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Magic rectangles, signed magic arrays and integer λ-fold relative Heffter arrays

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Abstract

Let m, n, s, k be integers such that $4 \leq s \leq n$, $4 \leq k \leq m$ and $ms = nk$. Let λ be a divisor of $2ms$ and let t be a divisor of $\frac{2ms}{\lambda}$. In this paper we construct magic rectangles $\mathrm{MR}(m, n; s, k)$, signed magic arrays $\mathrm{SMA}(m, n; s, k)$ and integer λ-fold relative Heffter arrays ${}^\lambda\mathrm{H}_t(m, n; s, k)$ where s, k are even integers. In particular, we prove that there exists an $\mathrm{SMA}(m, n; s, k)$ for all m, n, s, k satisfying the previous hypotheses. Furthermore, we prove that there exist an $\mathrm{MR}(m, n; s, k)$ and an integer ${}^\lambda\mathrm{H}_t(m, n; s, k)$ in each of the following cases: (i) $s, k \equiv 0 \pmod 4$; (ii) $s \equiv 2 \pmod 4$ and $k \equiv 0 \pmod 4$; (iii) $s \equiv 0 \pmod 4$ and $k \equiv 2 \pmod 4$; (iv) $s, k \equiv 2 \pmod 4$ and m, n both even.

1 Introduction

In this paper we study partially filled (pf, for short) arrays, with entries in $\mathbb{Z}$ and whose rows and columns have prescribed sums. In particular, we construct *magic rectangles*, *signed magic arrays* and *integer λ-fold relative Heffter arrays*.

Definition 1.1 A *signed magic array* $\mathrm{SMA}(m, n; s, k)$ is an $m \times n$ pf array with elements in $\Omega \subset \mathbb{Z}$, where $\Omega = \{0, \pm 1, \pm 2, \ldots, \pm(ms-1)/2\}$ if ms is odd and $\Omega = \{\pm 1, \pm 2, \ldots, \pm ms/2\}$ if ms is even, such that

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(a) each row contains s filled cells and each column contains k filled cells;
(b) every $x \in \Omega$ appears exactly once in the array;
(c) the elements in every row and column sum to 0.

The existence of an $\mathrm{SMA}(m, n; s, k)$ has been settled in the square case (i.e., when $m = n$ and so $s = k$) and in the tight case (i.e., when $k = m$ and $s = n$), by Khodkar, Schulz and Wagner [17].

Theorem 1.2 *[17] There exists an* $\mathrm{SMA}(n, n; k, k)$ *if and only if either* $n = k = 1$ *or* $3 \le k \le n$.

Theorem 1.3 *[17] There exists an* $\mathrm{SMA}(m, n; n, m)$ *if and only if one of the following cases occurs:*

(1) $m = n = 1$;
(2) $m = 2$ *and* $n \equiv 0, 3 \pmod 4$;
(3) $n = 2$ *and* $m \equiv 0, 3 \pmod 4$;
(4) $m, n > 2$.

Also the cases when each column contains two or three filled cells have been solved.

Theorem 1.4 *[13] There exists an* $\mathrm{SMA}(m, n; s, 2)$ *if and only if one of the following cases occurs:*

(1) $m = 2$ *and* $n = s \equiv 0, 3 \pmod 4$;
(2) $m, s > 2$ *and* $ms = 2n$.

Theorem 1.5 *[16] There exists an* $\mathrm{SMA}(m, n; s, 3)$ *if and only if* $3 \le m, s \le n$ *and* $ms = 3n$.

In this paper we settle the existence problem of an $\mathrm{SMA}(m, n; s, k)$ when s and k are both even, proving constructively the following.

Theorem 1.6 *Let* s, k *be two even integers with* $s, k \ge 4$. *Then there exists an* $\mathrm{SMA}(m, n; s, k)$ *if and only if* $4 \le s \le n$, $4 \le k \le m$ *and* $ms = nk$.

This result will be obtained by working in the more general context of the integer λ-fold relative Heffter arrays. In Figure 1 we give an $\mathrm{SMA}(5, 10; 8, 4)$ obtained thanks to our constructions.

In [1] Archdeacon introduced an important class of pf arrays, called *Heffter arrays*. One of the applications of these objects is that they allow, under suitable conditions, the construction of pairs of cyclic cycle decompositions of the complete graph K_v on v vertices. With the aim of extending this application to complete multipartite

1	−2		−7	8	11	−12		−17	18
20	3	−4		−9	10	13	−14		−19
−1	2	5	−6		−11	12	15	−16	
	−3	4	7	−8		−13	14	17	−18
−20		−5	6	9	−10		−15	16	19

Figure 1: An SMA$(5, 10; 8, 4)$.

graphs, in [8] the authors of the present paper, in collaboration with Costa and Pasotti, proposed a first generalization of Archdeacon's idea introducing pf arrays called *relative Heffter arrays*. A further generalization, that allows one to work with complete multipartite multigraphs, was introduced in [9] by Costa and Pasotti. These new objects are called *λ-fold relative Heffter arrays*. We recall here their definition, where we denote by $\mathcal{E}(A)$ the *list* of the entries of the filled cells of a pf array A.

Definition 1.7 Let m, n, s, k, t, λ be positive integers such that λ divides $2ms$ and t divides $\frac{2ms}{\lambda}$. Let J be the subgroup of order t of $\mathbb{Z}_v$, where $v = \frac{2ms}{\lambda} + t$. A *$\lambda$-fold Heffter array* over $\mathbb{Z}_v$ relative to J, denoted by ${}^\lambda\mathrm{H}_t(m, n; s, k)$, is an $m \times n$ pf array A with elements in $\Omega = \mathbb{Z}_v \setminus J$ such that:

(a) each row contains s filled cells and each column contains k filled cells;
(b) every element of Ω appears exactly λ times in the list $\mathcal{E}(A) \cup -\mathcal{E}(A)$;
(c) the elements in every row and column sum to 0.

Item (b) of the previous definition requires some explanation. The additive group $\mathbb{Z}_v$ contains an involution if and only if v is even; in this case, the unique involution $\iota \in \mathbb{Z}_v$ belongs to Ω if and only if t is odd. We observe that the assumption v even and t odd implies that λ is even and does not divide ms. So we can write (b) as follows: if Ω does not contain involutions, every $x \in \Omega$ appears in A, up to sign, exactly λ times; if Ω contains the involution ι, then every $x \in \Omega \setminus \{\iota\}$ appears, up to sign, exactly λ times, while ι appears exactly $\lambda/2$ times.

Some results on the existence of these objects are given in [9], mostly for the square case or for particular values of λ and/or t. Instead of working in a finite cyclic group, one can construct λ-fold relative Heffter arrays whose entries are integers. In this case, the previous definition becomes as follows.

Definition 1.8 Let m, n, s, k, t, λ be positive integers such that λ divides $2ms$ and t divides $\frac{2ms}{\lambda}$. Let

$$\Phi = \left\{1, 2, \ldots, \left\lfloor \frac{v}{2} \right\rfloor\right\} \setminus \left\{\ell, 2\ell, \ldots, \left\lfloor \frac{t}{2} \right\rfloor \ell\right\} \subset \mathbb{Z}, \quad \text{where } v = \frac{2ms}{\lambda} + t \text{ and } \ell = \frac{v}{t}.$$

An *integer* ${}^\lambda\mathrm{H}_t(m, n; s, k)$ is an $m \times n$ pf array with elements in Φ such that:

(a) each row contains s filled cells and each column contains k filled cells;

(b) if v is odd or if t is even, every element of Φ appears, up to sign, exactly λ times in the array; if v is even and t is odd, every element of $\Phi \setminus \{\frac{v}{2}\}$ appears, up to sign, exactly λ times while $\frac{v}{2}$ appears, up to sign, exactly $\frac{\lambda}{2}$ times;

(c) the elements in every row and column sum to 0.

Example 1.9 Consider the following arrays:

$A =$

1	−1		−5	5	1	−1		−5	5
7	2	−2		−7	7	2	−2		−7
−1	1	4	−4		−1	1	4	−4	
	−2	2	5	−5		−2	2	5	−5
−7		−4	4	7	−7		−4	4	7

,

$B =$

1	−1							−5	5
5	3	−3							−5
−1	1	1	−1						
	−3	3	3	−3					
		−1	1	1	−1				
			−3	3	3	−3			
				−1	1	1	−1		
					−3	3	3	−3	
						−1	1	5	−5
−5							−3	3	5

.

It is easy to see that A is an integer ${}^{8}\mathrm{H}_5(5, 10; 8, 4)$, where each entry $1, 2, 4, 5, 7$ appears, up to sign, exactly eight times. The array B is an integer ${}^{16}\mathrm{H}_5(10, 10; 4, 4)$, where each of the entries 1 and 3 appears, up to sign, exactly sixteen times, whereas the entry 5 appears, up to sign, exactly eight times.

Observe that when $\lambda = 1$ one retrieves the concept of an (integer) relative Heffter array. In particular, an (integer) ${}^{1}\mathrm{H}_1(m, n; s, k)$ is exactly a classical (integer) Heffter array, as defined by Archdeacon. The problem of the existence of *square* classical Heffter arrays has been completely solved in [3, 12] for the integer case, and in [5] for the general case. For the other cases (non-square or relative), partial results have been obtained in [2, 10, 18]. Applications of (relative) Heffter arrays to graph decompositions and biembeddings are described, for instance, in [4, 6, 7, 11].

Here, we prove the following result, where any admissible value of λ and t is considered.

Theorem 1.10 *Let m, n, s, k be integers such that $4 \leq s \leq n$, $4 \leq k \leq m$ and $ms = nk$. Let λ be a divisor of $2ms$ and let t be a divisor of $\frac{2ms}{\lambda}$. There exists an integer ${}^{\lambda}\mathrm{H}_t(m, n; s, k)$ in each of the following cases:*

(1) $s, k \equiv 0 \pmod 4$;

(2) $s \equiv 2 \pmod 4$ *and* $k \equiv 0 \pmod 4$;

(3) $s \equiv 0 \pmod 4$ *and* $k \equiv 2 \pmod 4$*;*
(4) $s, k \equiv 2 \pmod 4$ *and* m, n *both even.*

Looking at Definitions 1.1 and 1.8 the reader can easily see that, when ms is even, a signed magic array is a particular integer 2-fold relative Heffter array. In fact, the integer ${}^2\mathrm{H}_1(m, n; s, k)$ we construct in the following sections is actually a signed magic array $\mathrm{SMA}(m, n; s, k)$. So, Theorem 1.6 will follow from Theorem 1.10, except when $s, k \equiv 2 \pmod 4$ and m, n are odd. Nevertheless, for these exceptional values, we will construct an $\mathrm{SMA}(m, n; s, k)$ starting from *square* signed magic arrays, whose existence is assured by Theorem 1.2, and exploiting the flexibility of our constructions. Note that [9, Theorem 4.9], where the authors considered the particular case ${}^2\mathrm{H}_1(m, n; s, k)$ with s, k even, was actually proved using the previous Theorem 1.6.

Our results on signed magic arrays allow us also to build magic rectangles.

Definition 1.11 A *magic rectangle* $\mathrm{MR}(m, n; s, k)$ is an $m \times n$ pf array with elements in $\Omega = \{0, 1, \ldots, ms - 1\} \subset \mathbb{Z}$ such that

(a) each row contains s filled cells and each column contains k filled cells;
(b) every $x \in \Omega$ appears exactly once in the array;
(c) the sum of the elements in each row is a constant value c_1 and the sum of the elements in each column is a constant value c_2.

Clearly, in the previous definition we must have $c_1 = \frac{s(ms-1)}{2}$ and $c_2 = \frac{k(ms-1)}{2}$. The reader can find results on the existence of these objects in [14, 15] and in the references within. Here, we prove the following.

Theorem 1.12 *Let* m, n, s, k *be integers such that* $4 \leq s \leq n$*,* $4 \leq k \leq m$ *and* $ms = nk$*. There exists an* $\mathrm{MR}(m, n; s, k)$ *in each of the following cases:*

(1) $s, k \equiv 0 \pmod 4$*;*
(2) $s \equiv 2 \pmod 4$ *and* $k \equiv 0 \pmod 4$*;*
(3) $s \equiv 0 \pmod 4$ *and* $k \equiv 2 \pmod 4$*;*
(4) $s, k \equiv 2 \pmod 4$ *and* m, n *both even.*

2 Notation

In this paper, the arithmetic on the row (respectively, on the column) indices is performed modulo m (respectively, modulo n), where the set of reduced residues is $\{1, 2, \ldots, m\}$ (respectively, $\{1, 2, \ldots, n\}$), while the entries of the arrays are taken in $\mathbb{Z}$. Given two integers $a \leq b$, we denote by $[a, b]$ the interval consisting of the integers $a, a+1, \ldots, b$. If $a > b$, then $[a, b]$ is empty. We denote by (i, j) the cell in the i-th row and j-th column of an array A. The *support* of A, denoted by $\mathsf{supp}(A)$, is defined to be the set of the absolute values of the elements contained in A.

If A is an $m \times n$ pf array, for $i \in [1, n]$ we define the i-th diagonal as

$$D_i = \{(1, i), (2, i + 1), \ldots, (m, i + m - 1)\}.$$

Definition 2.1 A pf array with entries in $\mathbb{Z}$ is said to be *shiftable* if every row and every column contains an equal number of positive and negative entries.

Let A be a shiftable pf array and x be a nonnegative integer. Let $A \pm x$ be the (shiftable) pf array obtained by adding x to each positive entry of A and $-x$ to each negative entry of A. Observe that, since A is shiftable, the row and column sums of $A \pm x$ are exactly the row and column sums of A.

We denote by $\tau_i(A)$ and $\gamma_j(A)$ the sum of the elements of the i-th row and the sum of the elements of the j-th column, respectively, of a pf array A.

For a block B, we write $\mu(B) = \mu$ if every element of $\mathsf{supp}(B)$ appears exactly μ times in $\mathcal{E}(B) \cup -\mathcal{E}(B)$.

Given a sequence $S = (B_1, B_2, \ldots, B_r)$ of shiftable pf arrays and a nonnegative integer x, we write $S \pm x$ for the sequence $(B_1 \pm x, B_2 \pm x, \ldots, B_r \pm x)$. We set $\mathcal{E}(S) = \cup_i \mathcal{E}(B_i)$ and $\mathsf{supp}(S) = \cup_i \mathsf{supp}(B_i)$. We also write $\mu(S) = \mu$ if $\mu(B_i) = \mu$ for all i.

If $S_1 = (a_1, a_2, \ldots, a_r)$ and $S_2 = (b_1, b_2, \ldots, b_u)$ are two sequences, by $S_1 +\!\!+ S_2$ we mean the sequence $(a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_u)$ obtained by concatenation of S_1 and S_2. In particular, if S_1 is the empty sequence then $S_1 +\!\!+ S_2 = S_2$. Furthermore, given the sequences $S_1, \ldots, S_c$, we write $\mathop{+\!\!+}\limits_{i=1}^{c} S_i$ for $(\cdots((S_1 +\!\!+ S_2) +\!\!+ S_3) +\!\!+ \cdots) +\!\!+ S_c$.

Given a positive integer n and a sequence $S = (a_1, a_2, \ldots, a_r)$, we denote by $n * S$ the sequence obtained by concatenating n copies of S.

Finally, we recall that the support of an integer ${}^\lambda\mathrm{H}_t(m, n; s, k)$ is the set

$$\Phi = \left[1, \left\lfloor \frac{t\ell}{2} \right\rfloor\right] \setminus \left\{\ell, 2\ell, \ldots, \left\lfloor \frac{t}{2} \right\rfloor \ell\right\}, \quad \text{where } \ell = \frac{2ms}{\lambda t} + 1 = \frac{v}{t}.$$

Note that, if λ divides ms, then

$$\Phi = \left[1, \frac{ms}{\lambda} + \left\lfloor \frac{t}{2} \right\rfloor\right] \setminus \left\{\ell, 2\ell, \ldots, \left\lfloor \frac{t}{2} \right\rfloor \ell\right\}.$$

Also, every element of Φ appears in ${}^\lambda\mathrm{H}_t(m, n; s, k)$, up to sign, exactly λ times. If λ does not divide ms, in order to obtain an integer ${}^\lambda\mathrm{H}_t(m, n; s, k)$, we have to construct a pf array A such that

if ℓ is odd or if t is even, every element of Φ appears in A, up to sign, exactly λ times; otherwise, i.e, if ℓ is even and t is odd, every element of $\Phi \setminus \{\frac{t\ell}{2}\}$ appears in A, up to sign, exactly λ times, while the integer $\frac{t\ell}{2}$ appears, up to sign, $\frac{\lambda}{2}$ times. (2.1)

3 The case $s, k \equiv 0 \pmod 4$

In this section we prove the existence of an integer ${}^\lambda\mathrm{H}_t(m,n;s,k)$ when both s and k are divisible by 4. First of all, we set

$$d = \gcd(m,n), \quad m = d\bar{m}, \quad n = d\bar{n}, \quad s = 4\bar{s} \quad \text{and} \quad k = 4\bar{k}.$$

From $ms = nk$ we see that $\bar{n}$ divides $\bar{s}$ and $\bar{m}$ divides $\bar{k}$. Hence, we can write $\bar{s} = c\bar{n}$ and $\bar{k} = c\bar{m}$. Observe that $n = d\bar{n} \geq s = 4c\bar{n}$ implies $d \geq 4$.

Fix two integers $a, b \geq 0$ and consider the following shiftable pf array:

$$B = B_{a,b} = \begin{array}{|c|c|} \hline 1 & -(a+1) \\ \hline & \\ \hline -(b+1) & a+b+1 \\ \hline \end{array}\,.$$

Note that the sequences of the row/column sums are $(-a, a)$ and $(-b, b)$, respectively. We use this 3×2 block for constructing pf arrays whose rows and columns sum to zero. Start taking an empty $m \times n$ array A, fix $m\bar{n}$ nonnegative integers $y_0, y_1, \ldots, y_{m\bar{n}-1}$, and arrange the blocks $B \pm y_j$ in such a way that the element $1 + y_j$ fills the cell $(j+1, j+1)$ of A (recall that we work modulo m on row indices and modulo n on column indices). In this way, we fill the diagonals $D_{im-1}, D_{im}, D_{im+1}, D_{im+2}$ with $i \in [1, \bar{n}]$. In particular, every row has $4\bar{n}$ filled cells and every column has $4\bar{m}$ filled cells.

Looking at the rows, the elements belonging to the diagonals D_{im+1}, D_{im+2} sum to $-a$, while the elements belonging to the diagonals D_{im-1}, D_{im} sum to a. Looking at the columns, the elements belonging to the diagonals D_{im+1}, D_{im-1} sum to $-b$, while the elements belonging to the diagonals D_{im+2}, D_{im} sum to b. Then A has row/column sums equal to zero.

Applying this process c times (working with the diagonals $D_{im+3}, D_{im+4}, D_{im+5}, D_{im+6}$, and so on), we obtain a pf array A, whose rows have exactly $4\bar{n} \cdot c = s$ filled cells and whose columns have exactly $4\bar{m} \cdot c = k$ filled cells.

Example 3.1 For $a = 2$ and $b = 5$, fixing the integers $0, 1, 10, 11, 20, 21, 30, 31, 40, 41, 50, 51$, we can fill the diagonals $D_1, D_2, D_5, D_6, D_7, D_8, D_{11}, D_{12}$ of the following 6×12 pf array, where we highlighted the block $B_{2,5}$:

$A =$

1	−3			−26	28	31	−33			−56	58
59	2	−4			−27	29	32	−34			−57
−6	8	11	−13			−36	38	41	−43		
	−7	9	12	−14			−37	39	42	−44	
		−16	18	21	−23			−46	48	51	−53
−54			−17	19	22	−24			−47	49	52

Note that $\mathsf{supp}(A) = [1, 60] \setminus \{5j : j \in [1, 12]\}$. As the reader can verify, A is an integer ${}^1\mathrm{H}_{24}(6, 12; 8, 4)$: in this case $\ell = \frac{2\cdot 6\cdot 8}{24} + 1 = 5$.

The constructions we present in this section are obtained by following this procedure, so they all produce shiftable pf arrays of size $m \times n$ whose rows and columns sum to zero.

Here we always assume that $4 \leq s \leq n$, $4 \leq k \leq m$, $ms = nk$ and $s, k \equiv 0 \pmod 4$. Let λ be a divisor of $2ms$ and t be a divisor of $\frac{2ms}{\lambda}$; set

$$\ell = \frac{2ms}{\lambda t} + 1.$$

We first consider the case when λ divides ms. To obtain an integer ${}^\lambda\mathrm{H}_t(m,n;s,k)$ with $s,k \equiv 0 \pmod 4$, we only have to determine two integers $a, b \geq 0$ and a set $X = \{x_0, x_1, \ldots, x_{f-1}\} \subset \mathbb{N}$ such that $\mu(B_{a,b}) = \mu$ divides λ and $\bigcup\limits_{x \in X} \mathsf{supp}(B_{a,b} \pm x) = \Phi$, where $f = \frac{ms}{4}\frac{\mu}{\lambda}$. So we can take the sequence $Y = \frac{\lambda}{\mu} * (x_0, x_1, \ldots, x_{f-1})$. Writing $Y = (y_0, y_1, \ldots, y_{\frac{ms}{4}-1})$ we construct A using the blocks $B_{a,b} \pm y_j$. In this way, every element of $\mathsf{supp}(A)$ occurs, up the sign, λ times in A. For instance, we can arrange the blocks in such a way that the element $1 + y_j$ fills the cell $(j+1, 4q_j + j + 1)$, where q_j is the quotient of the division of j by $\mathrm{lcm}(m,n)$.

Lemma 3.2 *Let λ be a divisor of ms such that $\lambda \equiv 0 \pmod 4$. There exists an integer ${}^\lambda\mathrm{H}_t(m,n;s,k)$ for any divisor t of $\frac{2ms}{\lambda}$.*

Proof: Let $B = B_{0,0} =$

1	-1
-1	1

. Note that $\mu(B) = 4$. An integer ${}^\lambda\mathrm{H}_t(m,n;s,k)$, say A, can be obtained by following the construction described before, once we exhibit a suitable set X of size $\frac{ms}{\lambda}$, in such a way that $\mathsf{supp}(A) = \Phi$. Consider the set $X = \{i - 1 \mid i \in \Phi\}$ of size $\frac{ms}{\lambda}$: clearly, $\bigcup\limits_{x \in X} \mathsf{supp}(B \pm x) = \Phi$. Now we take $\frac{\lambda}{4}$ copies of every block $B \pm x$: the pf array A obtained by following our procedure is an integer ${}^\lambda\mathrm{H}_t(m,n;s,k)$. □

For instance, the integer ${}^8\mathrm{H}_5(5,10;8,4)$ given in Example 1.9 was obtained by following the proof of the previous lemma. In fact, $\lambda = 8$ and $t = 5$ divides $\frac{2 \cdot 5 \cdot 8}{8}$; note that $\ell = 3$ and $Y = 2 * (0,1,3,4,6)$.

Lemma 3.3 *Let λ be a divisor of ms such that $\lambda \equiv 2 \pmod 4$. There exists an integer ${}^\lambda\mathrm{H}_t(m,n;s,k)$ for any divisor t of $\frac{2ms}{\lambda}$.*

Proof: We first consider the case when ℓ is odd, which means that t divides $\frac{ms}{\lambda}$. Let $B = B_{1,0} =$

1	-2
-1	2

; note that $\mu(B) = 2$. We start considering the set $X_0 = \{0, 2, 4, \ldots, \ell - 3\}$ of size $\frac{\ell-1}{2} = \frac{ms}{\lambda t}$: it is easy to see that $\bigcup\limits_{x \in X_0} \mathsf{supp}(B \pm x) = [1, \ell] \setminus \{\ell\}$. Similarly, for any $i \in \mathbb{N}$, if $X_i = \{i\ell, i\ell + 2, i\ell + 4, \ldots, (i+1)\ell - 3\}$, then

$$\bigcup_{x \in X_i} \mathsf{supp}(B \pm x) = [i\ell + 1, (i+1)\ell] \setminus \{(i+1)\ell\}$$

and $X_{i_1} \cap X_{i_2} = \emptyset$ if $i_1 \neq i_2$.

If t is even, take $X = \bigcup_{i=0}^{t/2-1} X_i$: this is a set of size $\frac{t}{2} \cdot \frac{ms}{\lambda t} = \frac{ms}{2\lambda}$, as required. Furthermore,

$$\begin{aligned}\bigcup_{x\in X} \mathsf{supp}(B \pm x) &= \bigcup_{i=0}^{t/2-1} ([i\ell+1,(i+1)\ell] \setminus \{(i+1)\ell\}) \\ &= \left[1, \tfrac{t}{2}\ell\right] \setminus \left\{\ell, 2\ell, \ldots, \tfrac{t}{2}\ell\right\} = \left[1, \tfrac{ms}{\lambda} + \tfrac{t}{2}\right] \setminus \left\{\ell, 2\ell, \ldots, \tfrac{t}{2}\ell\right\}.\end{aligned}$$

Suppose now that t is odd, which implies that $\ell \equiv 1 \pmod 4$. Take

$$Z = \left\{\left(\tfrac{t-1}{2}\right)\ell, \left(\tfrac{t-1}{2}\right)\ell + 2, \left(\tfrac{t-1}{2}\right)\ell + 4, \ldots, \left(\tfrac{t-1}{2}\right)\ell + 2\tfrac{\ell-5}{4}\right\}.$$

Then $|Z| = \frac{\ell-1}{4} = \frac{ms}{2\lambda t}$ and $\bigcup_{z\in Z} \mathsf{supp}(B \pm z) = \left[\left(\frac{t-1}{2}\right)\ell + 1, \left(\frac{t-1}{2}\right)\ell + \frac{\ell-1}{2}\right]$. So, we can take $X = \left(\bigcup_{i=0}^{(t-3)/2} X_i\right) \cup Z$: this is a set of size $\frac{t-1}{2} \cdot \frac{ms}{\lambda t} + \frac{ms}{2\lambda t} = \frac{ms}{2\lambda}$, as required. In this case,

$$\begin{aligned}\bigcup_{x\in X} \mathsf{supp}(B \pm x) &= \bigcup_{i=0}^{\frac{t-3}{2}} ([i\ell+1,(i+1)\ell] \setminus \{(i+1)\ell\}) \cup \\ &\quad \left[\left(\tfrac{t-1}{2}\right)\ell + 1, \left(\tfrac{t-1}{2}\right)\ell + \tfrac{\ell-1}{2}\right] \\ &= \left(\left[1, \tfrac{t-1}{2}\ell\right] \setminus \left\{\ell, 2\ell, \ldots, \tfrac{t-1}{2}\ell\right\}\right) \cup \left[\left(\tfrac{t-1}{2}\right)\ell + 1, \tfrac{ms}{\lambda} + \tfrac{t-1}{2}\right] \\ &= \left[1, \tfrac{ms}{\lambda} + \left\lfloor \tfrac{t}{2} \right\rfloor\right] \setminus \left\{\ell, 2\ell, \ldots, \left\lfloor \tfrac{t}{2} \right\rfloor \ell\right\}.\end{aligned}$$

In both cases, considering $\frac{\lambda}{2}$ copies of the distinct blocks $B \pm x$ with $x \in X$, the pf array A obtained by following our procedure is an integer ${}^{\lambda}\mathrm{H}_t(m, n; s, k)$.

Finally, we consider the case when ℓ is even, which implies that $t \equiv 0 \pmod 4$. Let $B = B_{\ell,0} =$

1	$-(\ell+1)$
-1	$\ell+1$

; note that $\mu(B) = 2$. We start considering the set $X_0 = [0, \ell-2]$ of size $\ell - 1 = \frac{2ms}{\lambda t}$: it is easy to see that $\bigcup_{x\in X_0} \mathsf{supp}(B \pm x) = [1, 2\ell] \setminus \{\ell, 2\ell\}$. Similarly, for any $i \in \mathbb{N}$, if $X_i = [2i\ell, (2i+1)\ell - 2]$, then

$$\bigcup_{x\in X_i} \mathsf{supp}(B \pm x) = [2i\ell + 1, (2i+2)\ell] \setminus \{(2i+1)\ell, (2i+2)\ell\}$$

and $X_{i_1} \cap X_{i_2} = \emptyset$ if $i_1 \neq i_2$. Take $X = \bigcup_{i=0}^{t/4-1} X_i$: this is a set of size $\frac{t}{4} \cdot (\ell - 1) = \frac{ms}{2\lambda}$, as required. In this case,

$$\begin{aligned}\bigcup_{x\in X} \mathsf{supp}(B \pm x) &= \bigcup_{i=0}^{t/4-1} ([2i\ell+1,(2i+2)\ell] \setminus \{(2i+1)\ell, (2i+2)\ell\}) \\ &= \left[1, \tfrac{t}{2}\ell\right] \setminus \left\{\ell, 2\ell, \ldots, \tfrac{t}{2}\ell\right\} = \left[1, \tfrac{ms}{\lambda} + \tfrac{t}{2}\right] \setminus \left\{\ell, 2\ell, \ldots, \tfrac{t}{2}\ell\right\}.\end{aligned}$$

Now we take $\frac{\lambda}{2}$ copies of every block $B \pm x$: the pf array A obtained by following our procedure is an integer ${}^\lambda\mathrm{H}_t(m,n;s,k)$. □

We now deal with the case λ odd. This implies that λ divides $ms/4$.

Lemma 3.4 *Let λ be a positive odd integer. There exists an integer ${}^\lambda\mathrm{H}_t(m,n;s,k)$ for any divisor t of $\frac{2ms}{\lambda}$ such that $t \equiv 0 \pmod 8$.*

PROOF: Let $B = B_{\ell,2\ell} =$

1	$-(\ell+1)$
$-(2\ell+1)$	$3\ell+1$

, where $\ell = \frac{2ms}{\lambda t}+1$. Note that $\mu(B) = 1$. An integer ${}^\lambda\mathrm{H}_t(m,n;s,k)$, say A, can be obtained by following the construction described before, once we exhibit a suitable set X of size $\frac{ms}{4\lambda}$, in such a way that $\mathsf{supp}(A) = \left[1, \frac{ms}{\lambda}+\frac{t}{2}\right] \setminus \left\{\ell, 2\ell, \ldots, \frac{t}{2}\ell\right\}$.

Start considering the set $X_0 = [0, \ell-2]$ of size $\ell - 1 = \frac{2ms}{\lambda t}$: it is easy to see that $\bigcup_{x\in X_0} \mathsf{supp}(B \pm x) = [1, 4\ell] \setminus \{\ell, 2\ell, 3\ell, 4\ell\}$. Similarly, for any $i \in \mathbb{N}$, if $X_i = [4i\ell, (4i+1)\ell - 2]$, then

$$\bigcup_{x\in X_i} \mathsf{supp}(B \pm x) = [4i\ell+1, (4i+4)\ell] \setminus \{(4i+1)\ell, (4i+2)\ell, (4i+3)\ell, (4i+4)\ell\}.$$

Clearly, $X_{i_1} \cap X_{i_2} = \emptyset$ if $i_1 \neq i_2$. So, take $X = \bigcup_{i=0}^{t/8-1} X_i$: this is a set of size $\frac{t}{8}\cdot(\ell-1) = \frac{t}{8}\cdot\frac{2ms}{\lambda t} = \frac{ms}{4\lambda}$, as required. It is easy to see that

$$\begin{aligned}\bigcup_{x\in X} \mathsf{supp}(B \pm x) &= \bigcup_{i=0}^{t/8-1} ([4i\ell+1, (4i+4)\ell] \setminus \{(4i+1)\ell, (4i+2)\ell, (4i+3)\ell, \\ &\quad (4i+4)\ell\}) \\ &= \left[1, \tfrac{t}{2}\ell\right] \setminus \left\{\ell, 2\ell, \ldots, \tfrac{t}{2}\ell\right\} = \left[1, \tfrac{ms}{\lambda}+\tfrac{t}{2}\right] \setminus \left\{\ell, 2\ell, \ldots, \tfrac{t}{2}\ell\right\}.\end{aligned}$$

Now we take λ copies of every block $B \pm x$: the pf array A obtained by following our procedure is an integer ${}^\lambda\mathrm{H}_t(m,n;s,k)$. □

Lemma 3.5 *Let λ be a positive odd integer. There exists an integer ${}^\lambda\mathrm{H}_t(m,n;s,k)$ for any divisor t of $\frac{ms}{\lambda}$ such that $t \equiv 0 \pmod 4$.*

PROOF: Let $B = B_{1,\ell} =$

1	-2
$-(\ell+1)$	$\ell+2$

: note that $\mu(B) = 1$ and, since t divides $\frac{ms}{\lambda}$, $\ell = \frac{2ms}{\lambda t}+1$ is an odd integer. We start considering the set $X_0 = \{0, 2, 4, \ldots, \ell-3\}$ of size $\frac{\ell-1}{2} = \frac{ms}{\lambda t}$: it is easy to see that $\bigcup_{x\in X_0} \mathsf{supp}(B \pm x) = [1, \ell-1] \cup [\ell+1, 2\ell-1] =$

$[1, 2\ell] \setminus \{\ell, 2\ell\}$. Similarly, for any $i \in \mathbb{N}$, if $X_i = \{2i\ell, 2i\ell+2, 2i\ell+4, \ldots, (2i+1)\ell-3\}$, then

$$\bigcup_{x \in X_i} \mathsf{supp}(B \pm x) = [2i\ell + 1, 2(i+1)\ell] \setminus \{(2i+1)\ell, (2i+2)\ell\}$$

and $X_{i_1} \cap X_{i_2} = \emptyset$ if $i_1 \neq i_2$. So, take $X = \bigcup_{i=0}^{t/4-1} X_i$: this is a set of size $\frac{t}{4} \cdot \frac{\ell-1}{2} = \frac{t}{4} \cdot \frac{ms}{\lambda t} = \frac{ms}{4\lambda}$, as required. Hence,

$$\begin{aligned} \bigcup_{x \in X} \mathsf{supp}(B \pm x) &= \bigcup_{i=0}^{t/4-1} \left([2i\ell + 1, 2(i+1)\ell] \setminus \{(2i+1)\ell, (2i+2)\ell\}\right) \\ &= \left[1, \tfrac{t}{2}\ell\right] \setminus \left\{\ell, 2\ell, \ldots, \tfrac{t}{2}\ell\right\} = \left[1, \tfrac{ms}{\lambda} + \tfrac{t}{2}\right] \setminus \left\{\ell, 2\ell, \ldots, \tfrac{t}{2}\ell\right\}. \end{aligned}$$

Now we take λ copies of every block $B \pm x$: the pf array A obtained by following our procedure is an integer ${}^{\lambda}\mathrm{H}_t(m, n; s, k)$. □

For instance, to construct an integer ${}^{5}\mathrm{H}_4(5, 10; 8, 4)$ we can follow the proof of the previous lemma. In fact, $\lambda = 5$ and $t = 4$ divides $\frac{5 \cdot 8}{5}$; note that $\ell = 5$ and $Y = 5 * (0, 2)$.

${}^{5}\mathrm{H}_4(5, 10; 8, 4) =$

1	−2		−8	9	3	−4		−6	7
9	3	−4		−6	7	1	−2		−8
−6	7	1	−2		−8	9	3	−4	
	−8	9	3	−4		−6	7	1	−2
−4		−6	7	1	−2		−8	9	3

Lemma 3.6 *Let λ be a positive odd integer. There exists an integer ${}^{\lambda}\mathrm{H}_t(m, n; s, k)$ for any divisor t of $\frac{ms}{2\lambda}$.*

Proof: Let $B = B_{1,2} =$

1	−2
−3	4

. Note that $\mu(B) = 1$ and $\ell = \frac{2ms}{\lambda t} + 1 \equiv 1 \pmod 4$ since t divides $\frac{ms}{2\lambda}$. We start considering the set $X_0 = \{0, 4, 8, \ldots, \ell - 5\}$ of size $\frac{\ell-1}{4} = \frac{ms}{2\lambda t}$: clearly, $\bigcup_{x \in X_0} \mathsf{supp}(B \pm x) = [1, \ell] \setminus \{\ell\}$. Similarly, for any $i \in \mathbb{N}$, if $X_i = \{i\ell, i\ell + 4, i\ell + 8, \ldots, (i+1)\ell - 5\}$, then

$$\bigcup_{x \in X_i} \mathsf{supp}(B \pm x) = [i\ell + 1, (i+1)\ell] \setminus \{(i+1)\ell\}$$

and $X_{i_1} \cap X_{i_2} = \emptyset$ if $i_1 \neq i_2$.

If t is even, take $X = \bigcup_{i=0}^{t/2-1} X_i$: this is a set of size $\frac{t}{2} \cdot \frac{\ell-1}{4} = \frac{t}{2} \cdot \frac{ms}{2\lambda t} = \frac{ms}{4\lambda}$, as required. Hence,

$$\begin{aligned} \bigcup_{x \in X} \mathsf{supp}(B \pm x) &= \bigcup_{i=0}^{t/2-1} \left([i\ell + 1, (i+1)\ell] \setminus \{(i+1)\ell\}\right) \\ &= \left[1, \tfrac{t}{2}\ell\right] \setminus \left\{\ell, 2\ell, \ldots, \tfrac{t}{2}\ell\right\} = \left[1, \tfrac{ms}{\lambda} + \tfrac{t}{2}\right] \setminus \left\{\ell, 2\ell, \ldots, \tfrac{t}{2}\ell\right\}. \end{aligned}$$

Suppose now that t is odd. Notice that, in this case, $\ell \equiv 1 \pmod 8$. Take

$$Z = \left\{ \left(\frac{t-1}{2}\right)\ell, \left(\frac{t-1}{2}\right)\ell + 4, \left(\frac{t-1}{2}\right)\ell + 8, \ldots, \left(\frac{t-1}{2}\right)\ell + 4\frac{\ell-9}{8} \right\}.$$

Then $|Z| = \frac{\ell-1}{8} = \frac{ms}{4\lambda t}$ and $\bigcup_{z\in Z} \mathsf{supp}(B \pm z) = \left[\left(\frac{t-1}{2}\right)\ell + 1, \left(\frac{t-1}{2}\right)\ell + \frac{\ell-1}{2}\right]$. Take $X = \left(\bigcup_{i=0}^{(t-3)/2} X_i\right) \cup Z$: this is a set of size $\frac{t-1}{2}\cdot\frac{\ell-1}{4} + \frac{\ell-1}{8} = \frac{t-1}{2}\cdot\frac{ms}{2\lambda t} + \frac{ms}{4\lambda t} = \frac{ms}{4\lambda}$, as required. In this case,

$$\begin{aligned}
\bigcup_{x\in X} \mathsf{supp}(B \pm x) &= \bigcup_{i=0}^{\frac{t-3}{2}} \left([i\ell+1,(i+1)\ell] \setminus \{(i+1)\ell\}\right) \cup \\
&\quad \left[\left(\tfrac{t-1}{2}\right)\ell + 1, \left(\tfrac{t-1}{2}\right)\ell + \tfrac{\ell-1}{2}\right] \\
&= \left(\left[1, \tfrac{t-1}{2}\ell\right] \setminus \left\{\ell, 2\ell, \ldots, \tfrac{t-1}{2}\ell\right\}\right) \cup \left[\left(\tfrac{t-1}{2}\right)\ell + 1, \tfrac{ms}{\lambda} + \tfrac{t-1}{2}\right] \\
&= \left[1, \tfrac{ms}{\lambda} + \left\lfloor \tfrac{t}{2} \right\rfloor\right] \setminus \left\{\ell, 2\ell, \ldots, \left\lfloor \tfrac{t}{2} \right\rfloor \ell\right\}.
\end{aligned}$$

In both cases, we construct the pf array A using λ copies of every block $B \pm x$; so, the pf array A obtained by following our procedure is an integer ${}^{\lambda}\mathrm{H}_t(m,n;s,k)$. □

For instance, we can follow the proof of the previous lemma for constructing an integer ${}^{3}\mathrm{H}_3(9,9;8,8)$. In fact, $\lambda = 3$ and $t = 3$ divides $\frac{9\cdot 8}{2\cdot 3}$; note that $\ell = 17$ and $Y = 3 * (0,4,8,12,17,21)$.

${}^{3}\mathrm{H}_3(9,9;8,8) =$

1	−2	−20	21	13	−14		−7	8
12	5	−6	−24	25	18	−19		−11
−3	4	9	−10	−15	16	22	−23	
	−7	8	13	−14	−20	21	1	−2
−6		−11	12	18	−19	−24	25	5
9	−10		−15	16	22	−23	−3	4
8	13	−14		−20	21	1	−2	−7
−11	12	18	−19		−24	25	5	−6
−10	−15	16	22	−23		−3	4	9

.

We now consider the case when λ does not divide ms. We need to adjust our general strategy in order to satisfy (2.1).

Lemma 3.7 *Suppose that λ does not divide ms. Then, there exists an integer ${}^{\lambda}\mathrm{H}_t(m,n;s,k)$ for any divisor t of $\frac{2ms}{\lambda}$.*

Proof: Since λ divides $2ms$ but does not divide ms, from $s \equiv 0 \pmod 4$ we obtain $\lambda \equiv 0 \pmod 8$. We can easily adapt the proof of Lemma 3.2, using the block $B = B_{0,0} = \begin{array}{|c|c|}\hline 1 & -1 \\ \hline & \\ \hline -1 & 1 \\ \hline \end{array}$ and considering two possibilities. In both cases, an integer ${}^\lambda\mathrm{H}_t(m,n;s,k)$, say A, can be obtained by following the construction given at the beginning of this section and using the blocks $B \pm y_0, B \pm y_1, \ldots, B \pm y_{\frac{ms}{4}-1}$ for a suitable sequence $Y = (y_0, y_1, \ldots, y_{\frac{ms}{4}-1})$ in such a way that condition (2.1) is satisfied.

Suppose that ℓ is odd or t is even. It suffices to consider the sequence X obtained by taking the natural ordering $\leq$ of $\{i-1 \mid i \in \Phi\} \subset \mathbb{N}$, and define $Y = \frac{\lambda}{4} * X$.

Suppose that ℓ is even and t is odd. Let X_1 be the sequence obtained by taking the natural ordering $\leq$ of $\{i-1 \mid i \in \Psi\} \subset \mathbb{N}$, where $\Psi = \Phi \setminus \left\{\frac{t\ell}{2}\right\}$. Also, let $Y_1 = \frac{\lambda}{4} * X_1$ and let Y_2 be the sequence obtained by repeating $\frac{\lambda}{8}$ times the integer $\frac{t\ell}{2} - 1$. Define $Y = Y_1 + Y_2$ and note that $|Y| = \frac{\lambda}{4} \cdot \frac{2ms-\lambda}{2\lambda} + \frac{\lambda}{8} = \frac{ms}{4}$. □

For instance, the integer ${}^{16}\mathrm{H}_5(10,10;4,4)$ given in Example 1.9 was obtained by following the proof of the previous lemma. In fact, $\lambda = 16$ does not divide $ms = 40$; note that $\ell = 2$, $X_1 = (0,2)$ and $Y = (0,2,0,2,0,2,0,2,4,4)$.

Proposition 3.8 *Suppose* $4 \leq s \leq n$, $4 \leq k \leq m$, $ms = nk$ *and* $s,k \equiv 0 \pmod 4$. *Let* λ *be a divisor of* $2ms$. *There exists a shiftable integer* ${}^\lambda\mathrm{H}_t(m,n;s,k)$ *for every divisor* t *of* $\frac{2ms}{\lambda}$.

Proof: If λ does not divide ms, the statement follows from Lemma 3.7. So, suppose that λ divides ms. If $\lambda \equiv 0 \pmod 4$ or $\lambda \equiv 2 \pmod 4$, then we can apply Lemma 3.2 or Lemma 3.3, respectively. Now we assume λ odd. If $t \equiv 0 \pmod 8$, we apply Lemma 3.4. If $t \equiv 4 \pmod 8$, then t divides $\frac{ms}{\lambda}$ and hence we can apply Lemma 3.5. Finally, if $t \not\equiv 0 \pmod 4$, then t divides $\frac{ms}{2\lambda}$ and so the existence of an integer ${}^\lambda\mathrm{H}_t(m,n;s,k)$ follows from Lemma 3.6. In all these cases, the integer λ-fold Heffter array that we construct is shiftable. □

4 The case $s \equiv 2 \pmod 4$, k and m even

In this section, we will assume that s, m, k are positive even integers with $s \equiv 2 \pmod 4$ and $s \geq 6$. We need to distinguish two cases, according to the divisibility of ms by λ. In fact, if λ does not divide ms, from $ms \equiv 0 \pmod 4$ we obtain $\lambda \equiv 0 \pmod 8$. In this case, we have to construct pf arrays that satisfy (2.1).

If λ divides ms we write

$$\lambda = \lambda_1\lambda_2, \quad \text{where } \lambda_1 \text{ divides } \frac{m}{2} \text{ and } \lambda_2 \text{ divides } 2s. \tag{4.1}$$

Let t be a divisor of $\frac{2ms}{\lambda}$ and set

$$\ell = \frac{2ms}{\lambda t} + 1.$$

4.1 Construction of nice pairs of sequences

To obtain an integer ${}^{\lambda}\mathrm{H}_t(m,n;s,k)$, we first construct pairs of sequences, satisfying the following properties.

Definition 4.1 A pair $(\mathcal{B}_1,\mathcal{B}_2)$ of sequences is said to be *nice* if, for a fixed positive integer b, we have:

- the sequence $\mathcal{B}_1$ consists of blocks satisfying this condition:

$$\text{there exist } b \text{ integers } \sigma_1,\ldots,\sigma_b \text{ such that the elements of } \mathcal{B}_1 \text{ are shiftable blocks } B \text{ of size } 2\times 2b \text{ with } \tau_1(B)=\tau_2(B)=0 \text{ and } \gamma_{2i-1}(B)=-\gamma_{2i}(B)=\sigma_i \text{ for all } i\in[1,b]; \tag{4.2}$$

- the sequence $\mathcal{B}_2$ consists of blocks satisfying this condition:

$$\text{there exist } 2b \text{ integers } \sigma'_1,\ldots,\sigma'_{2b} \text{ with } \sum_{i=1}^{b}\sigma'_{2i-1}=\sum_{i=1}^{b}\sigma'_{2i}=0, \text{ such that the elements of } \mathcal{B}_2 \text{ are shiftable blocks } B' \text{ of size } 2\times 2b \text{ with } \tau_1(B')=\tau_2(B')=0 \text{ and } \gamma_i(B')=\sigma'_i \text{ for all } i\in[1,2b]; \tag{4.3}$$

- the sequences $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same length and, writing $\mathcal{B}_1=(B_1,B_2,\ldots,B_e)$ and $\mathcal{B}_2=(B'_1,B'_2,\ldots,B'_e)$, then $\mathcal{E}(B_i)=\mathcal{E}(B'_i)$ for all $i\in[1,e]$.

Observe that the sequences $\mathcal{B}_1,\mathcal{B}_2$ in the previous definition do not need to be distinct.

We construct these nice pairs of sequences, starting with the case when λ divides ms. In particular, our sequences $\mathcal{B}_i$, consisting of shiftable blocks of size $2\times s$, are of length $\frac{m}{2\lambda_1}$ and such that $\mu(\mathcal{B}_i)=\lambda_2$. We begin with the case when λ_2 is odd. Note that this implies that λ_2 divides $\frac{s}{2}$.

Lemma 4.2 *[18, Corollary 4.10 and Lemma 5.1] Let a and c be even integers with $a\geq 2$, $c\geq 6$ and $c\equiv 2 \pmod 4$. Let u be a divisor of $2ac$ and set $\rho=\frac{2ac}{u}+1$. There exists a nice pair $(\tilde{\mathcal{B}}_1,\tilde{\mathcal{B}}_2)$ of sequences of length $\frac{a}{2}$, where $\tilde{\mathcal{B}}_1$ and $\tilde{\mathcal{B}}_2$ consist of blocks of size $2\times c$, $\mu(\tilde{\mathcal{B}}_1)=\mu(\tilde{\mathcal{B}}_2)=1$ and*

$$\mathsf{supp}(\tilde{\mathcal{B}}_1)=\mathsf{supp}(\tilde{\mathcal{B}}_2)=[1,ac+\lfloor u/2\rfloor]\setminus\{j\rho : j\in[1,\lfloor u/2\rfloor]\}.$$

Corollary 4.3 *Let $\lambda=\lambda_1\lambda_2$ be as in (4.1). If $\lambda_2\neq\frac{s}{2}$ is odd, there exists a nice pair $(\mathcal{B}_1,\mathcal{B}_2)$ of sequences of length $\frac{m}{2\lambda_1}$, where $\mathcal{B}_1$ and $\mathcal{B}_2$ consist of blocks of size $2\times s$, $\mu(\mathcal{B}_1)=\mu(\mathcal{B}_2)=\lambda_2$ and*

$$\mathsf{supp}(\mathcal{B}_1)=\mathsf{supp}(\mathcal{B}_2)=\left[1,\frac{ms}{\lambda}+\left\lfloor\frac{t}{2}\right\rfloor\right]\setminus\left\{\ell,2\ell,\ldots,\left\lfloor\frac{t}{2}\right\rfloor\ell\right\}=\Phi.$$

Proof: Take $a = \frac{m}{\lambda_1}$, $c = \frac{s}{\lambda_2}$ and $u = t$. Since λ_1 divides $\frac{m}{2}$, a is a positive even integer; since $\lambda_2 \neq \frac{s}{2}$ is odd and divides $2s$, then c is an even integer such that $c \geq 6$ and $c \equiv 2 \pmod 4$. Note that t divides $2ac = \frac{2ms}{\lambda_1\lambda_2}$ and $\rho = \frac{2ac}{t} + 1 = \frac{2ms}{\lambda t} + 1 = \ell$. Hence, we can apply Lemma 4.2 obtaining a nice pair $(\tilde{\mathcal{B}}_1, \tilde{\mathcal{B}}_2)$ of sequences of length $\frac{m}{2\lambda_1}$ consisting of blocks of size $2 \times \frac{s}{\lambda_2}$ such that $\mu(\tilde{\mathcal{B}}_1) = \mu(\tilde{\mathcal{B}}_2) = 1$ and $\mathsf{supp}(\tilde{\mathcal{B}}_1) = \mathsf{supp}(\tilde{\mathcal{B}}_2) = \Phi$. Now, replace every block $\tilde{B}$ of $\tilde{\mathcal{B}}_i$, $i = 1, 2$, with the block B obtained by juxtaposing λ_2 copies of $\tilde{B}$. So, B is a block of size $2 \times s$ and $\mu(B) = \lambda_2$. Call $\mathcal{B}_1, \mathcal{B}_2$ the two sequences so obtained. It follows that the pair $(\mathcal{B}_1, \mathcal{B}_2)$ satisfies the required properties. □

Now we consider the case when $\lambda_2 = \frac{s}{2}$.

Lemma 4.4 *Let $\lambda = \lambda_1\lambda_2$ be as in (4.1) with $\lambda_2 = \frac{s}{2}$. There exists a nice pair $(\mathcal{B}_1, \mathcal{B}_2)$ of sequences of length $\frac{m}{2\lambda_1}$, where $\mathcal{B}_1$ and $\mathcal{B}_2$ consist of blocks of size $2 \times s$, $\mu(\mathcal{B}_1) = \mu(\mathcal{B}_2) = \frac{s}{2}$ and $\mathsf{supp}(\mathcal{B}_1) = \mathsf{supp}(\mathcal{B}_2) = \Phi$.*

Proof: We first consider the case when ℓ is odd. Consider the following shiftable blocks:

$$A = \begin{array}{|c|c|c|c|}\hline 1 & -2 & -3 & 4 \\ \hline -1 & 2 & 3 & -4 \\ \hline \end{array}, \qquad F = \begin{array}{|c|c|c|c|}\hline 1 & -2 & -4 & 5 \\ \hline -1 & 2 & 4 & -5 \\ \hline \end{array},$$

$$E = \begin{array}{|c|c|c|c|c|c|}\hline 1 & -1 & 3 & -4 & -3 & 4 \\ \hline -2 & 2 & -1 & 2 & 3 & -4 \\ \hline \end{array}, \qquad G = \begin{array}{|c|c|c|c|c|c|}\hline 4 & 2 & -2 & 2 & -1 & -5 \\ \hline -5 & -1 & 4 & -4 & 1 & 5 \\ \hline \end{array},$$

$$E' = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 3 & -1 & -4 & -3 & 4 \\ \hline -2 & -1 & 2 & 2 & 3 & -4 \\ \hline \end{array}, \qquad G' = \begin{array}{|c|c|c|c|c|c|}\hline 4 & -2 & 2 & 2 & -1 & -5 \\ \hline -5 & 4 & -1 & -4 & 1 & 5 \\ \hline \end{array}.$$

Note that A and F satisfy both (4.2) and (4.3); E and G satisfy (4.2); E' and G' satisfy (4.3). We first construct the sequence $\mathcal{B}_1$. To this purpose, take the block B obtained by juxtaposing the block E and $\frac{s-6}{4}$ copies of the block A. We obtain a block of size $2 \times s$ such that $\mathsf{supp}(B) = [1, 4]$ and $\mu(B) = \frac{s}{2}$. Also, let C be the block obtained by juxtaposing the block G and $\frac{s-6}{4}$ copies of the block F. Then C is a block of size $2 \times s$ such that $\mathsf{supp}(C) = \{1, 2, 4, 5\}$ and $\mu(C) = \frac{s}{2}$.

Assume $\ell \equiv 1 \pmod 4$. Let $S = \left(B, B \pm 4, B \pm 8, \ldots, B \pm 4\frac{\ell-5}{4}\right)$. Then $|S| = \frac{\ell-1}{4}$ and $\mathsf{supp}(S) = [1, \ell] \setminus \{\ell\}$. If t is even, take

$$\mathcal{B}_1 = S \mathbin{+\!\!+} (S \pm \ell) \mathbin{+\!\!+} (S \pm 2\ell) \mathbin{+\!\!+} \ldots \mathbin{+\!\!+} \left(S \pm \frac{t-2}{2}\ell\right).$$

If t is odd, then $\ell - 1 = 8\frac{m}{2\lambda_1 t} \equiv 0 \pmod 8$. Let

$$Y = \left(B, B \pm 4, B \pm 8, \ldots, B \pm \left(4\frac{\ell-9}{8}\right)\right)$$

and

$$\mathcal{B}_1 = S \mathbin{+\!\!+} (S \pm \ell) \mathbin{+\!\!+} (S \pm 2\ell) \mathbin{+\!\!+} \ldots \mathbin{+\!\!+} \left(S \pm \frac{t-3}{2}\ell\right) \mathbin{+\!\!+} \left(Y \pm \frac{t-1}{2}\ell\right).$$

In both cases, $\mathcal{B}_1$ is a sequence of length $\frac{(\ell-1)t}{8} = \frac{m}{2\lambda_1}$ such that $\mu(\mathcal{B}_1) = \frac{s}{2}$ and $\mathsf{supp}(\mathcal{B}_1) = \Phi$. The sequence $\mathcal{B}_2$ is obtained by replacing in $\mathcal{B}_1$ the block E with the block E'.

Assume $\ell \equiv 3 \pmod 4$. Note that, in this case, $8\frac{m}{2\lambda_1 t} \equiv 2 \pmod 4$ and so $t \equiv 0 \pmod 4$. Take $S = \left(B, B \pm 4, B \pm 8, \ldots, B \pm 4\frac{\ell-7}{4}, C \pm (\ell-3), B \pm (\ell+2), B \pm (\ell+6), B \pm (\ell+10), \ldots, B \pm (2\ell-5)\right)$. Then $|S| = \frac{\ell-1}{2}$ and $\mathsf{supp}(S) = [1, 2\ell] \setminus \{\ell, 2\ell\}$. Define

$$\mathcal{B}_1 = S \mathbin{+\!\!+} (S \pm 2\ell) \mathbin{+\!\!+} (S \pm 4\ell) \mathbin{+\!\!+} \ldots \mathbin{+\!\!+} \left(S \pm 2\frac{t-4}{4}\ell\right).$$

So, $\mathcal{B}_1$ is a sequence of length $\frac{(\ell-1)t}{8} = \frac{m}{2\lambda_1}$ such that $\mu(\mathcal{B}_1) = \frac{s}{2}$ and $\mathsf{supp}(\mathcal{B}_1) = \Phi$. The sequence $\mathcal{B}_2$ is obtained by replacing in $\mathcal{B}_1$ the block G with the block G'.

Finally, assume that ℓ is even. Note that, in this case, $t \equiv 0 \pmod 8$. Consider the shiftable blocks:

$H =$

1	$-(\ell+1)$	$-(2\ell+1)$	$3\ell+1$
-1	$\ell+1$	$2\ell+1$	$-(3\ell+1)$

,

$L =$

1	$3\ell+1$	$-(\ell+1)$	$\ell+1$	-1	$-(3\ell+1)$
$-(\ell+1)$	$-(2\ell+1)$	$2\ell+1$	$-(2\ell+1)$	1	$3\ell+1$

.

Note that the blocks H and L satisfy both (4.2) and (4.3). Let K be the block obtained by juxtaposing the block L and $\frac{s-6}{4}$ copies of the block H. Then K is a block of size $2 \times s$ such that $\mathsf{supp}(K) = \{1, \ell+1, 2\ell+1, 3\ell+1\}$ and $\mu(K) = \frac{s}{2}$. Let $S = (K, K \pm 1, K \pm 2, \ldots, K \pm (\ell-2))$. Then $|S| = \ell - 1$ and $\mathsf{supp}(S) = [1, 4\ell] \setminus \{\ell, 2\ell, 3\ell, 4\ell\}$. Define

$$\mathcal{B}_1 = \mathcal{B}_2 = S \mathbin{+\!\!+} (S \pm 4\ell) \mathbin{+\!\!+} (S \pm 8\ell) \mathbin{+\!\!+} \ldots \mathbin{+\!\!+} \left(S \pm 4\frac{t-8}{8}\ell\right).$$

So, $\mathcal{B}_i$ is a sequence of length $\frac{(\ell-1)t}{8} = \frac{m}{2\lambda_1}$ such that $\mu(\mathcal{B}_i) = \frac{s}{2}$ and $\mathsf{supp}(\mathcal{B}_i) = \Phi$. □

For instance, using the previous lemma with $m = 30$, $s = 10$, $\lambda_1 = 3$ and $t = 5$, we have $\ell = 9$. The sequence $\mathcal{B}_1$ consists of the following five shiftable blocks:

$B_1 =$

1	−1	3	−4	−3	4	1	−2	−3	4
−2	2	−1	2	3	−4	−1	2	3	−4

,

$B_2 =$

5	−5	7	−8	−7	8	5	−6	−7	8
−6	6	−5	6	7	−8	−5	6	7	−8

,

$B_3 =$

10	−10	12	−13	−12	13	10	−11	−12	13
−11	11	−10	11	12	−13	−10	11	12	−13

,

$B_4 =$

14	−14	16	−17	−16	17	14	−15	−16	17
−15	15	−14	15	16	−17	−14	15	16	−17

,

$B_5 =$

19	−19	21	−22	−21	22	19	−20	−21	22
−20	20	−19	20	21	−22	−19	20	21	−22

.

We now deal with the case $\lambda_2 \equiv 2 \pmod 4$.

Lemma 4.5 *Let $\lambda = \lambda_1\lambda_2$ be as in* (4.1) *with $\lambda_2 \equiv 2 \pmod 4$ and $\lambda_2 \geq 6$. There exists a nice pair $(\mathcal{B}, \mathcal{B})$, where $\mathcal{B}$ is a sequence of length $\frac{m}{2\lambda_1}$ consisting of blocks of size $2 \times s$ such that $\mu(\mathcal{B}) = \lambda_2$ and $\mathsf{supp}(\mathcal{B}) = \Phi$.*

Proof: We first consider the case when ℓ is odd. Consider the following shiftable blocks:

$$A = \begin{array}{|c|c|c|c|}\hline 1 & -1 & 2 & -2 \\ \hline -1 & 1 & -2 & 2 \\ \hline \end{array}, \qquad E = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & -1 & 1 & -1 & -2 \\ \hline -2 & -1 & 2 & -2 & 1 & 2 \\ \hline \end{array}.$$

Note that A and E satisfy both (4.2) and (4.3). To construct the sequence $\mathcal{B}$, first take the block C obtained by juxtaposing the block E and $\frac{\lambda_2-6}{4}$ copies of the block A. We obtain a block of size $2\times\lambda_2$ such that $\mathsf{supp}(C) = \{1,2\}$ and $\mu(C) = \lambda_2$. Consider the sequence $S = \left(C, C\pm 2, C\pm 4, \ldots, C \pm 2\frac{\ell-3}{2}\right)$. Then $|S| = \frac{\ell-1}{2}$, $\mu(S) = \lambda_2$ and $\mathsf{supp}(S) = [1,\ell] \setminus \{\ell\}$. If t is even, take

$$\tilde{\mathcal{B}} = S \mathbin{+\!\!+} (S \pm \ell) \mathbin{+\!\!+} (S \pm 2\ell) \mathbin{+\!\!+} \ldots \mathbin{+\!\!+} \left(S \pm \frac{t-2}{2}\ell\right).$$

If t is odd, then $\ell - 1 = 4\frac{\frac{m}{2\lambda_1}\cdot\frac{s}{\lambda_2}}{t} \equiv 0 \pmod 4$. Let

$$Y = \left(C, C\pm 2, C \pm 4, \ldots, C \pm \left(2\frac{\ell-5}{4}\right)\right)$$

and

$$\tilde{\mathcal{B}} = S \mathbin{+\!\!+} (S \pm \ell) \mathbin{+\!\!+} (S \pm 2\ell) \mathbin{+\!\!+} \ldots \mathbin{+\!\!+} \left(S \pm \frac{t-3}{2}\ell\right) \mathbin{+\!\!+} \left(Y \pm \frac{t-1}{2}\ell\right).$$

In both cases, $\tilde{\mathcal{B}}$ is a sequence of length $\frac{(\ell-1)t}{4} = \frac{ms}{2\lambda}$ such that $\mu(\tilde{\mathcal{B}}) = \lambda_2$ and $\mathsf{supp}(\tilde{\mathcal{B}}) = \Phi$.

Suppose now that ℓ is even. Note that, in this case, $t \equiv 0 \pmod 4$. Consider the shiftable blocks:

$$F = \begin{array}{|c|c|c|c|}\hline 1 & -1 & \ell+1 & -(\ell+1) \\ \hline -1 & 1 & -(\ell+1) & \ell+1 \\ \hline \end{array},$$

$$G = \begin{array}{|c|c|c|c|c|c|}\hline 1 & \ell+1 & -1 & 1 & -1 & -(\ell+1) \\ \hline -(\ell+1) & -1 & \ell+1 & -(\ell+1) & 1 & \ell+1 \\ \hline \end{array}.$$

Note that the blocks F and G satisfy both (4.2) and (4.3). Take the block H obtained by juxtaposing the block G and $\frac{\lambda_2-6}{4}$ copies of the block F. We obtain a block of size $2 \times \lambda_2$ such that $\mathsf{supp}(H) = \{1, \ell+1\}$ and $\mu(H) = \lambda_2$. Consider the sequence $S = (H, H \pm 1, H \pm 2, \ldots, H \pm (\ell - 2))$. Then $|S| = \ell - 1$, $\mu(S) = \lambda_2$ and $\mathsf{supp}(S) = [1, 2\ell] \setminus \{\ell, 2\ell\}$. Take

$$\tilde{\mathcal{B}} = S \mathbin{+\!\!+} (S \pm 2\ell) \mathbin{+\!\!+} (S \pm 4\ell) \mathbin{+\!\!+} \ldots \mathbin{+\!\!+} \left(S \pm 2\frac{t-4}{4}\ell\right).$$

Hence, $\tilde{\mathcal{B}}$ is a sequence of length $\frac{(\ell-1)t}{4} = \frac{ms}{2\lambda}$ such that $\mu(\tilde{\mathcal{B}}) = \lambda_2$ and $\mathsf{supp}(\tilde{\mathcal{B}}) = \Phi$.

Finally, for every ℓ, writing $\tilde{\mathcal{B}} = \big(K_1, K_2, \ldots, K_{\frac{ms}{2\lambda}}\big)$ and $q = \frac{s}{\lambda_2}$, for every $i \in \left[1, \frac{m}{2\lambda_1}\right]$ we construct the block B_i juxtaposing the q blocks $K_{1+(i-1)q}, K_{2+(i-1)q}, \ldots, K_{iq}$. The blocks B_i are of size $2 \times q\lambda_2$, that is, of size $2 \times s$. So, we can set $\mathcal{B} = \Big(B_1, B_2, B_3, \ldots, B_{\frac{m}{2\lambda_1}}\Big)$. □

For instance, using the previous lemma with $m = 84$, $s = 10$, $\lambda_1 = 7$, $\lambda_2 = 10$ and $t = 8$, we have $\ell = 4$. The sequence $\mathcal{B}$ consists of the following six shiftable blocks:

$B_1 =$

1	5	−1	1	−1	−5	1	−1	5	−5
−5	−1	5	−5	1	5	−1	1	−5	5

,

$B_2 =$

2	6	−2	2	−2	−6	2	−2	6	−6
−6	−2	6	−6	2	6	−2	2	−6	6

,

$B_3 =$

3	7	−3	3	−3	−7	3	−3	7	−7
−7	−3	7	−7	3	7	−3	3	−7	7

,

$B_4 =$

9	13	−9	9	−9	−13	9	−9	13	−13
−13	−9	13	−13	9	13	−9	9	−13	13

,

$B_5 =$

10	14	−10	10	−10	−14	10	−10	14	−14
−14	−10	14	−14	10	14	−10	10	−14	14

,

$B_6 =$

11	15	−11	11	−11	−15	11	−11	15	−15
−15	−11	15	−15	11	15	−11	11	−15	15

.

We now deal with the case $\lambda_2 = 2$.

Lemma 4.6 *Let $\lambda = \lambda_1\lambda_2$ be as in* (4.1) *with $\lambda_2 = 2$. Suppose that t divides $\frac{ms}{2\lambda_1}$. There exists a nice pair $(\mathcal{B}_1, \mathcal{B}_2)$ of sequences of length $\frac{m}{2\lambda_1}$, where $\mathcal{B}_1$ and $\mathcal{B}_2$ consist of blocks of size $2 \times s$, $\mu(\mathcal{B}_1) = \mu(\mathcal{B}_2) = 2$ and $\mathsf{supp}(\mathcal{B}_1) = \mathsf{supp}(\mathcal{B}_2) = \Phi$.*

Proof: Write $s = 4q + 6$ where $q \geq 0$ and take the following shiftable blocks:

$U_3 =$

1	−2	−4	5
−1	2	4	−5

, $U_5 =$

1	−2	−3	4
−1	2	3	−4

,

$V_1 =$

2	−2	−5	−6	4	7
−3	3	6	5	−4	−7

, $V_3 =$

1	−1	−5	−6	4	7
−2	2	6	5	−4	−7

,

$V_5 =$

6	−6	−2	−3	1	4
−7	7	3	2	−1	−4

, $V_7 =$

1	−1	−4	−5	3	6
−2	2	5	4	−3	−6

,

$Z =$

1	−1	4	−5	−7	8
−2	2	−4	5	7	−8

, $Z' =$

1	4	−1	−5	−7	8
−2	−4	2	5	7	−8

.

Note that, since t divides $\frac{ms}{2\lambda_1}$, ℓ is an odd integer.

If $\ell = 4x + 1 \geq 5$, take $\tilde{S} = (U_5, U_5 \pm 4, U_5 \pm 8, \ldots, U_5 \pm 4(x-1))$. Then $|\tilde{S}| = x$, $\mu(\tilde{S}) = 2$ and $\mathsf{supp}(\tilde{S}) = [1, \ell] \setminus \{\ell\}$. Let $\tilde{\mathcal{B}}$ be the sequence obtained by taking the first $\frac{mq}{2\lambda_1}$ blocks in $\mathop{+\!\!\!+}\limits_{c\geq 0}(\tilde{S}\pm \ell c)$. If $\ell = 4x+3 \geq 3$, take $\tilde{S} = (U_5, U_5\pm 4, U_5\pm 8, \ldots, U_5\pm 4(x-1), U_3\pm 4x, U_5\pm(4x+5), U_5\pm(4x+9), \ldots, U_5\pm(8x+1))$. Then $|\tilde{S}| = 2x+1$, $\mu(\tilde{S}) = 2$ and $\mathsf{supp}(\tilde{S}) = [1, 2\ell]\setminus\{\ell, 2\ell\}$. Let $\tilde{\mathcal{B}}$ be the sequence obtained by taking the first $\frac{mq}{2\lambda_1}$ blocks in $\mathop{+\!\!\!+}\limits_{c\geq 0}(\tilde{S}\pm 2\ell c)$. In both cases we obtain a sequence $\tilde{\mathcal{B}}$ of blocks of size 2×4 that satisfy both (4.2) and (4.3) and such that $\mathsf{supp}(\tilde{\mathcal{B}}) = [1, N]$ where $N = \frac{2mq}{\lambda_1} + \eta$ with $\eta = \left\lfloor \frac{2qt}{s}\right\rfloor$.

Now, we have to construct a sequence S' of shiftable blocks of size 2×6 satisfying condition (4.2) in such a way that $|S'| = \frac{m}{2\lambda_1}$ and

$$\mathsf{supp}(S') = \left[N+1, \frac{ms}{2\lambda_1} + \left\lfloor\frac{t}{2}\right\rfloor\right] \setminus \left\{ j\ell : j \in \left[\eta+1, \left\lfloor\frac{t}{2}\right\rfloor\right]\right\}.$$

If $\ell = 3$, then $t = \frac{ms}{2\lambda_1}$ and $N = 3\frac{mq}{\lambda_1} \equiv 0 \pmod 3$. We can take $S' = \mathop{+\!\!\!+}\limits_{c=0}^{\frac{m}{2\lambda_1}-1}(Z\pm(N+9c))$. If $\ell = 5$, then $t = \frac{ms}{4\lambda_1}$ and $N = 5\frac{mq}{2\lambda_1} \equiv 0 \pmod 5$. Define $T = (V_5, V_3\pm 7)$. If $\frac{m}{2\lambda_1}$ is even, we can take $S' = \mathop{+\!\!\!+}\limits_{c=0}^{\frac{m}{4\lambda_1}-1}(T\pm(N+15c))$. If $\frac{m}{2\lambda_1}$ is odd, we can take $S' = \left(\mathop{+\!\!\!+}\limits_{c=0}^{\frac{m-6\lambda_1}{4\lambda_1}}(T\pm(N+15c))\right) +\!\!\!+ \left(V_5 \pm\left(\frac{ms}{2\lambda_1}+\frac{t-15}{2}\right)\right)$.

Suppose now that $\ell\geq 7$: in this case, any set of 6 consecutive integers contains at most one multiple of ℓ. We start considering the interval $[N+1, N+6]$ and the first multiple of ℓ belonging to the interval $[N+1, \frac{ms}{2\lambda_1}+\lfloor t/2\rfloor]$. So, if $(\eta+1)\ell$ is an element of $[N+1, N+6]$ we take the block V_r where r must be chosen in such a way that $\mathsf{supp}(V_r\pm N)$ does not contain $(\eta+1)\ell$. Otherwise, we take the block V_7 and repeat this process considering the interval $[N+7, N+12]$.

It will be useful to define, for all $b\geq 1$, the sequence

$$H(b) = (V_7, V_7\pm 6, V_7\pm 12, \ldots, V_7\pm 6(b-1)).$$

Also, we set $H(0)$ to be the empty sequence: so, for all $b\geq 0$ the sequence $H(b)$ contains b elements and $\mathsf{supp}(H(b)) = [1, 6b]$.

Write $(\eta+1)\ell - N = 6h_0 + r_0$, where $0\leq r_0 < 6$, and define the sequence

$$S'_0 = (H(h_0), V_{r_0}\pm 6h_0).$$

Note that r_0 is odd, since ℓ is odd and $(\eta+1)\ell - N \equiv (\eta+1)\ell+\eta \equiv 1 \pmod 2$. Furthermore, $\mathsf{supp}(S'_0\pm N) = [N+1, N+6h_0+7]\setminus\{(\eta+1)\ell\}$.

Now, for all $j\in[1, \lfloor t/2\rfloor - \eta]$, write $\ell - 7 + r_{j-1} = 6h_j + r_j$, where $0\leq r_j < 6$, and define the sequence

$$S'_j = \left(H(h_j)\pm\left(7j + 6\sum_{i=0}^{j-1} h_i\right), V_{r_j}\pm\left(7j+6\sum_{i=0}^{j} h_i\right)\right).$$

Note that $(\eta+j+1)\ell - N = 6\sum_{i=0}^{j} h_i + 7j + r_j$ and

$$\mathsf{supp}(S'_j \pm N) = \left[N+1+7j+6\sum_{i=0}^{j-1} h_i,\ N+7(j+1)+6\sum_{i=0}^{j} h_i\right] \setminus \{(\eta+j+1)\ell\}.$$

The elements of S' are the first $\frac{m}{2\lambda_1}$ blocks in $\mathop{+\!\!+}_{c=0}^{\lfloor t/2\rfloor-\eta}(S'_c \pm N)$.

Finally, writing $\tilde{\mathcal{B}} = \left(A_1,\ldots,A_{\frac{mq}{2\lambda_1}}\right)$ and $S' = \left(G_1,\ldots,G_{\frac{m}{2\lambda_1}}\right)$, for all $i=1,\ldots,\frac{m}{2\lambda_1}$, let B_i be the block of size $2\times s$ obtained by juxtaposing the q blocks

$$A_{(i-1)q+1},\ A_{(i-1)q+2},\ A_{(i-1)q+3},\ \ldots, A_{iq}$$

and the block G_i. By construction, the sequence $\mathcal{B}_1 = (B_1,\ldots,B_{\frac{m}{2\lambda_1}})$ satisfies condition (4.2), has cardinality $\frac{m}{2\lambda_1}$, $\mu(\mathcal{B}_1)=2$ and $\mathsf{supp}(\mathcal{B}_1) = \mathsf{supp}(S)\cup\mathsf{supp}(S') = \Phi$. The sequence $\mathcal{B}_2$ is obtained from $\mathcal{B}_1$ by replacing the block Z with the block Z' (case $\ell=3$). □

Lemma 4.7 *Let $\lambda=\lambda_1\lambda_2$ be as in (4.1) with $\lambda_2=2$. Let p be an odd prime dividing s and suppose that t is a divisor of $\frac{ms}{\lambda_1}$ such that $t\equiv 0 \pmod{4p}$. There exists a nice pair $(\mathcal{B},\mathcal{B})$, where $\mathcal{B}$ is a sequence of length $\frac{m}{2\lambda_1}$ consisting of blocks of size $2\times s$ such that $\mu(\mathcal{B})=2$ and $\mathsf{supp}(\mathcal{B})=\Phi$.*

PROOF: Take the following blocks:

W_4 =

1	$-(\ell+1)$	$-(2\ell+1)$	$3\ell+1$
-1	$\ell+1$	$2\ell+1$	$-(3\ell+1)$

,

W_6 =

1	-1	$-(3\ell+1)$	$-(4\ell+1)$	$2\ell+1$	$5\ell+1$
$-(\ell+1)$	$\ell+1$	$4\ell+1$	$3\ell+1$	$-(2\ell+1)$	$-(5\ell+1)$

.

Then W_4 and W_6 satisfy both properties (4.2) and (4.3) with column sums $(0,0,0,0)$ and $(-\ell,\ell,\ell,-\ell,0,0)$, respectively. Furthermore, $\mu(W_4)=\mu(W_6)=2$ and

$$\mathsf{supp}(W_4) = \{j\ell+1 : j\in[0,3]\} \quad \text{and} \quad \mathsf{supp}(W_6) = \{j\ell+1 : j\in[0,5]\}.$$

Let V be the following $2\times 2p$ block:

V =

W_6	$W_4\pm 6\ell$	$W_4\pm 10\ell$	$\cdots$	$W_4\pm(2p-4)\ell$

.

Clearly, also V satisfies both (4.2) and (4.3) and its support is $\mathsf{supp}(V) = \{j\ell+1 : j\in[0,2p-1]\}$. We can use this block V for constructing our sequence $\mathcal{B}$: the $2\times s$ blocks of $\mathcal{B}$ are obtained simply by juxtaposing $h=\frac{s}{2p}$ blocks of type $V\pm x$, for $x\in X\subset\mathbb{N}$, following the natural ordering of $(X,\leq)$. So, we are left to exhibit a suitable set X of size $\frac{mh}{2\lambda_1}$ such that the support of the corresponding sequence $\mathcal{B}$ is Φ.

Let $X_0 = [0, \ell - 2]$. Then $\mathsf{supp}(V \pm x_{i_1}) \cap \mathsf{supp}(V \pm x_{i_2}) = \emptyset$ for each $x_{i_1}, x_{i_2} \in X_0$ such that $x_{i_1} \neq x_{i_2}$. Furthermore,

$$\bigcup_{x \in X_0} \mathsf{supp}(V \pm x) = [1, 2p\ell] \setminus \{j\ell : j \in [1, 2p]\}.$$

Similarly, for any $i \in \mathbb{N}$, if $X_i = [2pi\ell, (2pi+1)\ell - 2]$ then

$$\bigcup_{x \in X_i} \mathsf{supp}(V \pm x) = [1 + 2pi\ell, 2p\ell + 2pi\ell] \setminus \{j\ell : j \in [1 + 2pi, 2p + 2pi]\}.$$

Clearly, $X_{i_1} \cap X_{i_2} = \emptyset$ if $i_1 \neq i_2$. Therefore, take $X = \bigcup_{i=0}^{\frac{t}{4p}-1} X_i$: this is a set of size $\frac{t}{4p} \cdot (\ell - 1) = \frac{t}{4p} \cdot \frac{4mph}{2\lambda_1 t} = \frac{mh}{2\lambda_1}$. It follows that the sequence $\mathcal{B}$ obtained, as previously described, from the blocks $V \pm x$, with $x \in X$, has support equal to

$$\begin{aligned}\mathsf{supp}(\mathcal{B}) &= \bigcup_{i=0}^{\frac{t}{4p}-1} \left([1 + 2pi\ell, 2p\ell + 2pi\ell] \setminus \{j\ell : j \in [1 + 2pi, 2p + 2pi]\}\right) \\ &= \left[1, \tfrac{t}{2}\ell\right] \setminus \left\{j\ell : j \in \left[1, \tfrac{t}{2}\right]\right\} = \left[1, \tfrac{ms}{2\lambda_1} + \tfrac{t}{2}\right] \setminus \left\{\ell, 2\ell, \ldots, \tfrac{t}{2}\ell\right\},\end{aligned}$$

as required. □

Example 4.8 Using the previous lemma with $m = 18$, $s = 10$, $\lambda_1 = 3$ and $t = 20$, we can choose $p = 5$ so that $t \equiv 0 \pmod{20}$. Hence $\ell = 4$ and $\mathcal{B}$ consists of the following three shiftable blocks:

$B_1 =$

1	−1	−13	−17	9	21	25	−29	−33	37
−5	5	17	13	−9	−21	−25	29	33	−37

,

$B_2 =$

2	−2	−14	−18	10	22	26	−30	−34	38
−6	6	18	14	−10	−22	−26	30	34	−38

,

$B_3 =$

3	−3	−15	−19	11	23	27	−31	−35	39
−7	7	19	15	−11	−23	−27	31	35	−39

.

Lemma 4.9 *Let $\lambda = \lambda_1\lambda_2$ be as in* (4.1) *with $\lambda_2 = 2$. Let p be an odd prime dividing s and suppose that t is a divisor of $\frac{ms}{\lambda_1 p}$ such that $t \equiv 0 \pmod 4$. There exists a nice pair $(\mathcal{B}_1, \mathcal{B}_2)$ of sequences of length $\frac{m}{2\lambda_1}$, where $\mathcal{B}_1$ and $\mathcal{B}_2$ consist of blocks of size $2 \times s$, $\mu(\mathcal{B}_1) = \mu(\mathcal{B}_2) = 2$ and $\mathsf{supp}(\mathcal{B}_1) = \mathsf{supp}(\mathcal{B}_2) = \Phi$.*

Proof: By hypothesis we can write $\ell = py + 1$. Consider the following blocks:

$W_4 =$

$y+1$	$-(2y+1)$	$-((p+1)y+2)$	$(p+2)y+2$
$-(y+1)$	$2y+1$	$(p+1)y+2$	$-((p+2)y+2)$

,

$W_6 =$

$2y+1$	$-(2y+1)$	1	$-(y+1)$	$-((p+1)y+2)$	$(p+2)y+2$
$-(py+2)$	$py+2$	-1	$y+1$	$(p+1)y+2$	$-((p+2)y+2)$

,

$W_6' =$

$2y+1$	1	$-(2y+1)$	$-(y+1)$	$-((p+1)y+2)$	$(p+2)y+2$
$-(py+2)$	-1	$py+2$	$y+1$	$(p+1)y+2$	$-((p+2)y+2)$

.

Note that the block W_4 satisfies both conditions (4.2) and (4.3), while W_6 satisfies condition (4.2) and W_6' satisfies condition (4.3). Furthermore,

$$\begin{array}{rcl}\mathsf{supp}(W_4) & = & \{(jp+1)y+j+1, (jp+2)y+j+1 : j \in [0,1]\}, \\ \mathsf{supp}(W_6) = \mathsf{supp}(W_6') & = & \{jpy+j+1, (jp+1)y+j+1, (jp+2)y+j+1 : \\ & & j \in [0,1]\}.\end{array}$$

Let V be the following $2 \times 2p$ block:

$$V = \begin{array}{|c|c|c|c|c|}\hline W_6 & W_4 \pm 2y & W_4 \pm 4y & \cdots & W_4 \pm (p-3)y \\ \hline\end{array}\,.$$

Clearly, V satisfies (4.2) and its support is

$$\begin{array}{rcl}\mathsf{supp}(V) & = & \{iy+1, (p+i)y+2 : i \in [0, p-1]\} \\ & = & \{iy+1, \ell+(iy+1) : i \in [0, p-1]\}.\end{array}$$

We can use this block V for constructing the sequence $\mathcal{B}_1$ as done in Lemma 4.7: it suffices to exhibit a suitable set X of size $\frac{mh}{2\lambda_1}$, where $h = \frac{s}{2p}$, such that the support of the corresponding sequence $\mathcal{B}_1$ is Φ.

Let $X_0 = [0, y-1]$. Then $\mathsf{supp}(V \pm x_{i_1}) \cap \mathsf{supp}(V \pm x_{i_2}) = \emptyset$ for each $x_{i_1}, x_{i_2} \in X_0$ such that $x_{i_1} \neq x_{i_2}$. Furthermore,

$$\bigcup_{x \in X_0} \mathsf{supp}(V \pm x) = [1, py] \cup [\ell+1, \ell+py] = [1, 2\ell] \setminus \{\ell, 2\ell\}.$$

Similarly, for any $i \in \mathbb{N}$, if $X_i = [2i\ell, 2i\ell+y-1]$ then

$$\bigcup_{x \in X_i} \mathsf{supp}(V \pm x) = [1+2i\ell, (2i+2)\ell] \setminus \{(2i+1)\ell, (2i+2)\ell\}.$$

Clearly, $X_{i_1} \cap X_{i_2} = \emptyset$ if $i_1 \neq i_2$. Therefore, take $X = \bigcup_{i=0}^{\frac{t}{4}-1} X_i$: this is a set of size $\frac{t}{4} \cdot y = \frac{t}{4} \cdot \frac{\ell-1}{p} = \frac{t}{4} \cdot \frac{2mh}{\lambda_1 t} = \frac{mh}{2\lambda_1}$. It follows that the sequence $\mathcal{B}_1$ obtained from the blocks $V \pm x$, with $x \in X$, has support equal to

$$\begin{array}{rcl}\mathsf{supp}(\mathcal{B}_1) & = & \bigcup_{i=0}^{\frac{t}{4}-1} \left([1+2i\ell, 2\ell(i+1)] \setminus \{(2i+1)\ell, (2i+2)\ell\}\right) \\ & = & \left[1, \frac{t}{2}\ell\right] \setminus \left\{\ell, 2\ell, \ldots, \frac{t}{2}\ell\right\} = \Phi,\end{array}$$

as required. The sequence $\mathcal{B}_2$ is obtained by using W_6' instead of W_6. □

The last case we need is when $\lambda_2 \equiv 0 \pmod 4$.

Lemma 4.10 *Let $\lambda = \lambda_1\lambda_2$ be as in* (4.1) *with $\lambda_2 \equiv 0 \pmod 4$. There exists a nice pair $(\mathcal{B}, \mathcal{B})$, where $\mathcal{B}$ is a sequence of length $\frac{m}{2\lambda_1}$ consisting of blocks of size $2 \times s$ such that $\mu(\mathcal{B}) = \lambda_2$ and* $\mathsf{supp}(\mathcal{B}) = \Phi$.

Proof: Let Q be the $2 \times \frac{\lambda_2}{2}$ block obtained by juxtaposing $\frac{\lambda_2}{4}$ copies of the shiftable block

1	−1
−1	1

.

Clearly, Q satisfies both conditions (4.2) and (4.3). Furthermore, $\mathsf{supp}(Q) = \{1\}$ and $\mu(Q) = \lambda_2$. Take a partition of Φ into $\frac{m}{2\lambda_1}$ subsets X_i, each of cardinality $\frac{2s}{\lambda_2}$. Writing, for all $i \in \left[1, \frac{m}{2\lambda_1}\right]$, $X_i = \left\{x_{i,1}, x_{i,2}, \dots, x_{i,\frac{2s}{\lambda_2}}\right\}$, let B_i the block

$$B_i = \begin{array}{|c|c|c|c|c|}\hline Q \pm (x_{i,1} - 1) & Q \pm (x_{i,2} - 1) & Q \pm (x_{i,3} - 1) & \cdots & Q \pm \left(x_{i,\frac{2s}{\lambda_2}} - 1\right) \\ \hline \end{array}.$$

Then each B_i is a block of size $2 \times s$ such that $\mathsf{supp}(B_i) = X_i$ and $\mu(B_i) = \lambda_2$. Finally, it suffices to take the sequence $\mathcal{B} = \left(B_1, B_2, \dots, B_{\frac{m}{2\lambda_1}}\right)$. □

Example 4.11 Using the previous lemma with $m = 16$, $s = 10$, $\lambda_1 = 2$, $\lambda_2 = 4$ and $t = 5$, we have $\ell = 9$ and $\Phi = [1, 22] \setminus \{9, 18\}$. So, can take $X_1 = [1, 5]$, $X_2 = [6, 11] \setminus \{9\}$, $X_3 = [12, 16]$ and $X_4 = [17, 22] \setminus \{18\}$. Hence, the sequence $\mathcal{B}$ consists of the following four shiftable blocks:

$B_1 =$

1	−1	2	−2	3	−3	4	−4	5	−5
−1	1	−2	2	−3	3	−4	4	−5	5

,

$B_2 =$

6	−6	7	−7	8	−8	10	−10	11	−11
−6	6	−7	7	−8	8	−10	10	−11	11

,

$B_3 =$

12	−12	13	−13	14	−14	15	−15	16	−16
−12	12	−13	13	−14	14	−15	15	−16	16

,

$B_4 =$

17	−17	19	−19	20	−20	21	−21	22	−22
−17	17	−19	19	−20	20	−21	21	−22	22

.

Proposition 4.12 *Suppose that λ divides ms and write $\lambda = \lambda_1\lambda_2$ be as in (4.1). There exists a nice pair $(\mathcal{B}_1, \mathcal{B}_2)$ of sequences of length $\frac{m}{2\lambda_1}$, where $\mathcal{B}_1$ and $\mathcal{B}_2$ consist of blocks of size $2 \times s$, $\mu(\mathcal{B}_1) = \mu(\mathcal{B}_2) = \lambda_2$ and*

$$\mathsf{supp}(\mathcal{B}_1) = \mathsf{supp}(\mathcal{B}_2) = \left[1, \frac{ms}{\lambda} + \left\lfloor \frac{t}{2} \right\rfloor\right] \setminus \left\{\ell, 2\ell, \dots \left\lfloor \frac{t}{2} \right\rfloor \ell\right\} = \Phi.$$

Proof: If $\lambda_2 = \frac{s}{2}$, the statement follows from Lemma 4.4. If $\lambda_2 \neq \frac{s}{2}$ is odd, we apply Corollary 4.3. If $\lambda_2 \equiv 0 \pmod 4$, we use Lemma 4.10. So, we may assume $\lambda_2 \equiv 2 \pmod 4$. If $\lambda_2 \geq 6$, the statement follows from Lemma 4.5. Finally, suppose $\lambda_2 = 2$. Since $s \geq 6$ and $s \equiv 2 \pmod 4$, there exists an odd prime p that divides s. Now, our analysis depends on t; recall that t is a divisor of $\frac{ms}{\lambda_1}$. If t divides $\frac{ms}{2\lambda_1}$, we apply Lemma 4.6. Otherwise, we must have $t \equiv 0 \pmod 4$. If t divides $\frac{ms}{\lambda_1 p}$, the result follows from Lemma 4.9. If t does not divide $\frac{ms}{\lambda_1 p}$, then t is divisible by p. In particular, $t \equiv 0 \pmod{4p}$ and so we can apply Lemma 4.7. □

Proposition 4.13 *Suppose that λ does not divide ms. There exists a nice pair $(\mathcal{B}, \mathcal{B})$, where $\mathcal{B}$ is a sequence of length $\frac{m}{2}$ consisting of blocks of size $2 \times s$, such that* $\mathsf{supp}(\mathcal{B}) = \Phi$ *and condition* (2.1) *is satisfied.*

Proof: As previously observed, we have $\lambda \equiv 0 \pmod 8$. Let Q be the following shiftable block:

$$Q = \begin{array}{|c|c|}\hline 1 & -1 \\ \hline -1 & 1 \\ \hline \end{array}\,.$$

Clearly, Q satisfies both conditions (4.2) and (4.3). Furthermore, $\mathsf{supp}(Q) = \{1\}$ and $\mu(Q) = 4$.

Suppose that ℓ is odd or t is even. Consider the sequence X obtained by taking the natural ordering $\leq$ of $\{i-1 \mid i \in \Phi\} \subset \mathbb{N}$ and define $Y = \frac{\lambda}{4} * X$.

Suppose that ℓ is even and t is odd. Let X_1 be the sequence obtained by taking the natural ordering $\leq$ of $\{i-1 \mid i \in \Psi\} \subset \mathbb{N}$, where $\Psi = \Phi \setminus \left\{\frac{t\ell}{2}\right\}$. Also, let $Y_1 = \frac{\lambda}{4} * X_1$ and let Y_2 be the sequence obtained by repeating $\frac{\lambda}{8}$ times the integer $\frac{t\ell}{2} - 1$. Define $Y = Y_1 +\!\!+ Y_2$ and note that $|Y| = \frac{ms}{4}$.

In both cases, write $Y = (y_1, y_2, \ldots, y_{\frac{ms}{4}})$. For all $i \in \left[1, \frac{m}{2}\right]$, let B_i the block

$$B_i = \begin{array}{|c|c|c|c|}\hline Q \pm y_{1+(i-1)\frac{s}{2}} & Q \pm y_{2+(i-1)\frac{s}{2}} & \cdots & Q \pm y_{i\frac{s}{2}} \\ \hline \end{array}\,.$$

Then each B_i is a block of size $2 \times s$: it suffices to take the sequence $\mathcal{B} = (B_1, B_2, \ldots, B_{\frac{m}{2}})$. □

4.2 The subcase $k \equiv 0 \pmod 4$

Assuming $k \equiv 0 \pmod 4$, from $ms = nk$ it follows that m must be even. We now explain how to arrange the blocks of the sequences previously constructed, in order to build an integer ${}^{\lambda}\mathrm{H}_t(m, n; s, k)$. To this purpose, we define a 'base unit' that we will fill with the elements of the blocks.

Let $\mathcal{G} = (G_1, \ldots, G_d)$ be a sequence of blocks such that the following property is satisfied:

> there exist b integers $\sigma_1, \ldots, \sigma_b$ such that the elements of $\mathcal{G}$ are blocks G_r of size $2 \times 2b$ with $\gamma_{2i-1}(G_r) = -\gamma_{2i}(G_r) = \sigma_i$ for all $i \in [1, b]$. (4.4)

So, let $\mathcal{G}$ be a sequence satisfying (4.4), where the blocks $G_r = (g^{(r)}_{i,j})$ are all of size $2 \times 2b$, with $2b \leq d$. Let $P = P(\mathcal{G})$ be the pf array of size $2d \times d$ defined as follows. For all $i \in [1, b]$ and all $j \in [1, 2b]$, the cell $(i, i+j-1)$ of P is filled with the element $g^{(i)}_{1,j}$ and the cell $(d+i, i+j-1)$ is filled with the element $g^{(i)}_{2,j}$; here, the column indices are taken modulo d. The remaining cells of P are empty. An example of such construction is given in Figure 2.

We prove that P is a pf array whose columns all sum to zero. Observe that every row of P contains exactly $2b$ filled cells and every column contains exactly $4b$

$g_{1,1}^{(1)}$	$g_{1,2}^{(1)}$	$g_{1,3}^{(1)}$	$g_{1,4}^{(1)}$		
	$g_{1,1}^{(2)}$	$g_{1,2}^{(2)}$	$g_{1,3}^{(2)}$	$g_{1,4}^{(2)}$	
		$g_{1,1}^{(3)}$	$g_{1,2}^{(3)}$	$g_{1,3}^{(3)}$	$g_{1,4}^{(3)}$
$g_{1,4}^{(4)}$			$g_{1,1}^{(4)}$	$g_{1,2}^{(4)}$	$g_{1,3}^{(4)}$
$g_{1,3}^{(5)}$	$g_{1,4}^{(5)}$			$g_{1,1}^{(5)}$	$g_{1,2}^{(5)}$
$g_{1,2}^{(6)}$	$g_{1,3}^{(6)}$	$g_{1,4}^{(6)}$			$g_{1,1}^{(6)}$
$g_{2,1}^{(1)}$	$g_{2,2}^{(1)}$	$g_{2,3}^{(1)}$	$g_{2,4}^{(1)}$		
	$g_{2,1}^{(2)}$	$g_{2,2}^{(2)}$	$g_{2,3}^{(2)}$	$g_{2,4}^{(2)}$	
		$g_{2,1}^{(3)}$	$g_{2,2}^{(3)}$	$g_{2,3}^{(3)}$	$g_{2,4}^{(3)}$
$g_{2,4}^{(4)}$			$g_{2,1}^{(4)}$	$g_{2,2}^{(4)}$	$g_{2,3}^{(4)}$
$g_{2,3}^{(5)}$	$g_{2,4}^{(5)}$			$g_{2,1}^{(5)}$	$g_{2,2}^{(5)}$
$g_{2,2}^{(6)}$	$g_{2,3}^{(6)}$	$g_{2,4}^{(6)}$			$g_{2,1}^{(6)}$

Figure 2: This is a $P(G_1, \ldots, G_6)$, where $G_1, \ldots, G_6$ are arrays of size 2×4.

elements. The elements of the i-th column of P are

$$g_{1,1}^{(i)}, g_{1,2}^{(i-1)}, \ldots, g_{1,2b}^{(i+1-2b)}, \; g_{2,1}^{(i)}, g_{2,2}^{(i-1)}, \ldots, g_{2,2b}^{(i+1-2b)},$$

where the exponents must be read modulo d, with residues in $[1, d]$. Since the sequence $\mathcal{G}$ satisfies (4.4), we obtain

$$\gamma_i(P) = \sum_{j=1}^{2b} \gamma_j(G_{i+1-j}) = \sum_{j=1}^{2b} \gamma_j(G_i) = \sum_{u=1}^{b} (\sigma_u - \sigma_u) = 0.$$

Furthermore, notice that $\tau_j(P) = \tau_1(G_j)$ and $\tau_{d+j}(P) = \tau_2(G_j)$ for all $j \in [1, d]$.

Proposition 4.14 *Suppose $4 \le s \le n$, $4 \le k \le m$ and $ms = nk$. Let λ be a divisor of $2ms$ and let t be a divisor of $\frac{2ms}{\lambda}$. There exists a shiftable integer ${}^\lambda \mathrm{H}_t(m, n; s, k)$ in each of the following cases:*

(1) $s \equiv 2 \pmod 4$ *and* $k \equiv 0 \pmod 4$*;*
(2) $s \equiv 0 \pmod 4$ *and* $k \equiv 2 \pmod 4$.

Proof: (1) If λ divides ms, let $(\mathcal{B}_1, \mathcal{B}_2)$ be the nice pair of sequences constructed in Proposition 4.12 and set $\mathcal{B} = \lambda_1 * \mathcal{B}_1$. If λ does not divide ms, let $\mathcal{B}$ be the sequence constructed in Proposition 4.13. Write $d = \gcd(\frac{m}{2}, n)$ and $a = \frac{sd}{n}$. Note that a is even integer. In fact, write $m = 2\bar{m}d$ and $n = d\bar{n}$. Since $k \equiv 0 \pmod 4$, from $\frac{s}{2} \cdot \frac{m}{2} = n\frac{k}{4}$ we obtain $\bar{n}$ divides $\frac{s}{2}$.

Given a block $B_h \in \mathcal{B}$, define for every $j \in [1, \bar{n}]$ the block $T_j(B_h)$ of size $2 \times a$ consisting of the columns C_i of B_h with $i \in [a(j-1)+1, aj]$. So, the block B_h of size

1	−1		−13	−17		9	21		25	−29		−33	37	
	2	−2		−14	−18		10	22		26	−30		−34	38
−3		3	−19		−15	23		11	−31		27	39		−35
−5	5		17	13		−9	−21		−25	29		33	−37	
	−6	6		18	14		−10	−22		−26	30		34	−38
7		−7	15		19	−23		−11	31		−27	−39		35
1	−1		−13	−17		9	21		25	−29		−33	37	
	2	−2		−14	−18		10	22		26	−30		−34	38
−3		3	−19		−15	23		11	−31		27	39		−35
−5	5		17	13		−9	−21		−25	29		33	−37	
	−6	6		18	14		−10	−22		−26	30		34	−38
7		−7	15		19	−23		−11	31		−27	−39		35
1	−1		−13	−17		9	21		25	−29		−33	37	
	2	−2		−14	−18		10	22		26	−30		−34	38
−3		3	−19		−15	23		11	−31		27	39		−35
−5	5		17	13		−9	−21		−25	29		33	−37	
	−6	6		18	14		−10	−22		−26	30		34	−38
7		−7	15		19	−23		−11	31		−27	−39		35

Figure 3: An integer ${}^6\mathrm{H}_{20}(18, 15; 10, 12)$.

$2\times s$ is obtained by juxtaposing the blocks $T_1(B_h), T_2(B_h), \ldots, T_{\bar{n}}(B_h)$. Furthermore, for all $i \in [1, \bar{m}]$ and all $j \in [1, \bar{n}]$, each of the sequences

$$\left(T_j(B_{(i-1)d+1}), T_j(B_{(i-1)d+2}), \ldots, T_j(B_{id})\right),$$

of cardinality d, satisfies condition (4.4).

Let A be an empty array of size $\bar{m}\times\bar{n}$. For every $i \in [1, \bar{m}]$ and $j \in [1, \bar{n}]$, replace the cell (i, j) of A with the block $P\left(T_j(B_{(i-1)d+1}), T_j(B_{(i-1)d+2}), \ldots, T_j(B_{id})\right)$, according to the previous definition. Note that, for all $r \in [1, \frac{m}{2}]$, we have $\tau_r(A) = \tau_1(B_r) = 0$ and $\tau_{r+\frac{m}{2}}(A) = \tau_2(B_r) = 0$.

By construction, A is a pf array of size $m\times n$, $\mathsf{supp}(A) = \Phi$ and the rows and columns of A sum to zero. If λ divides ms, then every element of Φ appears, up to sign, exactly λ times. If λ does not divide ms, condition (2.1) is satisfied. Furthermore, each row contains $a\bar{n} = s$ elements and each column contains $2a\bar{m} = k$ elements. We conclude that A is a shiftable integer ${}^\lambda\mathrm{H}_t(m, n; s, k)$.
(2) This follows from (1). In fact, if $s \equiv 0 \pmod 4$ and $k \equiv 2 \pmod 4$, an integer ${}^\lambda\mathrm{H}_t(m, n; s, k)$ can be obtained simply by taking the transpose of an integer ${}^\lambda\mathrm{H}_t(n, m; k, s)$. □

The integer ${}^6\mathrm{H}_{20}(18, 15; 10, 12)$ shown in Figure 3 has been obtained by repeating $\lambda_1 = 3$ times each of the blocks of Example 4.8. In Figure 4 we give an integer ${}^8\mathrm{H}_5(16, 20; 10, 8)$, obtained by repeating $\lambda_1 = 2$ times each of the blocks of Example 4.11.

1	−1			2	−2			3	−3			4	−4			5	−5		
	6	−6			7	−7			8	−8			10	−10			11	−11	
		12	−12			13	−13			14	−14			15	−15			16	−16
−17			17	−19			19	−20			20	−21			21	−22			22
−1	1			−2	2			−3	3			−4	4			−5	5		
	−6	6			−7	7			−8	8			−10	10			−11	11	
		−12	12			−13	13			−14	14			−15	15			−16	16
17			−17	19			−19	20			−20	21			−21	22			−22
1	−1			2	−2			3	−3			4	−4			5	−5		
	6	−6			7	−7			8	−8			10	−10			11	−11	
		12	−12			13	−13			14	−14			15	−15			16	−16
−17			17	−19			19	−20			20	−21			21	−22			22
−1	1			−2	2			−3	3			−4	4			−5	5		
	−6	6			−7	7			−8	8			−10	10			−11	11	
		−12	12			−13	13			−14	14			−15	15			−16	16
17			−17	19			−19	20			−20	21			−21	22			−22

Figure 4: An integer ${}^{8}\mathrm{H}_5(16, 20; 10, 8)$.

4.3 The subcase $k \equiv 2 \pmod 4$

Here we only solve the case m even, which implies that also n is even.

Proposition 4.15 *Suppose $6 \leq s \leq n$, $6 \leq k \leq m$, $ms = nk$ and $s, k \equiv 2 \pmod 4$. Let λ be a divisor of $2ms$ and let t be a divisor of $\frac{2ms}{\lambda}$. If m is even, there exists a shiftable integer ${}^\lambda\mathrm{H}_t(m, n; s, k)$.*

PROOF: Without loss of generality, we may assume $m \geq n$ (and so $s \leq k$). If λ divides ms, let $(\mathcal{B}_1, \mathcal{B}_2)$ be the nice pair of sequences constructed in Proposition 4.12. Take $\mathcal{B}_1^* = \lambda_1 * \mathcal{B}_1$ and $\mathcal{B}_2^* = \lambda_1 * \mathcal{B}_2$. So, $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ have length $\frac{m}{2}$ and $\mu(\mathcal{B}_1^*) = \mu(\mathcal{B}_2^*) = \lambda$. If λ does not divide ms, let $(\mathcal{B}_1^*, \mathcal{B}_2^*)$ be the nice pair of sequences constructed in Proposition 4.13. In both cases, write $\mathcal{B}_1^* = (B_1, \ldots, B_{\frac{m}{2}})$ and $\mathcal{B}_2^* = (B'_1, \ldots, B'_{\frac{m}{2}})$, where $\mathcal{B}_1^*$ satisfies (4.2), $\mathcal{B}_2^*$ satisfies (4.3) and

$$\mathsf{supp}(\mathcal{B}_1^*) = \mathsf{supp}(\mathcal{B}_2^*) = \left[1, \left\lfloor \frac{t\ell}{2} \right\rfloor\right] \setminus \{j\ell : j \in [1, \lfloor t/2 \rfloor]\} \qquad \text{with } \ell = \frac{2ms}{\lambda t} + 1.$$

Set

$$\widetilde{\mathcal{B}}_1 = \left(B_{\frac{n}{2}+1}, \ldots, B_{\frac{m}{2}}\right) \quad \text{and} \quad \widetilde{\mathcal{B}}_2 = \left(B'_1, \ldots, B'_{\frac{n}{2}}\right).$$

Since, by construction, $\mathcal{E}(B_i) = \mathcal{E}(B'_i)$ for all $i \in [1, \frac{m}{2}]$, it follows that $\mathcal{E}(\widetilde{\mathcal{B}}_2 +\!\!+ \widetilde{\mathcal{B}}_1) = \mathcal{E}(\mathcal{B}_1^*) = \mathcal{E}(\mathcal{B}_2^*)$ and $\mathsf{supp}(\widetilde{\mathcal{B}}_2 +\!\!+ \widetilde{\mathcal{B}}_1) = \left[1, \lfloor \frac{t\ell}{2} \rfloor\right] \setminus \{j\ell : j \in [1, \lfloor t/2 \rfloor]\}$. Furthermore, if λ divides ms then $\mu(\widetilde{\mathcal{B}}_2 +\!\!+ \widetilde{\mathcal{B}}_1) = \lambda$; the same holds if λ does not divide ms, and ℓ is odd or t is even; if λ does not divide ms, ℓ is even and t is odd, then every element of $\Phi \setminus \left\{\frac{t\ell}{2}\right\}$ appears in $\mathcal{E}(\widetilde{\mathcal{B}}_2 +\!\!+ \widetilde{\mathcal{B}}_1)$, up to sign, exactly λ times, while the integer $\frac{t\ell}{2}$ appears, up to sign, $\frac{\lambda}{2}$ times.

Using the blocks of the sequence $\widetilde{\mathcal{B}}_2$, we first construct a square shiftable pf array A_1 of size n such that each row and each column contains s filled cells and such that the elements in every row and column sum to zero. Hence, take an empty array A_1 of size $n \times n$ and arrange the $\frac{n}{2}$ blocks $B'_r = (b_{i,j}^{(r)})$ of $\widetilde{\mathcal{B}}_2$ in such a way that the element $b_{1,1}^{(r)}$ fills the cell $(2r-1, 2r-1)$ of A_1. This process makes A_1 a pf array with s filled cells in each row and in each column. Since the rows of the blocks B'_r sum to zero, also the rows of A_1 sum to zero. Looking at the columns, the s elements of a column of A_1 are

$$b_{1,s}^{(r)}, b_{2,s}^{(r)},\ b_{1,s-2}^{(r+1)}, b_{2,s-2}^{(r+1)},\ b_{1,s-4}^{(r+2)}, b_{2,s-4}^{(r+2)},\ \ldots,\ b_{1,2}^{(r+s/2)}, b_{2,2}^{(r+s/2)}$$

or

$$b_{1,s-1}^{(r)}, b_{2,s-1}^{(r)},\ b_{1,s-3}^{(r+1)}, b_{2,s-3}^{(r+1)},\ b_{1,s-5}^{(r+2)}, b_{2,s-5}^{(r+2)},\ \ldots,\ b_{1,1}^{(r+s/2)}, b_{2,1}^{(r+s/2)},$$

where the exponents $r, \ldots, r+s/2$ must be read modulo $\frac{n}{2}$. Since $\widetilde{\mathcal{B}}_2$ satisfies condition (4.3), the sum of these elements is

$$\sum_{j=1}^{s/2} \sigma_{2j} = 0 \quad \text{or} \quad \sum_{j=1}^{s/2} \sigma_{2j-1} = 0, \qquad \text{respectively.}$$

By construction, $\mathcal{E}(A_1) = \mathcal{E}(\widetilde{\mathcal{B}}_2)$.

Now, if $m = n$, then A_1 is actually a shiftable integer ${}^\lambda\mathrm{H}_t(m, n; k, s)$. Suppose that $m > n$. If we arrange the blocks of the sequence $\widetilde{\mathcal{B}}_1$ mimicking what we did for the construction of an integer ${}^1\mathrm{H}_1(m-n, n; s, k-s)$ in the proof of Proposition 4.14, we obtain a shiftable pf array A_2 of size $(m-n) \times n$ such that $\mathcal{E}(A_2) = \mathcal{E}(\widetilde{\mathcal{B}}_1)$, rows and columns sum to zero, each row contains s filled cells and each column contains $k-s$ filled cells. Let A be the pf array of size $m \times n$ obtained by taking

$$A = \begin{array}{|c|}\hline A_1 \\ \hline A_2 \\ \hline \end{array}.$$

Each row of A contains s filled cells and each of its columns contains $s + (k-s) = k$ filled cells. By the previous properties of $\widetilde{\mathcal{B}}_2 + \widetilde{\mathcal{B}}_1$, it follows that A is a shiftable integer ${}^\lambda\mathrm{H}_t(m, n; s, k)$. □

An integer ${}^{28}\mathrm{H}_4(16, 16; 14, 14)$ is shown in Figure 5, choosing $\lambda_1 = 2$ and $\lambda_2 = 14$. In Figure 6 we give an integer ${}^{10}\mathrm{H}_3(20, 12; 6, 10)$, where $\lambda_1 = 5$ and $\lambda_2 = 2$.

1	2	−1	1	−1	−2	1	−1	2	−2	1	−1	2	−2		
−2	−1	2	−2	1	2	−1	1	−2	2	−1	1	−2	2		
		3	4	−3	3	−3	−4	3	−3	4	−4	3	−3	4	−4
		−4	−3	4	−4	3	4	−3	3	−4	4	−3	3	−4	4
7	−7			6	7	−6	6	−6	−7	6	−6	7	−7	6	−6
−7	7			−7	−6	7	−7	6	7	−6	6	−7	7	−6	6
8	−8	9	−9			8	9	−8	8	−8	−9	8	−8	9	−9
−8	8	−9	9			−9	−8	9	−9	8	9	−8	8	−9	9
2	−2	1	−1	2	−2			1	2	−1	1	−1	−2	1	−1
−2	2	−1	1	−2	2			−2	−1	2	−2	1	2	−1	1
3	−3	4	−4	3	−3	4	−4			3	4	−3	3	−3	−4
−3	3	−4	4	−3	3	−4	4			−4	−3	4	−4	3	4
−6	−7	6	−6	7	−7	6	−6	7	−7			6	7	−6	6
6	7	−6	6	−7	7	−6	6	−7	7			−7	−6	7	−7
−8	8	−8	−9	8	−8	9	−9	8	−8	9	−9			8	9
9	−9	8	9	−8	8	−9	9	−8	8	−9	9			−9	−8

Figure 5: An integer ${}^{28}\mathrm{H}_4(16, 16; 14, 14)$.

5 Conclusion

Thanks to the constructions of Sections 3 and 4, we can prove Theorem 1.10. In fact, case (1) follows from Proposition 3.8; cases (2) and (3) follow from Proposition 4.14; case (4) follows from Proposition 4.15. Unfortunately, we are not able to solve the existence of an integer ${}^\lambda\mathrm{H}_t(m, n; s, k)$ when $s, k \equiv 2 \pmod 4$ and m, n are odd. However, we can prove the existence of an SMA$(m, n; s, k)$ for this choice of m, n, s, k.

1	−1	−4	−5	3	6						
−2	2	5	4	−3	−6						
		7	−7	−11	−12	10	13				
		−8	8	12	11	−10	−13				
				1	−1	−4	−5	3	6		
				−2	2	5	4	−3	−6		
						7	−7	−11	−12	10	13
						−8	8	12	11	−10	−13
3	6							1	−1	−4	−5
−3	−6							−2	2	5	4
−11	−12	10	13							7	−7
12	11	−10	−13							−8	8
1	−1			−4	−5			3	6		
	7	−7			−11	−12			10	13	
		1	−1			−4	−5			3	6
−7			7	−12			−11	13			10
−2	2			5	4			−3	−6		
	−8	8			12	11			−10	−13	
		−2	2			5	4			−3	−6
8			−8	11			12	−13			−10

Figure 6: An integer ${}^{10}\mathrm{H}_3(20,12;6,10)$.

Proof of Theorem 1.6: If $s,k \equiv 0 \pmod 4$, the integer ${}^{2}\mathrm{H}_1(m,n;s,k)$ we construct in Lemma 3.3 is actually a (shiftable) $\mathrm{SMA}(m,n;s,k)$. Similarly, if $s \equiv 2 \pmod 4$ and m is even, the integer ${}^{2}\mathrm{H}_1(m,n;s,k)$ constructed in Propositions 4.14 and 4.15 are (shiftable) signed magic arrays. So, we are left to consider the case $s,k \equiv 2 \pmod 4$ with m,n odd.

Without loss of generality, we may assume $m \geq n$ (and so $s \leq k$). Let A_1 be an $\mathrm{SMA}(n,n;s,s)$, whose existence is assured by Theorem 1.2. Clearly if $m = n$ we have nothing to prove. So, suppose $m > n$. Since $m-n \geq 2$ is even and $k-s \equiv 0 \pmod 4$ with $k-s \geq 4$, by Proposition 4.14 there exists a shiftable $\mathrm{SMA}(m-n,n;s,k-s)$, say A_2. Let A be the pf array of size $m \times n$ obtained by taking

$$A = \begin{array}{|c|}\hline A_1 \\ \hline A_2 \pm ns/2 \\ \hline \end{array}.$$

Each row of A contains s filled cells and each of its columns contains $s + (k - s) = k$ filled cells. Also, note that $\mathcal{E}(A_1) = \{\pm 1, \pm 2, \ldots, \pm ns/2\}$ and $\mathcal{E}(A_2 \pm sn/2) = \{\pm(1+ns/2), \pm(2+ns/2), \ldots, \pm ms/2\}$. Since $\mathcal{E}(A) = \mathcal{E}(A_1) \cup \mathcal{E}(A_2) = \{\pm 1, \pm 2, \ldots, \pm ms/2\}$, A is an $\mathrm{SMA}(m,n;s,k)$. □

We can now prove the existence of magic rectangles.

Proof of Theorem 1.12: Let A be a shiftable $\mathrm{SMA}(m,n;s,k)$, whose existence was proved in Theorem 1.6, and let A^* be the pf array obtained by replacing every negative entry x of A with $x + \frac{ms}{2}$ and by replacing every positive entry y with

$y + \frac{ms}{2} - 1$. Since $\mathcal{E}(A) = \left\{-1, -2, \ldots, -\frac{ms}{2}\right\} \cup \left\{1, 2, \ldots, \frac{ms}{2}\right\}$, we obtain $\mathcal{E}(A^*) = \left\{0, 1, \ldots, \frac{ms}{2} - 1\right\} \cup \left\{\frac{ms}{2}, \frac{ms}{2} + 1, \ldots, ms - 1\right\}$. This means that every element of $[0, ms-1]$ appears just once in A^*. Obviously, every row of A^* contains exactly s filled cells and every column of A^* contains exactly k filled cells. Now, since A is shiftable, every row of A contains $\frac{s}{2}$ negative entries and $\frac{s}{2}$ positive entries. So, the elements of every row of A^* sum to $\frac{s}{2}\left(\frac{ms}{2} + \frac{ms}{2} - 1\right) = \frac{s(ms-1)}{2}$. Analogously, the elements of every column of A^* sum to $\frac{k(ms-1)}{2}$. We conclude that A^* is an $\mathrm{MR}(m, n; s, k)$. □

Example 5.1 Take the shiftable SMA$(5, 10; 8, 4)$ of Figure 1, whose construction is given in Lemma 3.3. Proceeding as described in the proof of Theorem 1.12, we obtain the following MR$(5, 10; 8, 4)$:

$A^* =$

20	18		13	27	30	8		3	37
39	22	16		11	29	32	6		1
19	21	24	14		9	31	34	4	
	17	23	26	12		7	33	36	2
0		15	25	28	10		5	35	38

.

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