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Weak king embeddings of oriented graphs

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Abstract

The concept of weak kings in oriented graphs, which extends the notion of kings in tournaments, was introduced by Pirzada and Shah in 2008. In this article we show that almost every oriented graph D is an induced subgraph of both an all-weak-kings oriented graph D' and an oriented graph having only the vertices of D as its weak kings. We also obtain lower bounds on the sizes of these supergraphs, where the one for all-weak-kings oriented graphs D' can be achieved for every D. These are analogues to the results proved by Reid (1982) for kings in tournaments.

1 Introduction

In 2008 Pirzada and Shah [7] introduced the concept of weak kings in oriented graphs, in some sense generalizing the notion of kings in tournaments [6]. They proved similar results in this setting as can be found in [2–6, 8, 9] for tournaments. In this article we generalize a result from [7] and investigate which (further) results from [9] hold in the context of weak kings.

An *oriented graph* $D = (V, A)$ is a digraph without loops and symmetric edges (we sometimes write $V(D) = V$); an n-oriented graph has n vertices. A vertex v in D is *dominated* by a vertex u if an arc directed from u to v exists, in that case u *dominates* v. We adopt the following notation from [7]. For any two vertices u and v in an oriented graph we write $u(1-0)v$ if u dominates v, $u(0-1)v$ if v dominates u, and $u(0-0)v$ if no arc exists between u and v; there are no other possibilities in an oriented graph. A *tournament* is an oriented graph that does not contain two vertices u, v with $u(0-0)v$.

A vertex v is *weakly reachable* (within two steps) from another vertex u if one of the following situations occurs for some vertex w: $u(1-0)v$, $u(0-0)v$, $u(1-0)w(1-0)v$, $u(1-0)w(0-0)v$, or $u(0-0)w(1-0)v$. If every vertex from $V \setminus \{u\}$ is weakly reachable from u, then u is called a *weak king* in D. If every vertex of D is a weak king, we call D an *all-weak-kings oriented graph.*

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For a vertex u the set $V \setminus \{u\}$ can be (naturally) partitioned into $O_D(u) = \{v \in V \setminus \{u\} : u(1-0)v\}, I_D(u) = \{v \in V \setminus \{u\} : u(0-1)v\}$, and $N_D(u) = \{v \in V \setminus \{u\} : u(0-0)v\}$; further we define $NO_D(u) = N_D(u) \cup O_D(u)$ and $NI_D(u) = N_D(u) \cup I_D(u)$. When the context is clear, we omit the index D. A vertex x with $I_D(x) = \emptyset$ is called a *transmitter* (note that unlike in the case of tournaments, an oriented graph can have more than one transmitter).

Two types of embeddings of an oriented graph D, i. e., finding an oriented graph D' with $D \subseteq D'$ and certain properties, will be considered here. The first one addresses the problem of finding a D' such that its weak kings are the vertices of D (a partial answer can be found in [7]). In the second case we want D' to be an all-weak-kings oriented graph. We characterize all oriented graphs for which the first problem is solvable (Section 2) and show that the second problem is only slightly different from the tournament case in [9] (Section 3).

2 Given graph induces set of weak kings

By adjusting the proof of the corresponding result (Lemma 3) in [6] we get:

Lemma 1. *For every vertex u in an oriented graph D that does not dominate every other vertex we find some $v \in NI(u)$ which is a weak king in D.*

We are now prepared to strengthen Theorem 2.8 from [7] using a similar idea (note that their example in Figure 4 is incorrect but can be fixed by inverting the edges (u_2, x) and (u_3, x)).

Theorem 1. *An oriented graph D is contained in an oriented graph whose weak kings are the vertices of D if and only if D does not contain a transmitter which dominates every other vertex of D.*

Proof. As in the proof of Theorem 4 from [9] the necessity follows from Lemma 1, since every oriented graph whose set of weak kings contains the vertices of D needs to have a vertex that is not dominated by the transmitter, therefore it has a weak king outside of D.

For the other direction first of all note that the case where D does not contain a transmitter is dealt with in Theorem 2.8 from [7] and that if D is an all-weak-kings oriented graphs there is nothing to show.

Let T be the set of transmitters of D. We distinguish two cases.

- If $|T| \geq 2$, say $T = \{x_1, x_2, \ldots, x_t\}$, let $V(D) \setminus T = \{u_1, u_2, \ldots, u_s\}$ and D_1 be an isomorphic copy of D with vertex set $\{x'_1, x'_2, \ldots, x'_t, u'_1, u'_2, \ldots, u'_t\}$ where x'_i (resp. u'_i) corresponds to x_i (resp. u_i). Note that $s \geq 1$ holds.

  Consider the graph $D_2 = D \cup D_1$ with (additionally) $x_i(1-0)x'_j$ for $i \neq j$, $x_i(1-0)u'_j$, $u_i(1-0)x'_j$, $u_i(1-0)u'_j$ for $i \neq j$, $u_i(0-1)u'_i$, and $x_i(0-0)x'_i$, for all (possible) $i, j \geq 1$.

Then every element of $V(D)$ is a weak king in D_2. Since no x_j is weakly reachable from x'_i for $i \neq j$ and no element from $I_D(u_i)$ is weakly reachable from u'_i, no element of $V(D_1)$ is a weak king in D_2.

- If $|T| = 1$, say $T = \{x\}$, we consider $O_D(x) = \{u_1, u_2, \ldots, u_r\}$ and $N_D(x) = \{v_1, v_2, \ldots, u_s\}$ (note that $r + s \geq 2$ and $s \geq 1$ hold). Further, let D_1 be an isomorphic copy of D with vertex set $\{x', u'_1, u'_2, \ldots, u'_r, v'_1, v'_2, \ldots, v'_s\}$ where x' (resp. u'_i, v'_i) corresponds to x (resp. u_i, v_i).

  Consider the oriented graph $D_2 = D \cup D_1$ with (additionally) $x(0-0)x'$, $x(1-0)u'_i$, $x(1-0)v'_i$, $u_i(1-0)x'$, $u_i(1-0)v'_j$, $u_i(1-0)u'_j$ for $i \neq j$, $u_i(0-1)u'_i$, $v_i(1-0)x'$, $v_i(1-0)u'_j$, $v_i(1-0)v'_j$ for $i \neq j$, and $v_i(0-1)v'_i$, for all (possible) $i, j \geq 1$, and further, we have (changing D_1) $v'_i(1-0)x'$ for $1 \leq i \leq s$.

  Then every element of $V(D)$ is a weak king in D_2. Further, no v_i is weakly reachable from x' (because of $v'_i(1-0)x'$) and x is not weakly reachable from u'_i. Finally, every element v' of $N_{D_1}(x')$ is dominated by some u'_i or v'_i, therefore u_i or v_i is not weakly reachable from v'.

In every case we have constructed an oriented graph with the desired property. □

Problem. Determine the smallest number $m(D)$ of vertices of an oriented graph D' which has the vertices of the n-oriented graph D (having no vertex dominating all others) as its weak kings. Apparently for all-weak-kings oriented graphs $m(D)$ is n and in general it is at most $2n$.

A lower bound for $m(D)$ (in terms of $l_2(D)$ which is defined next) if D is a tournament is given in [9]. We show that this bound does not apply to oriented graphs in general. Nevertheless, a slightly weaker lower bound can be proven for oriented graphs.

The set of vertices $V(D)$ of an oriented graph D can be partitioned (we call this the *weak king partition*) in the following way (analogous to [9]): Let V_1 denote the set of weak kings in D, and, inductively, let $V_i = \{u \in V \setminus (V_1 \cup \cdots \cup V_{i-1}) :$ every vertex $v \in V \setminus (V_1 \cup \cdots \cup V_{i-1}), v \neq u,$ is weakly reachable from u in $D\}$, $2 \leq i \leq n$. Let $p = p(D)$ denote the largest index such that V_p is non-empty; further define $l_i(D) = \lceil \log_i p(D) \rceil$ for $i \in \{2, 3\}$, where $\lceil x \rceil = \min\{a \in \mathbb{Z} : a \geq x\}$. Note that $p(D) = 1$ holds if and only if D is an all-weak-kings oriented graph.

The following lemma can be proved similarly to the corresponding result in [9] (Lemma 5). Since the proof is not given explicitly, parts of it will be included here.

Lemma 2. *Let D be an n-oriented graph, and let V_i, $1 \leq i \leq p = p(D)$, be as above. Then for each i, $2 \leq i \leq p$, and for each v_i in V_i there exist vertices v_j in V_j, $1 \leq j \leq i-1$, such that*

- $NO(v_i) \subseteq NO(v_{i-1}) \subseteq \cdots \subseteq NO(v_1)$,
- $O(v_i) \subseteq O(v_{i-1}) \subseteq \cdots \subseteq O(v_1)$.

Proof. Let $v_i \in V_i$ for i with $2 \leq i \leq p$. Then there exists some $v_{i-1} \in I(v_i) \cap V_{i-1}$ which is not weakly reachable from v_i, otherwise v_i would lie in V_{i-1}. It follows $v_{i-1}(1-0)x$ for every $x \in O(v_i)$ which shows $O(v_i) \subseteq O(v_{i-1})$, and, further, $v_{i-1}(1-0)x$ or $v_{i-1}(0-0)x$ for every $x \in N(v_i)$ which shows $NO(v_i) \subseteq NO(v_{i-1})$. The lemma can now easily be proved by induction. □

In [9] the corresponding result (Lemma 5) is used to prove the lower bound $m(D) \geq n + l_2(D)$ for a tournament D. Unfortunately, this bound is not true for oriented graphs in general, as the following example (which has been constructed by Laura Gerken [1]) shows (the main problem being that a non existing arc from u to v does not imply an arc from v to u in oriented graphs).

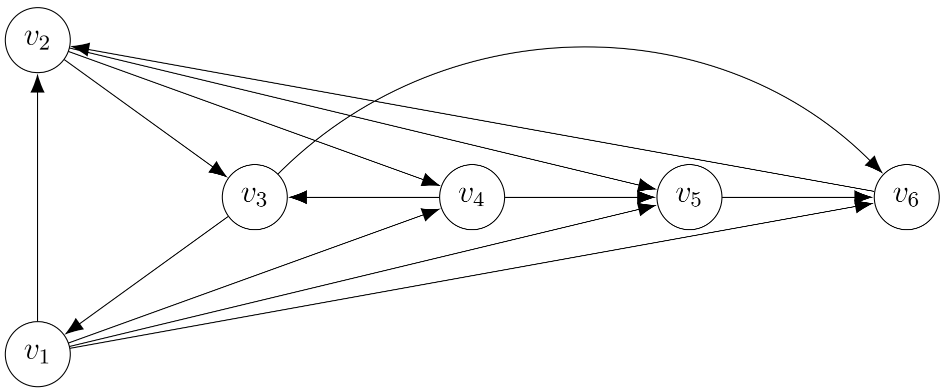

Figure 1: Graph D' with $p(D' \setminus \{v_6\}) = 3$ and $l_2(D' \setminus \{v_6\}) = 2$, but $m(D' \setminus \{v_6\}) = 1$.

The graph $D = D' \setminus \{v_6\}$ in Figure 2 can be decomposed into $V_1 = \{v_1, v_2, v_3\}$, $V_2 = \{v_4\}$ and $V_3 = \{v_5\}$. Adding v_6 and the displayed edges yields a graph with $\{v_1, v_2, \ldots, v_5\}$ as its set of weak kings.

But by changing the proof of Theorem 6 from [9] appropriately, it is possible to show the following.

Theorem 2. *For every n-oriented graph D, we have* $m(D) \geq n + l_3(D)$.

Proof. The result apparently holds if $p(D) = 1$. So, suppose we have $D = (V, A)$ with $p = p(D) > 1$ and an oriented graph $D' = (V', A')$ with set of weak kings equal to $V(D)$. Let v_p be a vertex in V_p and let $v_1, v_2, \ldots, v_{p-1}$ be as in Lemma 2. Note that $v_i(1-0)v_j$ holds for $1 \leq i < j \leq p-1$.

We show that we cannot have $N_{D'}(v_i) \cap (V' \setminus V) = N_{D'}(v_j) \cap (V' \setminus V)$ and $O_{D'}(v_i) \cap (V' \setminus V) = O_{D'}(v_j) \cap (V' \setminus V)$ (at the same time) for $i \neq j$.

Assume those two equalities would hold for some i and j with $i < j$. Since v_j is a weak king in D' and $v_i(1-0)v_j$ holds, we find some $z \in V'$ with $v_j(1-0)z(1-0)v_i$ or $v_j(1-0)z(0-0)v_i$ or $v_j(0-0)z(1-0)v_i$. Because of our assumption z cannot lie in $V' \setminus V$. The first two constellations are impossible because of $O_D(v_j) \subseteq O_D(v_i)$ and, since $NO_D(v_j) \subseteq NO_D(v_i)$, so is the last one.

Therefore, every ordered partition of $V' \setminus V$ into three subsets corresponds to at most one v_i. Thus, we have $p \leq 3^{m-n}$ which implies $m \geq n + l_3(D)$. □

3 All-weak-kings oriented graphs

For the second embedding problem the idea from [9] can be directly applied to get an all-weak-kings oriented graph D' on $n+p(D)-1$ vertices that contains the n-oriented graph D as a (induced) subgraph. Since in the proof of Theorem 2 we do not make any assumption on the vertices in D' which do not lie in D, the lower bound $n+l_3(D)$ on $|V(D')|$ does also apply in this problem.

Theorem 8 from [9] states that for any tournament D some all-(weak)-kings tournament D' on $n + l_2(D)$ vertices with $D \subseteq D'$ can be found (and this is best possible). By a similar construction we can show an analogous result:

Theorem 3. *Every n-oriented graph D is a (induced) subgraph of an all-weak-kings oriented graph with $n + l_3(D)$ vertices.*

Proof. We prove the theorem by induction on $l_3(D)$. If $l_3(D) = 0$, we have $p(D) = 1$ implying that D is an all-weak-kings oriented graph. Assume that the statement is true for all oriented graphs Z with $l_3(Z) < k$, where $k \geq 1$, and consider an n-oriented graph D with $l_3(D) = k$.

Let $U_i = \{x \in V_i : x \text{ does not dominate any vertex in } V_{i+1}\}$ for $1 \leq i \leq p-1$, where $p = p(D)$ and V_i are defined as above. Since no vertex from V_{i+1} can weakly reach every vertex from V_i, no vertex from U_i can dominate every vertex in $V_i \setminus U_i$.

We construct an $(n+1)$-oriented graph D_1 by adjoining to T a new vertex z such that Z dominates exactly the vertices in

$$\bigcup\{V_i \setminus U_i : 1 \leq i \leq p-1, i \equiv 1 \bmod 3\} \cup W,$$

where

$$W = \begin{cases} V_p, & \text{if } p \equiv 1 \bmod 3 \\ \emptyset, & \text{else} \end{cases}$$

and is dominated by exactly the vertices in

$$\bigcup\{V_i : 1 \leq i \leq p-1, 3 \mid i\} \cup W,$$

where

$$W = \begin{cases} V_p, & \text{if } 3 \mid p \\ \emptyset, & \text{else.} \end{cases}$$

We have $z(1-0)v$ or $z(0-0)v$ for every $v \in V_i$ with $3 \nmid i$. Further, for every $v \in V_i$ with $3 \mid i$ we find some $w \in V_{i-1}$ such that $w(1-0)v$. Since $z(0-0)w(1-0)v$, the vertex z is a weak king in D_1.

Further, z is weakly reachable (in D_1) from every element of U_i and V_{i+1} directly and from every element of $V_i \backslash U_i$ via some vertex in V_{i+1} for $i \equiv 1 \bmod 3$, $1 \leq i \leq p-1$. Every element from V_i is weakly reachable (in D_1) from every element of V_{i+1} directly or via z for $i \equiv 1 \bmod 3$, $1 \leq i \leq p-1$. Finally, every element in V_i weakly reaches every element in V_{i-2} and V_{i-1} via z for $3 \mid i$, $1 \leq i \leq p$.

This (and previously mentioned properties of the sets V_i) shows that $V(D_1)$ has the weak king partition (using $V_i(D_1)$ to distinguish the different oriented graphs)

$$\underbrace{(\{z\} \cup V_1 \cup V_2 \cup V_3)}_{V_1(D_1)}, \underbrace{(V_4 \cup V_5 \cup V_6)}_{V_2(D_1)}, \underbrace{(V_7 \cup V_8 \cup V_9)}_{V_3(D_1)}, \ldots,$$

and, therefore, fulfils $p(D_1) = \lceil p(D)/3 \rceil$ and $l_3(D_1) = k-1$. By induction the statement of the theorem follows. □

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