Spiralling and Folding: The Topological View

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Abstract

For every n, we construct two arcs in the four-punctured sphere that have at least n intersections and which do not form spirals. This is accomplished in several steps: we first exhibit closed curves on the torus that do not form double spirals, then arcs on the four-punctured torus that do not form spirals, and finally arcs in the four-punctured sphere which do not form spirals.

1 Introduction

In a surface, draw a pair of curves with a large number of intersections. Must the drawing contain any particular sub-structures? And if so, can these structures be used to simplify the drawing? Affirmative answers to these questions can be used to bound the complexity of certain drawings, such as those yielding string graphs. A *string graph* is the intersection graph of curves in the plane or other surface. Recognition for string graphs is an old problem [1, 2] that was recently solved in the planar case [6, 10] by bounding the complexity of a minimal drawing realizing the string graph.

Which sub-structures might we expect to see in a drawing of two curves with a large number of intersections? Starting with any drawing, one can increase the number of intersections, by taking a section of one curve and dragging it over the other curve. This creates a *bigon*, see Figure 1. Of course, there is an obvious simplification. Moreover, it is not hard to create a drawing with many intersections and no bigons.

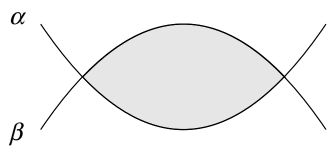

Figure 1: α and β form a bigon.

Another candidate structure is a *fold*, pictured in Figure 2. Kratochvíl and Matoušek [5] use folds to create string graphs whose drawings require $\Omega(2^n)$ intersections. This figure also indicates a bigon, but it can easily be eliminated by puncturing the surface or adding genus in the middle of the bigon. So while a fold without a bigon does not yield an obvious simplification, it does indicate topological complexity of the surface.

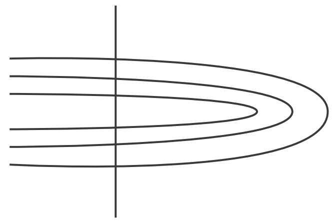

Figure 2: A fold. The vertical arc is a section of the curve α all others are portions of the curve β.

The focus here is a third structure, the *spiral*, pictured in Figure 3. Two spanning arcs, β and γ, of an annulus A *form a* d*-spiral in* A if they have minimal intersection (no bigons) in A and they intersect $d+2$ times in A. If $d \geq 1$, we say that β *and* γ *form a spiral in* A. Similarly, we will say that two curves, $\beta \subset F$ and $\gamma \subset F$, form a d*-spiral in* F if there is an embedded annulus $A \subset F$ so that sub-arcs of β and γ form a d-spiral in A. Again, if $d \geq 1$, we say that β *and* γ *form a spiral in* F.

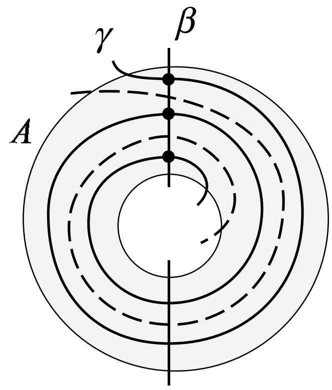

Figure 3: β and γ form a 1-spiral.

Pach and Tóth [6] showed that spirals can be used to simplify certain string drawings. In this and a companion paper [9], we exhibit drawings with an arbitrary number of intersections and no spirals. The companion paper shows that certain drawings in the torus are

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spiral-free by analyzing intersection words. Here we use topological techniques to construct spiral-free drawings, both in the torus and in punctured spheres. All the drawings are bigon free and the torus drawings are also fold-free. Thus, eliminating spirals is not sufficient to bound the complexity of drawings in surfaces of higher genus. This result casts doubt on the hope that a complex drawing must necessarily contain a complex sub-drawing in a simple subsurface. In particular, one cannot expect to find a complex sub-drawing in a disk (a bigon) or an annulus (a spiral).

2 Arcs and Spirals

We will assume that curves in a surface are *properly embedded*, that is, they do not intersect themselves and arc-components have their endpoints in the boundary of the surface. Any endpoint not in the boundary can be easily fixed by puncturing the surface at the endpoint.

Disjoint spanning arcs will have very similar behavior in an annulus A. This can be seen in the following lemma, which gives bounds on $i(\cdot,\cdot)$, the number of intersections between a pair of curves.

Lemma 1 *Suppose that β, β', γ and γ' are spanning arcs for an annulus A, all pairs have reduced intersection, and $i(\beta,\beta') = i(\gamma,\gamma') = 0$. Then $|i(\beta,\gamma) - i(\beta',\gamma')| \leq 2$.*

We also show that parallel disjoint spirals imply the existence of deeper spirals:

Lemma 2 *Suppose that β and γ form a d-spiral in F, β' and γ' form a d'-spiral in F, $i(\beta,\beta') = i(\gamma,\gamma') = 0$, and that all pairs have reduced intersection in F. If the defining annuli A and A' are disjoint in F but have boundary curves that are isotopic in F then β and γ form a $(d+d')$-spiral in F.*

3 Fibonacci curves: 2-spiral free curves on the torus

For our first example, we construct pairs of curves on the torus that while containing spirals, do not contain deep spirals.

Fortunately, it is easy to represent closed curves on the torus, see for example [8]. First, one chooses a (non-unique) basis, any pair of curves, μ and λ, such that $i(\mu,\lambda) = 1$. We can think of the longitude, λ, as the curve that goes the "long" way around the torus once and the meridian, μ, as a curve that goes once around the "short" way. (This is a highly prejudicial view, but one that won't lead us astray). Then any closed multi-curve on the torus, γ, can be represented by an ordered pair of integers, p measuring the number of times the curve wraps longitudinally and q the number of times the curve wraps meridionally, i.e., $\gamma = p\lambda + q\mu = (p,q)$.

The curve indicated by (F_n, F_{n-1}), a pair of sequential Fibonacci numbers, does not form a deep spiral with the basis curve $\mu = (0,1)$:

Theorem 3 *The curves $\gamma_n = (F_n, F_{n-1}), n \geq 1$ and $\mu = (0,1)$ do not form a d-spiral on the torus for $d \geq 2$.*

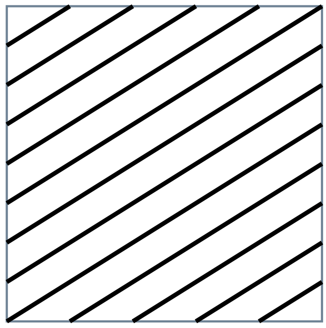

Figure 4: The curve $(F_n, F_{n-1}) = (8,5)$ on the torus.

4 Train Tracks

Train tracks give us a shorthand for drawing complicated multi-curves on a surface. They are very useful in studying diffeomorphisms of surfaces [7] and were used by Hass, Snoeyink and Thurston to construct unknots with large spanning disks [3]. A *train track* is a graph T embedded in the surface F so that each vertex of T has a well-defined tangent and degree either one or three. (See the figures in later sections.) The vertices of degree one are called *terminals*, the vertices of degree three are called *switches*, and the edges are called *branches.* Because of the well-defined tangent, there are two branches coming into one side of each switch and one branch on the other.

We specify an embedded multi-curve in an arbitrarily small neighborhood of the track T by adding a *weight* w_i, a non-negative integer, to each branch. The *weight vector* $\overrightarrow{w}$ will denote the collection of weights for each branch. The weighted track $\{T, \overrightarrow{w}\}$, determines a unique multi-curve in the neighborhood of the track, where the weights indicate the number of times the multi-curve travels along each branch. In order that the multi-curve respect the tangency conditions at the switches, the set of weights $\{w_i\}$ is required to satisfy a *matching equation* at each switch. That is, the sum of the weights of the branches entering the switch from one side is equal to the weight of the branch on the other. For example, in the track in Figure 5, the switches yield two matching equations, each being $p = q + r$. We will also adopt the convention that a branch meeting a terminal has weight one.

We say that T *carries* a multi-curve $\beta \subset F$, if there is a set of weights on T satisfying the matching equations that induces β.

We will say that a multi-curve carried by a train track contains a *d-spiral* if some component of the multi-curve forms a d-spiral in F with a *basis arc* for a branch of T, that is, an arc embedded in F that meets T transversely in a single point contained in that branch.

5 Spiral-free on the torus

Figure 5 indicates a train track on a torus. This train track carries all closed multi-curves on the torus whose coordinates (p, q) satisfy $p \geq 0, q \geq 0$.

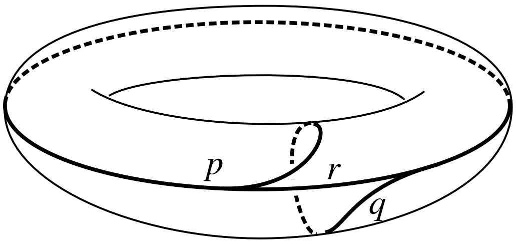

Figure 5: The train track T_0 on a torus.

It follows from Theorem 3 that we can weight track T_0 so that it does not contain deep spirals.

Corollary 4 *The weighted train track* $\{T_0|p = F_n, q = F_{n-1}, r = F_{n-2}\}$ *does not contain a d-spiral for* $d \geq 2$.

We now consider the train track T_1 pictured in Figure 6(a). We identify opposite edges of the square so this train track is embedded in the torus. In this section we sketch a proof that that $\{T_1|p = P = F_n - 1, q = Q = F_{n-1} - 1, r = R = F_{n-2} - 1,$ terminals have weight $1\}$ is spiral-free.

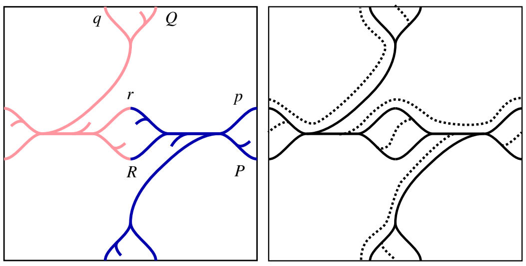

Figure 6: a) The train track T_1. The weighted track (T_1, $p = P = F_n - 1, q = Q = F_{n-1} - 1, r = R = F_{n-2} - 1$) has no 1-spirals, b) embedding curves from T_1 in T_0,

The most interesting property of this track is that it admits a map $\pi : T_1 \to T_1$, defined by rotating the square about its center through the angle π. Moreover, π is an involution, that is, it is its own inverse. The involution π is orientation-preserving and exchanges branches with lower-case labels with branches with upper-case labels. The only fixed points of the involution π as it acts on the torus are: the center of the square, the corners of the square (all identified to a single point), and the midpoint of each edge (opposite edges identified). Since the track T_1 avoids all of these points, $\pi : T_1 \to T_1$ is fixed-point free. We have:

Lemma 5 *$\pi : T_1 \to T_1$ has no fixed points.*

Lemma 6 *π does not preserve any closed connected curve γ carried by T_1.*

There is also a natural embedding $\epsilon : T_1 \to T_0$ of any curves carried by T_1 into the track T_0. This is accomplished by connecting the terminals and then collapsing nearby parallel branches into a single branch, see Figure 6. For closed curves ϵ can be thought of as a map that converts to lower-case. If γ is a closed curve carried by T_1, then $\epsilon(\gamma)$ will be carried by T_0, where its weights are the sum of the upper- and lower-case weights of T_1: i.e., $p' = p + P, q' = q + Q, r' = r + R$. If γ contains arcs, and the weight of each terminal is 1, then $\epsilon(\gamma)$ consists of sub-arcs of a closed multi-curve carried by T_0. In this case, the weights of the closed multi-curve carried by T_0 will be $p' = p + P + 2, q' = q + Q + 2, r' = r + R + 2$.

The involution π will preserve a multi-curve carried by T_1 if and only if the curve's weights are preserved by π, that is $p = P, q = Q, r = R$, and terminals of equal weight are exchanged.

Lemma 7 *Let γ be a closed curve carried by T_1. Then ϵ maps γ and $\pi(\gamma)$ to isotopic curves in T_0.*

Proposition 8 *Suppose that the weighted train track $\{T_1|\overrightarrow{w}\}$ is invariant under π and contains a spiral. Then $\{T_1|\overrightarrow{w}\}$ contains two disjoint spirals.*

Note that the weighted track $\{T_1|p = P = F_n - 1, q = Q = F_{n-1} - 1, r = R = F_{n-2} - 1,$ terminals have weight $1\}$ is invariant under π. If it contains a 1-spiral, then Proposition 8 implies that we can actually find two disjoint spirals. By Lemma 7, ϵ maps these to disjoint parallel spirals in $\{T_0|p' = 2F_n, q = 2F_{n-1}, r = 2F_{n-2}\}$. But then, Lemma 2 tells us that this weighted track has a 2-spiral, in direct contradiction to Theorem 3. We have:

Theorem 9 *The weighted track* $\{T_1|p = P = F_n - 1, q = Q = F_{n-1} - 1, r = R = F_{n-2} - 1,$ *terminals have weight* $1\}$ *does not contain a d-spiral for* $d \geq 1$.

The multi-curve in T_1 specified by these weights consists of four arcs. Let α be the one with the highest p weight (its weight on the track labeled p). Then α and β, a basis arc for the p branch, are both properly embedded in a small neighborhood of the track T_1, a surface homeomorphic to the 4-punctured sphere. Moreover, they have at least $\frac{F_n - 1}{4}$ intersections without forming a spiral, fold or bigon.

Corollary 10 *For every n, there are two arcs in the 4-punctured torus which have at least n intersections and which do not form a bigon, fold, or spiral.*

6 Spiral-free curves in the plane

We are now ready to prove our ultimate goal, that the planar weighted track pictured below is spiral-free.

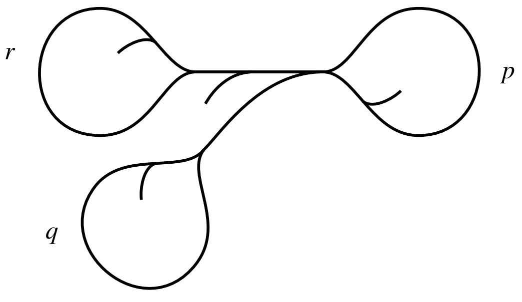

Figure 7: The weighted track $\{T_2 | p = F_n - 1, q = F_{n-1} - 1, r = F_{n-2} - 1, \text{terminals have weight } 1\}$ is spiral-free.

There is a strong similarity between the (weighted) tracks T_1 and T_2. This similarity is induced by π: We can view T_1 as formed by cutting each of the loops of T_2, taking two copies, and then gluing them together. Or alternatively, the track T_2 can be seen as a copy of T_1 where each point has been identified with its image under π: $x = \pi(x)$. There is a natural projection $\rho : T_1 \to T_2$ which sends each point in T_1 to its corresponding identified point: $\rho(x) = \rho(\pi(x))$. In fact, ρ is a *local homeomorphism*, and restricts to a homeomorphism of any neighborhood $U \subset T_1$ that is disjoint from its image $\pi(U) \subset T_1$. Together, the space T_1 and map ρ are referred to as a *covering space* of T_2. See [4]. Covering spaces have very strong properties. Here we will use only the fact that any simply-connected region $U \subset T_2$ *lifts to* T_1, that is, the inverse image $\rho^{-1}(U)$ consists of n disjoint regions $\{U_1, U_2, ..., U_n\} \subset T_1$, each homeomorphic to U. In our case, T_1 is a double-cover of T_2, i.e. $n = 2$. Points, arcs, and disks are all simply connected, so each will lift to two disjoint identical copies in T_1. While a spiral is not simply-connected, it is a consequence of Lemma 6 that it will also lift to two disjoint spirals.

Because the lower-case and upper-case weights are equal, ρ can also be regarded as a covering map between the weighted tracks $\rho : \{T_1 | p = P = F_n - 1, q = Q = F_{n-1} - 1, r = R = F_{n-2} - 1$, terminals have weight $1\ \} \to \{T_2 | p = F_n - 1, q = F_{n-1} - 1, r = F_{n-2} - 1$, terminals have weight $1\}$, carrying induced curves to induced curves. Since spirals lift, Theorem 9 implies that the weights on T_2 do not determine a spiral.

Theorem 11 *The weighted train track $\{T_2 | p = F_n - 1, q = F_{n-1} - 1, r = F_{n-2} - 1$, terminals have weight $1\}$ does not contain a spiral.*

These weights specify two arcs (and no closed curves) in T_2. The arc with the largest p weight and a basis arc for the p branch are properly embedded in a regular neighborhood of the track T_2, have at least $\frac{F_n - 1}{2}$ intersections, and do not form a bigon or spiral. Since the neighborhood of T_2 is homeomorphic to a 4-punctured sphere, we have:

Corollary 12 *For every n, there are two arcs in the 4-punctured sphere which have at least n intersections and which do not form a bigon or spiral.*

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