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The diameter of total domination and independent domination vertex-critical graphs

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Abstract

We show that the diameter of a total domination vertex-critical graph is at most $5(\gamma_t - 1)/3$, and that the diameter of an independent domination vertex-critical graph is at most $2(i-1)$. For all values of $\gamma_t \equiv 2 \pmod 3$ there exists a total domination vertex-critical graph with the maximum possible diameter. For all values of $i \geq 2$ there exists an independent domination vertex-critical graph with the maximum possible diameter.

1 Introduction and Definitions

Domination critical graphs originated from Sumner and Blitch [10] in 1983 and have since received much attention ([9] and elsewhere). The original idea was to investigate graphs where the addition of any edge lowers the domination number. A popular variation is where the deletion of any vertex lowers the domination number [2]. A summary of early domination criticality results involving edge addition and vertex deletion has been compiled [11] and is now a standard reference. We study graphs where the deletion of any vertex reduces the total domination number. In particular we study the maximum diameter of such graphs in Section 2. In fact, for every $k \equiv 2$ (mod 3) there exists a graph of maximum diameter with total domination number equal to k such that the deletion of any vertex reduces the total domination number. In Section 3 we study graphs where the deletion of any vertex lowers the independent domination number. There we provide a bound on the maximum diameter of such graphs and show that for all $k \geq 2$ there is a graph of maximum diameter with independent domination number equal to k such that the deletion of any vertex reduces the independent domination number.

A set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G) - S$ is adjacent to a vertex in S. The set S is called a *total dominating set* if every vertex in $V(G)$ is

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adjacent to a vertex in S. An *independent dominating set* is a dominating set that is also an independent set. The minimum cardinality of a dominating set of G is called the *domination number of* G, and is denoted by $\gamma(G)$. If $S \subseteq V(G)$ is a dominating set with cardinality $\gamma(G)$, then S is called a $\gamma(G)$*-set*. In cases where the graph G is understood by the context, we write γ-set. Similar definitions follow for the total domination number, $\gamma_t(G)$, and the independent domination number, $i(G)$.

For a vertex $v \in V(G)$, the graph $G - v$ denotes the graph created from G by deleting v as well as all edges incident with v. A graph G is *domination vertex-critical*, or γ-vertex-critical, if $\gamma(G - v) < \gamma(G)$ for every vertex $v \in V(G)$. There are many other types of domination criticality, but since we only discuss vertex-criticality here we write γ-critical for brevity. If G is γ-critical and $\gamma(G) = k$, then we say that G is k-γ*-critical*. Notice that in general it is possible to have $\gamma(G - v)$ arbitrarily larger than $\gamma(G)$. Again, similar definitions follow for total domination vertex-criticality and independent domination vertex-criticality.

Vertex-criticality for different types of domination has been investigated; for example, γ_t-critical graphs were first studied in work by Goddard et al. [7]. Paired domination vertex-critical graphs (or γ_{pr}-vertex-critical graphs) first appeared in work by Edwards and Hou [3] and independent domination edge-criticality and independent domination vertex-criticality first appeared in the Master's thesis by Ao [1].

An active area in the study of domination criticality is with results involving diameter. Results on vertex-criticality and diameter first appeared in work by Brigham et al. [2], where the authors conjectured that if G is k-γ-vertex-critical, then $\text{diam}(G) \leq 2(k - 1)$. Fulman et al. [6] provided a proof for this conjecture and showed that the bound is sharp. Paired domination vertex-critical graphs were investigated by Henning and Mynhardt [8] who provided a construction which shows that for every even $k \geq 4$ there exists a k-γ_{pr}-vertex-critical graph with diameter equal to $3(k - 2)/2$. Edwards and Hou [5] used the method of Fulman et al. [6] to verify that if G is k-γ_{pr}-vertex-critical, then $\text{diam}(G) \leq 3(k - 2)/2$. We exploit this same method in Section 2 and Section 3.

In Section 2 we show that if G is k-γ_t-critical, then in fact $\text{diam}(G) \leq 5(k - 1)/3$. The best previous diameter bounds for γ_t-vertex-critical graphs are credited to Goddard et al. [7] who showed that if G is k-γ_t-vertex-critical, then $\text{diam}(G) \leq 2k - 3$. In Section 3 we show that if G is k-i-critical, then $\text{diam}(G) \leq 2(k - 1)$.

2 The Diameter of γ_t-critical Graphs

In a graph G, an *end-vertex* is a vertex of degree one. Notice that if G has an isolated vertex, $\gamma_t(G)$ is undefined. Goddard et al. [7] characterized the γ_t-critical graphs G with end-vertices. This characterization yields a result on diameter.

Proposition 1 [7] *If G is a connected k-γ_t-critical graph with at least one end-vertex, then* $\text{diam}(G) \leq k$ *if* $k \in \{3, 4\}$ *and* $\text{diam}(G) \leq k - 1$ *if* $k \geq 5$*, and these bounds are sharp.*

Thus in what follows, we consider γ_t-critical graphs without end-vertices. To create vertex-critical graphs from smaller vertex-critical graphs we make use of one particular construction technique here and in Section 3. This construction has appeared in many articles ([1], [3], [5], [6], [7], [8]).

Let G and H be disjoint graphs, and let $x \in V(G)$ and $y \in V(H)$. The *coalescence of G and H with respect to x and y* is the graph $G \cdot_{xy} H$ with vertex set $V(G \cdot_{xy} H) = (V(G) - \{x\}) \cup (V(H) - \{y\}) \cup \{v\}$ and edge set $E(G \cdot_{xy} H) = E(G - x) \cup E(H - y) \cup \{vw : xw \in E(G) \text{ or } yw \in E(H)\}$. We call v the *vertex of identification* of $G \cdot_{xy} H$. We consider $V(G)$ and $V(H)$ to be subsets of $V(G \cdot_{xy} H)$ and regard v as an element of both $V(G)$ and $V(H)$. Informally, $G \cdot_{xy} H$ is the graph obtained from $G \cup H$ by identifying x and y. If the context is clear, or if the vertices x and y are not important, we simply write $G \cdot H$. The graph $G_1 \cdot G_2 \cdot \cdots \cdot G_k$ is defined recursively by $G_1 \cdot G_2 \cdot \cdots \cdot G_k = (G_1 \cdot G_2 \cdot \cdots \cdot G_{k-1}) \cdot G_k$.

Observation 2 [7] *If G is a γ_t-critical graph without end-vertices, then $\gamma_t(G - v) = \gamma_t(G) - 1$ for every $v \in V(G)$. Furthermore, a γ_t-set of $G - v$ contains no neighbour of v.*

We now consider the maximum diameter of k-γ_t-critical graphs.

Proposition 3 [7] *The diameter of a k-γ_t-critical graph G is at most $2k - 3$.*

Theorem 4 [7] *For all $k \equiv 2 \pmod 3$, there exists a k-γ_t-critical graph of diameter $(5k - 7)/3$.*

The graphs from Theorem 4 are constructed as follows (see Figures 1 and 2 on page 259 of [7]): Let F be the graph obtained from $P_4 \cup \overline{P_4}$ by adding all edges between P_4 and $\overline{P_4}$ except for the perfect matching between corresponding vertices, and then adding a vertex x adjacent to every vertex of P_4 and a vertex y adjacent to every vertex of $\overline{P_4}$. Thus F is a 3-γ_t-critical graph with diameter 3. Let Q be the graph obtained from two copies of F, call them F_1 and F_2, by deleting y from F_1 and x from F_2 and adding all edges between the the four neighbours of y in F_1 and the four neighbours of x in F_2. Notice that $\gamma_t(Q) = 4$, $\mathrm{diam}(Q) = 5$, and that Q is not γ_t-critical. Let FQ^nF be the graph $F \cdot_{yx} Q \cdot_{yx} Q \cdot_{yx} \cdots \cdot_{yx} Q \cdot_{yx} F$ with n copies of Q. From Theorem 4, $\gamma_t(FQ^nF) = 3n + 5$, $\mathrm{diam}(FQ^nF) = 5n + 6$, and FQ^nF is γ_t-critical for every $n \geq 1$.

It is interesting to note that, unlike with γ-critical and i-critical graphs, not every block of a γ_t-critical graph G needs to be γ_t-critical. In addition, if G is a γ_t-critical graph, then $G \cdot G$ need not be γ_t-critical. Consider $C_6 \cdot C_6$. Notice that $\gamma_t(C_6) = 4$, $\gamma_t(C_6 \cdot C_6) = 6$, but $C_6 \cdot C_6$ is not γ_t-critical (the vertex of identification is not a critical vertex). The exact conditions under which $G_1 \cdot_{xy} G_2$ is γ_t-critical are known [4].

For the following result, we employ the technique used for γ-critical graphs [6] and γ_{pr}-critical graphs [5]. This technique proves useful in attaining a tight upper bound for the diameter of vertex-critical graphs. However, by the discussion in the previous paragraph, we use a slight modification of the technique in the proof of the following result.

Theorem 5 *The diameter of a connected γ_t-critical graph G without end-vertices satisfies* $\mathrm{diam}(G) \le 5(\gamma_t(G)-1)/3$.

Proof. Let G be a γ_t-critical graph with $\mathrm{diam}(G) = d$, and let x be a vertex of maximum eccentricity. We define the *level sets* $X_0, X_1, \ldots, X_d$ by $X_j = \{y \in V(G) : d(x,y) = j\}$, $0 \le j \le d$. For $0 \le j \le d$, the set U_j is defined by $U_j = X_0 \cup X_1 \cup \ldots \cup X_j$, and let $\langle U_j \rangle$ be the graph induced by U_j.

From Proposition 3, we know that if $\gamma_t(G) = 3$, then $d \le 3$ hence we assume that $\gamma_t(G) \ge 4$ in what follows. Let D by any γ_t-set of G. For $j \ge 1$, we say that $\langle U_j \rangle$ is *D-sufficient* if $j \le (5|D \cap U_j| - 8)/3$. If $G = \langle U_d \rangle$ is D-sufficient, then $d \le (5\gamma_t(G) - 8)/3 \le 5(\gamma_t(G) - 1)/3$.

We first show that there exists a γ_t-set D such that $\langle U_2 \rangle$ is D-sufficient. Let $y \in X_1$ and consider a γ_t-set S of $G - y$. By Observation 2, $x \notin S$. Since S totally dominates $G - y$, we have that $|S \cap U_2| \ge 2$. But $S \cup \{x\}$ is a total dominating set of G with cardinality $\gamma_t(G)$ and so let $D = S \cup \{x\}$. Therefore $|D \cap U_2| \ge 3$, and for any γ_t-critical graph G there exists a j and a γ_t-set D such that $\langle U_j \rangle$ is D-sufficient. Let m be the maximum value of j such that $\langle U_j \rangle$ is D-sufficient. If $m = d$, we are finished, so suppose that $m < d$. Then for all $j > m$, $\langle U_j \rangle$ is not D-sufficient. Notice that the value of m may differ for different choices of D. Consider a γ_t-set D that maximizes the value of m.

We first show a restriction on m, modulo 5. We have that $|D \cap U_m| \ge 3m/5 + 8/5$ while $|D \cap U_{m+1}| < 3m/5 + 11/5$. Suppose that $m = 5t + 1$ for some $t \in \mathbb{Z}$. Then $|D \cap U_m| \ge 3t + 3$ and $|D \cap U_{m+1}| < 3t + 3$, a contradiction. Suppose that $m = 5t + 3$ for some $t \in \mathbb{Z}$. Then $|D \cap U_m| \ge 3t + 4$ and $|D \cap U_{m+1}| < 3t + 4$, a contradiction. Therefore we have that $m = 5t$, $m = 5t + 2$, or $m = 5t + 4$ for some $t \in \mathbb{Z}$. Suppose that $m < d - 1$.

If $m = 5t$, then $|D \cap U_m| \ge 3t + 2$ and $|D \cap U_{m+1}| \le 3t + 2$, and $|D \cap U_{m+2}| \le 3t + 2$ which implies that $|D \cap U_m| = 3t + 2$, $D \cap X_{m+1} = \emptyset$, and $D \cap X_{m+2} = \emptyset$. In addition, $|D \cap U_{m+3}| \le 3t + 3$ which, implies that $|D \cap X_{m+3}| \le 1$. Since D is a total dominating set, we have that $|D \cap X_{m+3}| \ge 1$, and so let $D \cap X_{m+3} = \{w\}$. But then $|D \cap U_{m+4}| \le 3t + 3$ and so $D \cap X_{m+4} = \emptyset$, a contradiction to D being a total dominating set.

If $m = 5t + 2$, then $|D \cap U_m| \ge 3t + 3$, $|D \cap U_{m+1}| \le 3t + 3$, and $|D \cap U_{m+2}| \le 3t + 3$, which implies that $|D \cap U_m| = 3t + 3$, $D \cap X_{m+1} = \emptyset$, and $D \cap X_{m+2} = \emptyset$. In addition, $|D \cap U_{m+3}| \le 3t + 4$ which implies that $|D \cap X_{m+3}| \le 1$. Since D is a total dominating set, we have that $|D \cap X_{m+3}| \ge 1$, and so let $D \cap X_{m+3} = \{w\}$.

If $m = 5t + 4$, then $|D \cap U_m| \ge 3t + 4$ and $|D \cap U_{m+1}| \le 3t + 4$, which implies that $|D \cap U_m| = 3t + 4$ and $D \cap X_{m+1} = \emptyset$. In addition, $|D \cap U_{m+2}| \le 3t + 5$ and $|D \cap U_{m+3}| \le 3t + 5$, which implies that $|D \cap (X_{m+2} \cup X_{m+3})| \le 1$. Since D is a total dominating set we can conclude that $D \cap X_{m+2} = \emptyset$ and $D \cap X_{m+3} = \{w\}$.

In all cases, we have that $D \cap X_{m+1} = \emptyset$, $D \cap X_{m+2} = \emptyset$, $D \cap X_{m+3} = \{w\}$, and so w dominates all of X_{m+2}. Consider D_w, a γ_t-set of $G - w$. By Observation 2, $D_w \cap X_{m+2} = \emptyset$. Let $y \in X_{m+2}$. Then $D_w^y = D_w \cup \{y\}$ is a γ_t-set of G. In all cases, if $|D_w \cap U_{m+1}| > |D \cap U_{m+1}|$, then $\langle U_{m+1} \rangle$ is D_w^y-sufficient, a contradiction of the maximality of m. If $|D_w \cap U_{m+1}| < |D \cap U_{m+1}|$, then $|D_w - U_{m+1}| \geq |D - U_{m+1}|$ and so $(D_w \cap U_{m+1}) \cup (D - U_{m+1})$ is a total dominating set of G with smaller cardinality than D, a contradiction. Therefore suppose that $|D_w \cap U_{m+1}| = |D \cap U_{m+1}|$. If $m = 5t + 2$, then $|D_w^y \cap U_{m+2}| = 3t + 4$, a contradiction of the maximality of m. If $m = 5t + 4$, then $|D_w^y \cap U_{m+2}| = 3t + 5$. Recall that $D_w \cap X_{m+2} = \emptyset$ and that D_w dominates X_{m+3} in $G - w$. Therefore $|D_w \cap (X_{m+3} \cup X_{m+4} \cup X_{m+5})| \geq 2$, and so $|D_w^y \cap U_{m+5}| \geq 3t + 5 + 2 = 3t + 7$. But then $\langle U_{m+5} \rangle$ is D_w^y-sufficient, a contradiction of the maximality of m. We can thus conclude that $m \geq d - 1$.

We now have that either $d = m$ (and so $d \leq (5\gamma_t(G) - 8)/3$) or that $d = m + 1$ with $m = 5t$, $m = 5t + 2$, or $m = 5t + 4$ (so that $d = 5t + 1$, $d = 5t + 3$, or $d = 5t + 5$). Furthermore, if $d = m + 1$, $m = 5t$ or $m = 5t + 2$ or $m = 5t + 4$, and $\langle U_m \rangle$ is D-sufficient, then $D \cap X_{m+1} = \emptyset$ and so $|D \cap U_m| = \gamma_t(G)$. Hence if $d = m + 1$, the above argument gives $d = m + 1 \leq (5\gamma_t(G) - 8)/3 + 1 = 5(\gamma_t(G) - 1)/3$ as desired. ■

The following result is an immediate consequence of Proposition 1 and Theorem 5.

Corollary 6 *The diameter of a connected γ_t-critical graph G satisfies* $\operatorname{diam}(G) \leq 5(\gamma_t(G) - 1)/3$.

Notice that if $\gamma_t(G) \equiv 2 \pmod 3$, then $\lfloor 5(\gamma_t(G) - 1)/3 \rfloor = (5\gamma_t(G) - 7)/3$. Therefore the graph FQ^nF achieves equality in the bound from Theorem 5 for all $n \geq 1$. For $\gamma_t \equiv 1 \pmod 3$, it is straightforward to show that $G = F \cdot_{yx} FQ^nF$ is γ_t-critical with $\gamma_t(G) = 3(n+1) + 4$ and $\operatorname{diam}(G) = 5(n+1) + 4 = \lfloor 5(\gamma_t(G) - 1)/3 \rfloor - 1$. For $\gamma_t(G) \equiv 0 \pmod 3$, it is straightforward to show that $G = F \cdot_{yx} F \cdot_{yx} FQ^nF$ is γ_t-critical with $\gamma_t(G) = 3(n+2) + 3$ and $\operatorname{diam}(G) = 5(n+2) + 2 = \lfloor 5(\gamma_t(G) - 1)/3 \rfloor - 1$.

3 The Diameter of i-critical Graphs

We open with a series of results on i-critical graphs.

Observation 7 *If G is i-critical, then for every i-set S of $G - v$, $u \notin S$ for all $u \in V(G)$ with $uv \in E(G)$. Furthermore, $i(G - v) = i(G) - 1$ for every $v \in V(G)$ and for any vertex v, there exists an i-set S such that $v \in S$.*

It is known that the only 2-γ-critical graphs are K_{2n} less a perfect matching [2] and the following is an easy consequence of this result.

Observation 8 *The only 2-i-critical graphs are K_{2n} less a perfect matching.*

Methods to create i-critical graphs from smaller i-critical graphs have been investigated [1]. Again we make use of the coalescence here.

Theorem 9 [1] *The graph $G \cdot H$ is i-critical if and only if both G and H are i-critical. Furthermore, $i(G \cdot H) = i(G) + i(H) - 1$ if $G \cdot H$ is i-critical.*

Since the proof of the following theorem closely follows that of Theorem 5 and the analogous theorem for γ-critical graphs [6], we omit some calculations.

Theorem 10 *If G is i-critical, then* $\operatorname{diam}(G) \leq 2(i(G) - 1)$.

Proof. Let G be an i-critical graph with diameter d, and let x be a vertex of maximum eccentricity. We define the level sets $X_0, X_1, \ldots, X_d$ and the sets $U_0, U_1, \ldots, U_d$ as before.

From Observation 8, the only 2-i-critical graphs are K_{2n} less a perfect matching, thus we assume $i \geq 3$ for the remainder of the proof. Let D be any i-set of G. We say that $\langle U_j \rangle$ is *D-sufficient* if $j \leq 2(|D \cap U_j| - 1)$. If $G = \langle U_d \rangle$ is D-sufficient for some i-set D, then $d \leq 2(i(G) - 1)$. Let D_x be an i-set of $G - x$ and let $D_x^x = D_x \cup \{x\}$. Notice that $\langle U_2 \rangle$ is D_x^x-sufficient. Let m be the maximum value of j such that $\langle U_j \rangle$ is D-sufficient. If $m = d$, we are finished, so suppose $m < d$. Then for all $j > m$, $\langle U_j \rangle$ is not D-sufficient. Notice that the value of m may differ for different choices of D, consider an i-set D that maximizes the value of m.

Following straight from the definition for $\langle U_m \rangle$ to be D-sufficient it is easy to show that $m = 2t$ for some $t \in \mathbb{Z}$, $|D \cap U_m| = t + 1$, and $D \cap X_{m+1} = \emptyset$. Suppose that $d > m + 1$. Again it is easy to show that $D \cap X_{m+2} = \emptyset$, $D \cap X_{m+3} = \{w\}$, $D \cap X_{m+4} = \emptyset$ (if X_{m+4} exists), and so w dominates X_{m+2} and X_{m+3}.

Now consider D_w, an i-set of $G - w$ and let $D_w^w = D_w \cup \{w\}$. Notice that $D_w \cap (X_{m+2} \cup X_{m+3}) = \emptyset$. If $|D_w^w \cap U_{m+1}| > |D \cap U_{m+1}|$, then $\langle U_{m+1} \rangle$ is D_w^w-sufficient, a contradiction of the maximality of m. If $|D_w^w \cap U_{m+1}| < |D \cap U_{m+1}|$, then $(D_w^w \cap U_{m+1}) \cup (D - U_{m+1})$ is an independent dominating set of G with cardinality less than $i(G)$, a contradiction. Thus $|D_w^w \cap U_{m+1}| = |D \cap U_{m+1}| = t + 1$. But then $|D_w^w \cap U_{m+4}| \geq t + 3$ and so $\langle U_{m+4} \rangle$ is D_w^w-sufficient, a contradiction of the maximality of m. Thus it follows that either $d \leq 2(i(G) - 1)$ or $d = m + 1 = 2t + 1$. In particular, the theorem is true for all i-critical graphs of even diameter.

Now suppose $d = m + 1 = 2t + 1$. Then $G \cdot_{xx} G$ is i-critical with diameter equal to $2d$ and $i(G \cdot_{xx} G) = 2i(G) - 1$ and so $2d \leq 2(2i(G) - 2)$. Therefore $d \leq 2(i(G) - 1)$ as desired. ■

We close by showing that the bound in Theorem 10 is sharp. Notice that the cycle on four vertices, C_4, is 2-i-critical with diameter 2, and that C_4 is a graph which reaches equality for the bound in Theorem 10. Now $\operatorname{diam}(C_4 \cdot C_4) = 4$, $i(C_4 \cdot C_4) = 3$ by Theorem 9, and so $\operatorname{diam}(C_4 \cdot C_4)$ also reaches equality in our bound. This construction can be continued by identifying a vertex of maximum eccentricity in $C_4 \cdot C_4$ with any vertex in C_4. The resulting graph has diameter 6 and independent domination number 4, again achieving equality in the bound. Thus by creating a chain of 4-cycles where the identified vertices are independent, we have an infinite family of graphs that reach equality in Theorem 10. In fact, these graphs are all γ-critical and they reach equality in the diameter bound for γ-critical graphs [6].

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