A new algorithm and improved lower bound for point placement on a line in two rounds

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Abstract

In this paper we show how to construct in 2 rounds a line-rigid point placement graph of size $4n/3+O(1)$ from small graphs called 6:6 jewels, an extension of the 4:4 jewel of [3]. This improves a result reported in [2] that uses 5-cycles. More significantly, we improve their lower bound on 2-round algorithms from $17n/16$ to $14n/13$.

1 Introduction

Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of n distinct points on a line L. In this paper, we consider the problem of learning P (up to translation and reflection) by making distance queries between pairs of points.

For starters, here's a simple algorithm [3]. Query the distance between two points, say p_1 and p_2. The position of each of the remaining points $p_i \geq 3$ is determined by querying the distances from p_i to p_1 and p_2; p_i lies between p_1 and p_2 if the sum of the distances is equal to $|p_1p_2|$, and to the left of p_1 or to the right of p_2 if the difference of the distances is equal to $|p_1p_2|$. The number of queries made is $2n-3$, which is of the form $\alpha n + \beta$.

We can represent the $2n-3$ queries above in the form of a point placement graph (*ppg*) on P, with an edge between p_i and p_j if we have queried the distance between them. As there is a unique placement on L of the vertices of this graph we call it line-rigid. Structurally, the *ppg* is made up of $n-2$ triangles, with a common edge. Each individual triangle is line-rigid as long as the length of one side is the sum or difference of the lengths of the other two.

This provides the cue to more efficient algorithms - find larger graphs than triangles which are either intrinsically line-rigid (such as the triangle or the jewel of Damaschke [3]) or are line-rigid under some constraints on their edge lengths (thus a quadrilateral is line-rigid as long as it is not a parallelogram [3]). We glue together line-rigid quadrilaterals into a line-rigid *ppg*. Using quadrilaterals as the basic line-rigid elements, brings down α to $3/2$ as we query 3 edges per 2 points. To meet the line-rigidity condition for quadrilaterals, we must choose point-pairs carefully. Here is a simple 2-round algorithm due to Damaschke [3].

In round one we query all the edges incident at the end points of the edge xy (Fig. 1). In round two we form line-rigid quadrilaterals by querying edges joining pairs of points p and q that makes the quadrilateral $xpqy$ line-rigid. We can make sure that $|xp| \neq |yq|$ by having 2 extra edges at y, in view of the following basic observation:

Observation. At a point p on a line there can be at most two edges incident that have the same length.

We complete this round by making 2 triangles with x and each of the 2 residual edges.

The motivation for this problem comes from diverse areas - biology, learning theory, computational geometry. Early research on this problem was reported in [4]. It was shown in [3] that the 4:4 jewel (see next section for a definition) and $K_{2,3}$ are both line-rigid, and also how to build large rigid graphs of density 8/5 (this is an asymptotic measure of the number of edges per point as the number of points go to infinity) using the jewel as a basic line-rigid component. Chin et al [2] improved many of the results of [3], their principal contribution being the 3-round construction of rigid graphs of density 5/4 from 6-cycles and a lower bound of $17n/16$ on α in any 2-round algorithm.

In [1] we proposed a 2-round algorithm that queries $10n/7 + O(1)$ edges to construct a line-rigid *ppg* on n points, using a 5:5 jewel as the basic component. In this paper we propose a 2-round algorithm that queries $4n/3 + O(1)$ edges to construct a line-rigid *ppg* on n points, using a 6:6 jewel as the basic component, bettering a result of [2] that uses 5-cycles. More significantly, we improve their lower bound on any 2-round algorithm from $17n/16$ to $14n/13$.

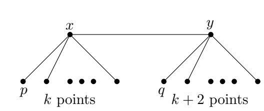

Figure 1: Creating a ppg with quadrilaterals.

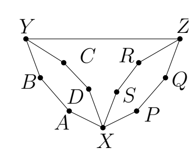

Figure 2: 6:6 jewel.

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2 A two-round algorithm

A generic $m : n$ jewel consists of an m-cycle, C_1, and an n-cycle, C_2, that are hinged at a common vertex with a strut between two other vertices (Fig. 2 shows a 6:6 jewel). This is a generalization of Damaschke's 4:4-jewel. We use 6:6 jewels to build a line-rigid *ppg*.

First we determine a set of sufficient conditions that make them line-rigid by drawing it as a *layer graph* [2]. A layer graph has (i) all its edges parallel to one of the two orthogonal directions, x and y; (ii) the length of an edge is equal to its weight; (iii) not all edges are along the same direction (implying a two-dimensional extent); (iv) when collapsed (i.e., rigidly folded onto a line) along either side of either axis, no two vertices coincide.

The importance of a layer-graph drawing lies in the fact proved by Chin *et al.* [2] that a *ppg* is line-rigid iff it cannot be drawn as a layer graph.

Since by making the individual 6-cycles C_1 and C_2 line-rigid we reduce the 6:6 jewel to a triangle that cannot be drawn as a layer graph, the union of the conditions that make these cycles individually line-rigid are sufficient to make the 6:6 jewel line-rigid.

A 6-cycle can be drawn as a layer graph in 16 different configurations giving rise to 16 different rigidity conditions. For the 6-cycle $XABYCD$ the conditions are: $|YB| \neq |XA|$, $|XD| \neq |AB|$, $|XA| \neq |CD|$, $|YC| \neq |XD|$, $|YB| \neq |CD|$, $|YC| \neq |AB|$, $|YB| \neq |XD|$, $|YC| \neq |XA|$, $|AB| \neq |CD|$, $|YB| \neq |XA| \pm |XD|$, $|YB| \neq |YC| \pm |XA|$, $|XA| \neq |YC| \pm |CD|$, $|YB| \neq |YC| \pm |XD|$, $|YC| \neq |XA| \pm |XD|$, $|YB| \neq |XD| \pm |CD|$ and $|XA| \neq |YB| \pm |CD|$.

For the 6-cycle $XPQZRS$ they are: $|ZQ| \neq |XP|$, $|XS| \neq |PQ|$, $|XP| \neq |SR|$, $|ZR| \neq |XS|$, $|ZQ| \neq |RS|$, $|ZR| \neq |PQ|$, $|ZQ| \neq |XS|$, $|ZR| \neq |XP|$, $|PQ| \neq |RS|$, $|ZQ| \neq |XP| \pm |XS|$, $|ZQ| \neq |ZR| \pm |XP|$, $|XP| \neq |ZR| \pm |RS|$, $|ZQ| \neq |ZR| \pm |XS|$, $|ZR| \neq |XP| \pm |XS|$, $|ZQ| \neq |XS| \pm |RS|$ and $|XP| \neq |ZQ| \pm |RS|$.

Hence a total of 32 conditions that make a 6:6 jewel line-rigid. Several of the above conditions involve the edges AB, CD, PQ and RS. These come in the way of the line-rigid *ppg* we want to build. So we reformulate the 14 conditions involving these distances with other suitable ones.

For the 6-cycle $XABYCD$ we can replace $|AB| \neq |CD|$ with $|YB| \pm |YC| \neq |XA| \pm |XD|$ (Fig. 3a). Similarly, for the 6 cycle $XPQZRS$ we can replace $|PQ| \neq |RS|$ with $|ZQ| \pm |ZR| \neq |XP| \pm |XS|$ (Fig. 3b).

As for the rest, we draw the layer graph of the whole jewel in such a way that those conditions are violated. Then we replace them with the conditions that make the whole jewel line-rigid. Let us consider the 6 cycle $XABYCD$ and reformulate the condition $|XA| \neq |CD|$ (Fig. 3b). YZ may be horizontal or vertical. First we consider different configurations when YZ is horizontal. Since ZX is diagonal there will be 4 different configu-

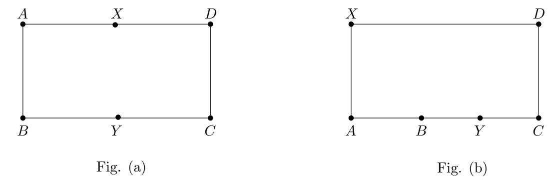

Figure 3: Replacing conditions (a)$|AB| \neq |CD|$ and (b)$|XA| \neq |CD|$.

rations for P and Q (Fig. 4a - 4d). In these figures we do not draw the edges XS, SR and RZ. If we make these configurations line-rigid $XPQZ$ will be line-rigid. So, we shall also need $|XS| \neq |ZR|$ to make the whole jewel line-rigid for all these configurations.

To make these configurations line-rigid we need: $|ZQ| \neq |XA| \pm |XP|$ (Fig. 4a), $|YC| \pm |YZ| \neq |XD| \pm |XP|$ (Fig. 4b), $|ZQ| \neq |XA|$ (Fig. 4c) and $|ZQ| \pm |YC| \pm |YZ| \neq |XD| \pm |XP|$ (Fig. 4d). The conditions for Fig. 4b and Fig. 4d are obvious. The conditions for the other two figures will make $XPQZA$ line-rigid. Since YZ is an edge $XPQZA$ will collapse on YZ. Now X and C are fixed, and XD and CD are known. So, D will also be unique. Hence, the whole configuration will be line-rigid.

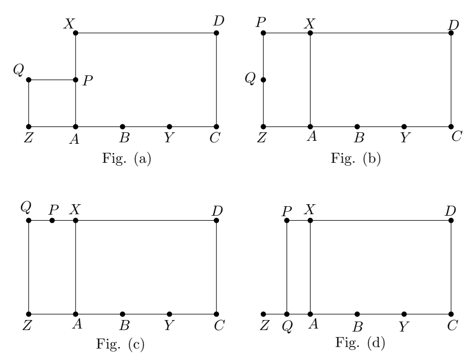

Figure 4: Replacing condition $|XA| \neq |CD|$ when YZ is horizontal.

When YZ is vertical and $|YZ| = |XA|$ there is only one layer graph (Fig. 5). For this case $|XA| \neq |CD|$ can be replaced with $|YZ| \neq |XA|$ and $|XP| \neq |ZQ|$. $|XP| \neq |ZQ|$ will make $XPQZ$ line-rigid. Then for the cycle $XABYZQP$ we need $|YZ| \neq |XA|$ to make it line-rigid. Thus, X and Y are fixed. For the 4-cycle $XYCD$ it is evident that $|XD| \neq |YC|$. This makes the 4-cycle line-rigid, and hence make C and D unique. As before we also need $|XS| \neq |ZR|$ to make the whole jewel line-rigid.

Similarly, we can replace the other conditions involving AB, CD, PQ and RS. We summarize the results in the following lemma.

Lemma 1 *A 6:6 jewel with vertices* X, Y, Z, A, B,

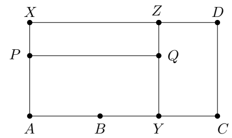

Figure 5: Replacing condition $|XA| \neq |CD|$ when YZ is vertical and $|YZ| = |XA|$.

C, D, P, Q, R, S and edges $|XA|$, $|AB|$, $|BY|$, $|YC|$, $|CD|$, $|DX|$, $|XP|$, $|PQ|$, $|QZ|$, $|ZR|$, $|RS|$, $|SX|$, $|YZ|$ *is line-rigid if its edges satisfy a set of 74 sufficient conditions.*

Proof. Omitted. □

This brings us to the construction of a line-rigid *ppg* that has a structure shown in Fig. 6 - a whole lot of 6:6 jewels whose strut vertices Y and Z are chosen from a complete graph on 40 points. Each strut YZ should satisfy the following conditions on its length:

$$|YZ| \neq \begin{array}{l}\{|XA|, |XD|, |XP|, |XS|, |XA| \pm |XP|, \\ |XD| \pm |XP|, |XA| \pm |XS|\}.\end{array}$$

These are 10 in number; in view of the **Observation** stated in the Introduction section, at Y there can be, in the worst case, 2 edges that are equal to each of the edges $|XA|$, $|XD|$ etc. Thus to pick a Z so that $|YZ|$ satisfies all of the above conditions, we have to know the pairwise distances from Y to 21 other points from which we can pick a Z.

But if we use 22 fixed points for the selection of Z for a particular Y it may happen that all the 6:6 jewels get attached to one fixed point chosen as Z. We need to attach the 6:6 jewels evenly to all the fixed points so that the same number of edges can be attached to them in the first round and all of them, except a constant number of them, are used to attach the 6:6 jewels. To specify the number of 6:6 jewels attached to a fixed point we shall use the term valence.

Lemma 2 *A set S of 40 fixed points is sufficient to attach 6:6 jewels uniformly to them.*

Proof. Omitted. □

In addition to the extra 216 edges needed at each of Z's to satisfy the conditions on ZQ and ZR we need 2 more edges to accommodate this difference of 1 6:6 jewel that can be attached to them. Thus, we need a total of 218 extra edges at each of the 40 fixed points.

Algorithm For convenience we change the labels as follows: $X \to X_i$, $A \to A_i$, $B \to B_j$, $C \to B_k$, $D \to D_i$, $P \to P_i$, $Q \to Q_m$, $R \to Q_l$ and $S \to S_i$. Let the total number of points be n. We shall attach b number of jewels to each of 20 fixed points and $b+1$ to each of the rest 20 fixed points. There will be a total of $20b + 10$ jewels.

In the first round, we choose the distance queries represented by the edges of the graph in Fig. 6. All the nodes Y_i or Z_i $(i = 1, ..., 40)$ are mutually connected to form a complete graph. There are 780 edges in the subgraph. Each of Y_i/Z_i $(i = 1, ..., 40)$ has $2b + 218$ leaves to attach b or $b + 1$ jewels. Extra 216 leaves are needed to have the latitude to satisfy the conditions on Z_iQ_l. We query the distances Y_iB_j and Z_iQ_l $(j, l = 1, ..., 2b + 218)$. Since there will be $20b + 10$ extended jewels we have $20b + 10$ groups of 5 nodes $(A_i, D_i, S_i, P_i, X_i)$ $(i = 1, ..., 20b + 10)$. We query the distances $|A_iX_i|$, $|D_iX_i|$, $|S_iX_i|$ and $|P_iX_i|$, $(i = 1, ..., 20b + 10)$. In total there will be $160b + 9540$ pairwise distance queries in the first round for the placement of a total of $n = 180b + 8810$ points.

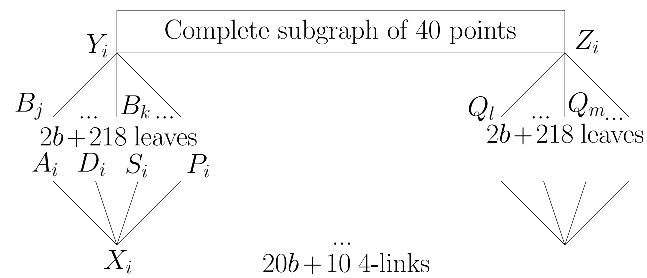

Figure 6: Queries in the first round for 6:6 jewel.

In the second round, for each 4-link $(A_i, D_i, S_i, P_i, X_i)$ we find edges Y_iB_j, Y_iB_k, Y_iZ_i rooted at Y_i, and Z_iQ_l, Z_iQ_m rooted at Z_i to form a 6:6 jewel which satisfy the conditions for line-rigidity. Then we query the distances $|A_iB_j|$, $|D_iB_k|$, $|S_iQ_l|$ and $|P_iQ_m|$. The 216 additional children at Y_i and Z_i provide us with the latitude to choose edges that satisfy the above conditions for line-rigidity. So, for each 4-link $(A_i, D_i, S_i, P_i, X_i)$ we can always find edges YB_j, YB_k, YZ_i, ZQ_l and ZQ_m for the 6:6 jewel such that the conditions for line-rigidity are satisfied. For each of the unused 216 or 218 leaves B_j of node Y_i or Q_l of node Z_i we query its distance from a fixed point other than its parent. Total number of queries in the second round will be $80b + 8760$.

In both the rounds a total of $240b + 18300$ pairwise distances are to be queried for the placement of $180b + 8810$ points. Thus, $4n/3 + O(1)$ queries are sufficient to place n distinct points on a line using two rounds.

3 Lower Bound for Two Rounds

We push the adversarial argument given in the lower bound proof of [2] very much farther.

Let the set of edges queried in the first and second round be E_1 and E_2 respectively, the query graph in the first round be $G_1 = (V, E_1)$, and the *ppg* be $G_2 =$

$(V, E_1 \cup E_2)$.

In the first round the adversary will fix the length of the edges according to the following strategy to keep open options to make the *ppg* ambiguous:

1. Fixes the layout of all nodes of degree 3 or more and reports the lengths of the edges incident on these nodes.
2. For maximal paths of length 3 or more formed by degree 2 nodes, say $p_1, p_2, ..., p_k (k \geq 3)$, let p_0 and p_{k+1} be the nodes of degree not equal to 2 which are adjacent to p_1 and p_k. The adversary sets the edge lengths as follows: $|p_i p_{i+1}| = |p_{i-1} p_{i+2}|$ and $|p_{i-1} p_i| = |p_{i+1} p_{i+2}|, i = 1(\text{mod } 3)$. Nodes p_i and p_{i+1}, $i = 1(\text{mod } 3)$, are treated as special node pairs [2].
3. For maximal paths of length 2 formed by one node of degree 2 and another node of degree 1 the adversary sets the length of the edge common to both the nodes same for all such paths.

Now we consider second round. Strategy 2 of the adversary warrants that for each special node pair (p_i, p_{i+1}) in G_2, there must be at least one edge from E_2 incident on either p_i or p_{i+1} [2]. This means that in G_2 any maximal path of degree 2 can have at most 2 consecutive edges from E_1. Together with this requirement strategy 3 of the adversary requires the following property for the *ppg*:

Lemma 3 *The number of nodes in any maximal path of degree 2 in G_2 is at most 3.*

Proof. Suppose number of degree 2 nodes in a maximal path is 4. Let the nodes be p_1, p_2, p_3 and p_4. Let p_0 and p_5 be nodes of degree at least 3 that are adjacent to p_1 and p_4 respectively. Since any maximal path of degree 2 in G_2 can have at most 2 consecutive edges from E_1 we can have the following 5 combinations of the E_1 and E_2 edges for the edges $|p_0 p_1|$, $|p_1 p_2|$, $|p_2 p_3|$, $|p_3 p_4|$ and $|p_4 p_5|$:

E_2, E_1, E_2, E_1, E_1; E_2, E_1, E_1, E_2, E_1; E_1, E_2, E_1, E_2, E_1; E_1, E_1, E_2, E_2, E_1; E_1, E_1, E_2, E_1, E_1.

For combination 1, since there are two edges in E_2 lengths of those edges can be set in such a way that $|p_0 p_5| = |p_1 p_3|$ and $|p_0 p_1| = |p_3 p_5|$, and the graph G_2 becomes non-rigid. Similarly, for combinations 2-4 the adversary can make the graph ambiguous. As for combination 5, the adversary can set $|p_1 p_2| = |p_3 p_4|$ in the first round by strategy 3 and can set the length of $p_2 p_3$ in round 2 in such a way that $|p_2 p_3| = |p_4 p_5| + |p_5 p_0| + |p_0 p_1|$. Then the cycle $p_0, p_1, p_2, p_3, p_4, p_5$ will not be line-rigid. □

Theorem 4 *The minimum density of any line-rigid ppg for two round queries is at least* $14/13$.

Proof. We determine the minimum of the average numbers of edges for all types of nodes. For this the *ppg* is divided into pieces each of which consists of one node and fractions of edges incident on it. To split the edges and allocate their parts between their corresponding adjacent nodes the nodes are categorized as light and heavy nodes. If an edge joins two light nodes or two heavy nodes then the edge is divided equally. Otherwise, the light node owns $1/2 + g$ and the heavy node owns $1/2 - g$, where $0 \leq g < 1/2$.

By construction there are three types of nodes that are analyzed below for their average density:

a. **Special node pairs:** They are considered as heavy nodes. Since each special node pair has at least one edge in E_2 incident on one of them, the total edge allocated to the pair will be at least $1/2 + 2(1/2 - g)$ and $1/2 + (1/2 - g)$. Average density for each node is at least $(5 - 6g)/4$.

b. **Normal nodes of degree at least 3:** They are also heavy nodes. Each node has at least $3(1/2 - g)$ edges.

c. **Nodes in the maximal path formed by degree two normal nodes:** These are light nodes. By Lemma 1 each maximal path of degree two has length k where $k \leq 3$. The total edge for a path is $2(1/2 + g) + (k - 1)$. The average density for each node in a path is $1 + 2g/k$. It is minimum when $k = 3$. Thus, each node has at least $1 + 2g/3$ edges.

Minimum average density for all nodes in G_2 will be $\max\min\{(5 - 6g)/4, 3/2 - 3g, 1 + g/2\} = 14/13$ when $g = 3/26$. □

An important open problem is to further refine this argument to obtain a better lower bound. We believe that this may be possible.

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