G)$ be the set of paths P in $\mathcal{P}_1(G)$ such that both ends of P belong to $V'_t(G)$.

Let H denote the graph $G' - \cup_{C \in \mathcal{C}(G)} E(C)$ and $\mathcal{T}(H)$ the family of non-trivial components of H. It is clear that H contains no cycles, i.e., each $T \in \mathcal{T}(H)$ is a tree.

Observe that

(i) For each $P \in \mathcal{P}'_1(G)$, if x_i is one end of P, then the path formed by P and the edge $w_i x_i$, denoted by $P + w_i x_i$, is a path belonged to $\cup_{T \in \mathcal{T}(H)} \mathcal{P}_2(T)$. It is clear that P is even if and only if $P + w_i x_i$ is odd.

(ii) For each $P \in \mathcal{P}_1''(G)$, if x_i and x_j are the two ends of P, then the path formed by P and the edges w_ix_i and w_jx_j, denoted by $P+w_ix_i+w_jx_j$, is a component of H. It is clear that P is even if and only if $P+w_ix_i+w_jx_i$ is even.

(iii) $\mathcal{P}_1(G) \setminus (\mathcal{P}_1'(G) \cup \mathcal{P}_1''(G)) = \cup_{T\in\mathcal{T}(H)}\mathcal{P}_1(T)$.

(iv) For each $P \in \mathcal{P}_2'(G)$, if x_i is one end of P, then the path formed by P and the edge w_ix_i, denoted by $P+w_ix_i$, is a component of H. It is clear that P is odd if and only if $P+w_ix_i$ is even.

(v) $\cup_{T\in\mathcal{T}(H)}\mathcal{P}_2(T) = (\mathcal{P}_2(G)\setminus\mathcal{P}_2'(G))\cup\{P+w_ix_i : P \in \mathcal{P}_1'(G), x_i \text{ is one end of } P\}$.

By the above observations, G satisfies conditions (a) and (b) in Theorem 3.7 if and only if each component of H is either an even path or satisfies conditions (a) and (b) in Lemma 3.8. Hence, by Lemma 3.8, the result holds. □

By Lemma 3.9, we have

Corollary 3.10. *Let G be a connected cycle-e-disjoint graph which is not a tree. Then*

$$\gamma_w(G) \geq \frac{v(G) - v_1(G) + 1 - n_c(G) - n_{oc}(G)}{2}, \tag{8}$$

where the equality holds if and only if each component of the graph $G' - \cup_{C\in\mathcal{C}(G)}E(C)$ is contained in $\{K_1\}\cup\mathcal{R}$ and $F_{C,G}$ is a subset of some MWCDS of C for every cycle $C \in \mathcal{C}(G)$.

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