Existence of SBIBD($4k^2, 2k^2 \pm k, k^2 \pm k$) and Hadamard matrices with maximal excess

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Abstract

It is shown that SBIBD($4k^2, 2k^2 \pm k, k^2 \pm k$) and Hadamard matrices with maximal excess exist for $k = qs, q \in \{q : q \equiv 1 \pmod 4 \text{ is a prime power}\}$, $s \in \{1, \ldots, 33, 37, \ldots, 41, 45, \ldots, 59\} \cup \{2g + 1, g \text{ the length of a Golay sequence}\}$.

This leaves the following odd $k < 250$ undecided 47,71,77,79,103,107,127,131,133,139, 141,151,163,167,177,179,191,199,209,...,217,223,227, 231,233,237,239,243,249.

There is also a proper n dimensional Hadamard matrix of order $(4k^2)^n$.

Regular symmetric Hadamard matrices with constant diagonal are obtained for orders $4k^2$ whenever complete regular 4-sets of regular matrices of order k^2 exist.

1 Introduction

We refer the reader to J. Wallis [8] and A.V. Geramita and J. Seberry [2] for undefined terms.

The *excess* of an Hadamard matrix is the sum of its elements. The maximal excess of all Hadamard matrices of order $4k^2$ is $8k^3$ and this is equivalent to the existence of an SBIBD($4k^2, 2k^2 \pm k, k^2 \pm k$) (see Seberry [6], Best [1]).

Theorem 1 *Suppose there exist* $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ *supplementary difference sets. Then*

(i) there is an Hadamard matrix of order $4q^2$ *with maximal excess* $8q^3$*,*

(ii) there is an $SBIBD(4q^2, 2q^2 \pm q, q^2 \pm q)$*,*

(iii) there is a proper n *dimensional Hadamard matrix of order* $(4q^2)^n$*.*

Proof: Form the group ± 1 incidence matrix (type 1) for each of the sets. Then each row has $\frac{1}{2}q(q-1)$ elements plus one and $\frac{1}{2}q(q+1)$ elements minus one. Thus the row sum is $-q$. Negate each matrix to form four matrices A, B, C, D each with row sum q and thus excess q^3.

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Then, recalling that the $1,-1$ incidence matrices of $n-\{v,k,\lambda\}$ supplementary difference sets have inner product, given by Lemma 1.20 [8], $4(nk-\lambda)I+(nv-4nk+4\lambda)J=4q^2I$ in this case, form

$$H=\begin{bmatrix} -A & BR & CR & DR \\ BR & A & D^TR & -C^TR \\ CR & -D^TR & A & B^TR \\ DR & C^TR & -B^TR & A \end{bmatrix},$$

where R transforms each matrix into its type 2 form (see Geramita and Seberry [2]). H is an Hadamard matrix of order $4q^2$ with excess $8q^3$.

We note each row of H has $3q(q+1)/2+q(q-1)/2=2q^2+q$ elements $+1$ and since H has constant inner product zero the underlying (0,1) matrix (mapping -1 to 0) has constant inner product q^2+q. Thus we have the required SBIBD.

Let 0,1,2,3 be the cyclic group of order 4. Then the elements (i,d_j), $i=0,1,2,3$, $d_j\in D_j$, $j=1,2,3,4$ the supplementary difference sets (in this case with the extra properties of the D_j), form an abelian group difference set and so satisfy Theorem 4 of [3]. This gives the proper higher dimensional Hadamard matrix. □

Now M. Xia and G. Liu [9] have reported the existence of these supplementary difference sets for every $q\equiv 1 \pmod 4$ a prime power. Thus we have

Theorem 2 *Let $q\equiv 1 \pmod 4$ be a prime power. Then there is an $SBIBD(4q^2, 2q^2\pm q, q^2\pm q)$.*

Combining these results with those of Koukouvinos and Seberry [5] we have, noting that Koukouvinos, Kounias and Sotirakoglou [4] have now found T-sequences of lengths 55 and 57 which correspond to $s^2=s^2+0^2+0^2+0^2$,

Corollary 3 *$SBIBD(4k^2, 2k^2\pm k, k^2\pm k)$ and Hadamard matrices with maximal excess exist for*

(i) $k\in\{1,\dots,45,49,\dots,69,73,75,81,\dots,101,105,109,\dots,125,129,135,137,143,\dots,149,153,\dots,161,165,169,\dots,175,181,\dots,189,193,\dots,197,201,\dots,207,219,221,225,229,235,241,245,247\}\cup\{p: p\equiv 1 \pmod 4$ *is a prime power*$\}\cup\{2s+1:$ *s the length of Golay sequences*$\}$,

(ii) $k\in\{5(2s+1), 5.3^i(2s+1),\ i>0,$ *s the length of Golay sequences*$\}$,

(iii) $k\in\{qs: q\equiv 1 \pmod 4$ *is a prime power and* s (odd) $=1,\dots,33,37,\dots,41,45,\dots,59$ *or* $s=2g+1$, *g the length of Golay sequences*$\}$.

We recall Theorem 6 of [5] which uses T-sequences of length s^2 corresponding to the decomposition

$$s^2=s^2+0^2+0^2+0^2$$

and Williamson-type matrices of order $4w^2$ corresponding to the decomposition

$$4w^2=w^2+w^2+w^2+w^2$$

to form Hadamard matrices with maximal excess of order $4k^2=4s^2w^2$ and an SBIBD$(4k^2, 2k^2\pm k, k^2\pm k)$.

Hence we have

Corollary 4 *Let $q \equiv 1 \pmod 4$ be a prime power. Then there exists an Hadamard matrix with maximal excess of order $4k^2 = 4q^2s^2$ and an $SBIBD(4k^2, 2k^2 \pm k, k^2 \pm k)$ for s (odd) $\in \{1, ..., 33, 37, ..., 41, 45, ..., 59\} \cup \{2g+1 : g$ the length of a Golay sequence$\}$.*

Proof: The required T-sequences are found in §3 of [5] and [4]. □

2 Regular matrices

We now define a *complete regular 4-set of regular matrices* of order q^2 as four $1, -1$ matrices satisfying

$$A_i^T = A_i, \quad A_iA_j = aJ, \quad i \neq j,\ i,j = 1,2,3,4, \text{ a constant}$$
$$A_iJ = qJ,$$
$$\textstyle\sum_{i=1}^4 A_i^2 = 4q^2 I.$$

Then we have:

Theorem 5 *If there exist complete regular 4-sets of regular matrices of orders s^2 and t^2 respectively then there exists a complete regular 4-set of regular matrices of order s^2t^2.*

Proof: Let the sets of order s^2 and t^2 be A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 respectively, so $A_iJ = sJ$, $B_iJ = tJ$, $i = 1,2,3,4$. Then

$$\begin{aligned} C_1 &= \frac{1}{2}[A_1 \times (B_1 + B_2) + A_2 \times (B_1 - B_2)] \\ C_2 &= \frac{1}{2}[-A_1 \times (B_3 - B_4) + A_2 \times (B_3 + B_4)] \\ C_3 &= \frac{1}{2}[A_3 \times (B_1 + B_2) - A_4 \times (B_1 - B_2)] \\ C_4 &= \frac{1}{2}[A_3 \times (B_3 - B_4) + A_4 \times (B_3 + B_4)] \end{aligned}$$

is a complete regular 4-set of regular matrices of order s^2t^2. □

A complete regular 4-set of regular matrices may be constructed from the following $4 - \{9; 6; 15\}$

$$\{x, x+1, x+2, 2x, 2x+1, 2x+2\}$$
$$\{1, 2, x, x+2, 2x, 2x+1\}$$
$$\{1, 2, x+1, x+2, 2x+1, 2x+2\}$$
$$\{1, 2, x, x+1, 2x, 2x+2\}$$

given by A.L. Whiteman (see also [7]). So we have

Corollary 6 *Complete regular 4-sets of regular matrices exist for orders 9^i, $i = 1, 2, ...,$.*

If we could establish the existence of complete regular 4-sets of regular matrices of orders $q_1^2, q_2^2, ...$ with row sums $q_1, q_2, ...$ respectively we would have

Theorem 7 *Let $q_1^2, q_2^2, ...$ be the orders of complete regular 4-sets of regular matrices with row sums $q_1, q_2, ...$ respectively. Write $w^2 = q_1^2.q_2^2. ....$ Then there is* *(i) a complete regular 4-set of matrices of order* w^2

(ii) Williamson-type matrices of order w^2

(iii) SBIBD$(4w^2, 2w^2 \pm w, w^2 \pm w)$

(iv) a regular symmetric Hadamard matrix with constant diagonal of order $4w^2$ *with maximal excess* $8w^3$

(v) a proper higher dimensional Hadamard matrix of order $(4w^2)^n$.

Proof: The previous theorem gives (i). Any complete regular 4-set of regular matrices of order w^2 (and row sum $q_1.q_2. \ldots$) may be used as Williamson matrices giving (ii). Let the matrices be A, B, C, D then

$$H = \begin{bmatrix} A & B & C & -D \\ B & A & -D & C \\ C & -D & A & B \\ -D & C & B & A \end{bmatrix}$$

using the $A_iA_j = aJ,\ i \neq j$ property is a regular symmetric Hadamard matrix with constant diagonal of order $4w^2$ and row sum $2w$. Hence H has excess $8w^3$ which is maximal for the order giving (iii) and (iv). The higher dimensional Hadamard matrix is constructed as in the proof of Theorem 1. □

3 Decomposition into squares

Complete regular 4-sets of regular matrices of order w^2 give Williamson-type matrices of order $4w^2$ corresponding to the decomposition

$$4w^2 = w^2 + w^2 + w^2 + w^2.$$

So recalling Theorem 6 of [5] which uses T-sequences of length s^2 corresponding to the decomposition

$$s^2 = s^2 + 0^2 + 0^2 + 0^2$$

we have

Lemma 8 *Suppose there exist regular 4-sets of regular matrices of order* w^2*. Then there exists an Hadamard matrix with maximal excess of order* $4k^2 = 4w^2s^2$ *and an SBIBD*$(4k^2, 2k^2 \pm k, k^2 \pm k)$ *for* $s(\text{odd}) \in \{1, \ldots, 33, 37, \ldots, 41, 45, \ldots, 59\} \cup \{2g+1 :$ *g the length of a Golay sequence*$\}$.

Remark: Hadamard matrices of order $4k^2$ are quite large but the limited knowledge of the family SBIBD$(4k^2, 2k^2 \pm k, k^2 \pm k)$ makes these results worth pursuing.

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