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AN EXTENDED VIEW OF THE CHU-CONSTRUCTION

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Abstract. The cyclic Chu-construction for closed bicategories with pullbacks, which generalizes the original Chu-construction for symmetric monoidal closed categories, turns out to have a non-cyclic counterpart. Both use so-called Chu-spans as new 1-cells between 1-cells of the underlying bicategory, which form the new objects. Chu-spans may be seen as a natural generalization of 2-cell-spans in the base bicategory that no longer are confined to a single hom-category. This view helps to clarify the composition of Chu-spans.

We consider various approaches of linking the underlying bicategory with the newly constructed ones, for example, by means of two-dimensional generalizations of bifibrations. In the quest for a better connection, we investigate, whether Chu-spans form a double category. While this turns out not to be the case, we are led to considering a generalization of the construction to paths of 1-cells in the base, leading to two hierarchies of closed bicategories, one for linear paths and one for loops. The possibility of moving beyond paths, respectively, loops of the same length is indicated.

Finally, Chu-spans in ***rel*** are identified as bipartite state transition systems. Even though their composition may fail here due to the lack of pullbacks in ***rel***, basic game-theoretic constructions can be performed on cyclic Chu-spans. These are available in all symmetric monoidal closed categories with finite products. If pullbacks exist as well, the bicategory of cyclic Chu-spans inherits a monoidal structure that on objects coincides with the categorical product.

1. Introduction

The Chu-construction's original purpose was to build $*$-autonomous categories out of autonomous (= symmetric monoidal closed) categories. But although successful at that, it has long been regarded as a slightly obscure technical trick. Here we wish step back and analyze the real core of the construction at the level of closed bicategories. A wider scope of the construction then becomes discernible and the connection with $*$-autonomy turns out to hinge on the restriction to cyclic chains of what we call *Chu-spans*.

In his late-1970s Master's Thesis [Chu79], supervised by Michael Barr, Po-Hsiang Chu constructed a new category $\mathcal{A}_a = \boldsymbol{chu}\langle \mathcal{V}, a\rangle$ from an autonomous category $\mathcal{V}$ and a $\mathcal{V}$-object a. Besides being autonomous, $\mathcal{A}_a$ contained a so-called *dualizing object* $\perp$ (depending on a) that by means of its internal hom-functor $[-,\perp]$ induced an equivalence between $\mathcal{A}_a^{\mathrm{op}}$ and $\mathcal{A}_a$. While $*$-autonomous categories had been discovered by Barr in the realm of functional analysis [Bar79], they also turned out to provide nice models for certain fragments of linear logic [See87]. Hence interest for such categories grew among computer scientists and logicians

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during the 1980s. One open question concerned models for non-symmetric linear logic.

The early 1990s saw an extension of the categorical ideas to the non-symmetric but closed setting: Dualizing objects with examples were discussed in Street's lecture notes [Str91]. Independently, Barr [Bar95] defined $*$-autonomy and described a Chu-construction in that setting, provided the base category $\mathcal{V}$ has finite limits. In [Bar96] he restructured his original approach by first considering the Chu-construction for the terminal object t. Certain monads in $\boldsymbol{chu}\langle\mathcal{V}, t\rangle$, which may be identified with $\mathcal{V}\times\mathcal{V}^{\mathrm{op}}$, then generate the (generalized) $*$-autonomous categories $\boldsymbol{chu}\langle\mathcal{V}, a\rangle$, $a \in \boldsymbol{ob}(\mathcal{V})$, as categories of endo-modules.

Recall that a monoidal category may be viewed as a bicategory with one object, the tensor corresponding to the composition of 1-cells. Barr's revised approach together with the well-known construction of the bicategory of monads from a bicategory with local coequalizers prompted the author to consider the Chu-construction in closed bicategories $\mathcal{B}$ with local pullbacks [Kos01]. In particular, this raised the question of formulating a 2-dimensional generalization of $*$-autonomy, which tied in with research by Robin Cockett and Robert Seely on linearly distributive categories (loosely speaking "$*$-autonomous categories without dualizing objects"), ultimately resulting in the notion of a *linear bicategory* [CKS00]. In the inherently non-symmetric setting of a bicategory's 1-cell composition one should in fact expect to have two negations, that is, two ways of reversing 1-cells. These may agree, however, and in the presence of "dualizing 1-cells" this leads to the notion of *cyclic $*$-autonomous bicategory*.

The present paper concerns the other important lesson of [Kos01], not exploited at the time: to view the Chu-construction as a genuinely bicategorical construction, independent from the construction of the bicategory of monads. In fact, and perhaps surprisingly, the construction per se is not even concerned with $*$-autonomy. Initially, it uses all 1-cells of a closed bicategory $\mathcal{B}$ with local pullbacks as objects for a new closed bicategory $\boldsymbol{Chu_1}(\mathcal{B})$. This contains a non-full cyclic $*$-autonomous sub-bicategory $\boldsymbol{cChu_1}(\mathcal{B})$ with the endo-1-cells as objects.

Both the cyclic and the non-cyclic variant of the construction employ the same type of new 1-cells between the objects, which we call *(cyclic) Chu-spans*, compare Section 2. If $\mathcal{B}$ is (the suspension of) a symmetric monoidal closed category $\mathcal{V}$, for $a \in \mathcal{V}$ the classical Chu-category $\mathcal{A}_a = \boldsymbol{chu}\langle\mathcal{V}, a\rangle$ appears as full subcategory of the hom-category of $\boldsymbol{cChu_1}(\mathcal{V})$ at a, compare [Kos01]. In general, endo-Chu-spans on a need not be constrained by symmetry (as required in the classical case), and there also will be Chu-spans between *different* objects of $\mathcal{V}$.

The composition of Chu-spans (Section 3) utilizes the closedness of $\mathcal{B}$ and the existence of local pullbacks. Modulo exponential transposition it can actually be viewed as the composition of genuine spans (hence the terminology), which eliminates the technical obscurity often associated with the classical Chu-construction.

In order to ascertain the actual scope of the construction, in Section 4 we first consider two canonical (strict) functors $\boldsymbol{cChu_1}(\mathcal{B}) \longrightarrow \mathcal{B}$ and $\boldsymbol{Chu_1}(\mathcal{B}) \longrightarrow \mathcal{B} \times \mathcal{B}$ connecting the new bicategories with the base $\mathcal{B}$. While both may be viewed as two-dimensional generalizations of the notion of bifibration, the resulting possibilities of embedding either $\mathcal{B}\times\mathcal{B}$ into $\boldsymbol{Chu_1}(\mathcal{B})$ or $\mathcal{B}$ into $\boldsymbol{cChu_1}(\mathcal{B})$ come across as somewhat artificial. Clearly, a better way of relating $\mathcal{B}$ to the Chu-bicategories is called for.

To this end, we then explore the question, whether Chu-spans admit second mode of composition, orthogonal to the one discussed in Section 3, possibly giving rise to a double category. This turns out not to be the case, but short of composing Chu spans in this fashion, we may still "chain" them together.

This opens up the possibility of considering other domains and codomains for Chu-spans besides single 1-cells of $\mathcal{B}$, which we explore in Section 5. In particular, 1-cell paths in $\mathcal{B}$ of fixed length $n \in \mathbb{N}$ in $\mathcal{B}$ straightforwardly induce a closed bicategory $\boldsymbol{Chu_n}(\mathcal{B})$. In addition, infinite paths indexed either by $\mathbb{N}$ or by $\mathbb{Z}$ may be considered as objects of still further closed bicategories based on $\mathcal{B}$. This suggests using finite 1-cell loops in $\mathcal{B}$ of length $n > 0$ as objects of a cyclic $*$-autonomous bicategory $\boldsymbol{cChu_n}(\mathcal{B})$. But the existence of non-trivial automorphisms of such loops has to be taken into account. A similar construction produces $\boldsymbol{cChu_{\mathbb{Z}}}(\mathcal{B})$, into which the other cyclic $*$-autonomous bicategories $\boldsymbol{cChu_n}(\mathcal{B})$, $n > 0$ may be embedded, albeit not fully.

Finally, the notion of Chu-span may even be useful in cases where composition is not always possible. Section 6 shows that Chu-spans in the category $\boldsymbol{rel}$ (with trivial 2-cells) can be viewed as "bipartite" labeled transition systems (LTSs) that serve as a basis for strictly alternating two-party interactions and games. Three important operations on games have direct interpretations in terms of Chu-operations. Unfortunately, $\boldsymbol{rel}$ does not have pullbacks. But the game operations carry over to any symmetric monoidal closed category with finite products. If pullbacks exist as well, cyclic Chu-span morphisms into $\boldsymbol{R} \multimap \boldsymbol{S}$ provide arrows $\boldsymbol{R} \longrightarrow \boldsymbol{S}$ for a $*$-autonomous category.

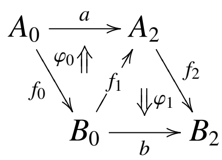
Some of these results were presented at the Workshop on Chu Spaces at the University of California, Santa Barbara, 2000-06-25.

2. Chu-spans and Chu-morphisms, the basic building blocks

In case of a symmetric monoidal closed category $\mathcal{V} = \langle \mathcal{V}, \otimes, \top \rangle$ with pullbacks the Chu-category $\boldsymbol{chu}\langle \mathcal{V}, a\rangle$ has $\mathcal{V}$-morphisms of the form $f_1 \otimes f_0 \xrightarrow{\varphi} a$ as objects.

The monoidal category $\boldsymbol{chu}\langle \mathcal{V}, a\rangle$ can be viewed as a bicategory in the sense of Bénabou [Ben67], compare also [Bor94], with just one 0-cell, which may be identified with the $\mathcal{V}$-object a. Then $\langle f_0, \varphi, f_1\rangle$ specifies an endo-1-cell on a. This suggests considering all $\mathcal{V}$-objects simultaneously as 0-cells of a new bicategory, which raises the question, how to generalize the endo-1-cells above to 1-cells between possibly different 0-cells. We can even try to use the 1-cells of an arbitrary bicategory $\mathcal{B} = \langle \mathcal{B}, \otimes, \top \rangle$ as 0-cells of a new structure (here $\top$ maps objects to identity 1-cells).

2.1. Definition. A *Chu-span* $\boldsymbol{\varphi} = \langle f_0, \varphi_0, f_1, \varphi_1, f_2\rangle$ from $A_0 \xrightarrow{a} A_2$ to $B_0 \xrightarrow{b} B_2$ in $\mathcal{B}$ consists of two independent 2-cells

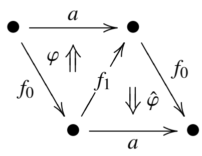


(2-01)

We call $\boldsymbol{\varphi}$ *simple*, if a and b are terminal in their hom-categories, *trivial*, if f_0 and f_2 are isomorphisms, and *cyclic*, if $f_0 = f_2$. In this case the *opposite Chu-span* from b to a is given by $\boldsymbol{\varphi}^* = \langle f_1, \varphi_1, f_0 = f_2, \varphi_0, f_1 \rangle$.

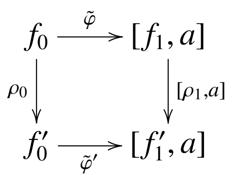
The new terminology (in [Kos01] we called these gadgets "Chu-cells") was inspired by the observation that a trivial Chu-span is just an ordinary span in the hom-category $\boldsymbol{B}\langle A_0, A_2 \rangle$. Hence Chu-spans may be viewed as generalizations of spans with domains and codomains in possibly different hom-categories of $\mathcal{B}$.

In a symmetric monoidal category $\mathcal{V}$ any $f_1 \otimes f_0 \xrightarrow{\varphi} a$ specifies half of a Chu-span

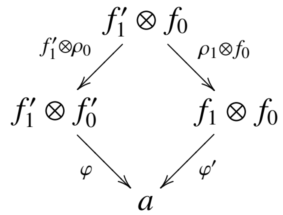


where the other half is induced by symmetry. If $\mathcal{V}$ is not symmetric, $f_1 \otimes f_0 \xrightarrow{\varphi} a$ need not have a canonical counterpart unless some additional structure is present (for example, a shift operation in case that $\mathcal{V}$ consists of graded objects). Our approach sidesteps the issue of canonical counterpart by pairing $f_1 \otimes f_0 \xrightarrow{\varphi} a$ with every 2-cell $f_0 \otimes f_1 \xrightarrow{\psi} a$.

If $\mathcal{V}$ is also closed, classically a morphism from $f_1 \otimes f_0 \xrightarrow{\varphi} a$ to $f_1' \otimes f_0' \xrightarrow{\varphi'} a$ in $\boldsymbol{chu}\langle \mathcal{V}, a \rangle$ consists of $\mathcal{V}$-morphisms $f_0 \xrightarrow{\rho_0} f_0'$ and $f_1' \xrightarrow{\rho_1} f_1$ such that



or equivalently

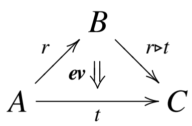


(2-02)

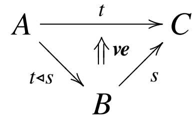
Here $[-,-]$ denotes the closed structure of $\mathcal{V}$ and $\tilde{}$ indicates exponential transposes. While the first presentation superficially resembles a coalgebra homomorphism, the second one identifies ρ_0 and ρ_1 as "formal adjoints" with respect to φ and φ', denoted by $\rho_0 \dashv^{\varphi}_{\varphi'} \rho_1$ (think of $\mathcal{V} = \boldsymbol{set}$, where $\otimes$ is cartesian product).

Even though morphisms between Chu-spans can be defined in any bicategory $\mathcal{B}$, closedness of $\mathcal{B}$ (with respect to 1-cell composition) allows for a more concise formulation and will shortly be needed to define the composition of Chu-spans.

2.2. Assumption. We require $\mathcal{B}$ to be *right-closed* in the sense of [SW78]: every 1-cell $A \xrightarrow{t} C$ admits a *right extension* $\langle \boldsymbol{ev}, r \triangleright t \rangle$ along any $A \xrightarrow{r} B$, and a *right lifting* through any $B \xrightarrow{s} C$



and



in the sense that for the 2-cells pasting at $r \triangleright t$, respectively, at $t \triangleleft s$ is bijective. The transition from $s \otimes r \xRightarrow{\chi} t$ to $s \xRightarrow{\chi^{\triangleright}} r \triangleright t$ or to $r \xRightarrow{\chi^{\triangleleft}} t \triangleleft s$ is known as "(exponential) transposition" or, alternatively, as "currying".

2.3. Definition. Given Chu-spans $a \xrightarrow{\boldsymbol{\varphi}} b$ and $a \xrightarrow{\boldsymbol{\varphi}'} b$, a *Chu-morphism* $\boldsymbol{\varphi} \overset{\rho}{\Longrightarrow} \boldsymbol{\varphi}'$ consists of 2-cells $\langle f_0 \overset{\rho_0}{\Longrightarrow} f_0', f_1 \overset{\rho_1}{\Longleftarrow} f_1', f_2 \overset{\rho_2}{\Longrightarrow} f_2' \rangle$ subject to

$$\begin{array}{ccccc} f_0 \triangleright a & \overset{\varphi_0^{\triangleright}}{\Longleftarrow} & f_1 & \overset{\varphi_1^{\triangleleft}}{\Longrightarrow} & b \triangleleft f_2 \\ {\scriptstyle \rho_0 \triangleright a}\Uparrow & & \Uparrow{\scriptstyle \rho_1} & & \Uparrow{\scriptstyle b \triangleleft \rho_2} \\ f_0' \triangleright a & \underset{\varphi_0'^{\triangleright}}{\Longleftarrow} & f_1' & \underset{\varphi_1'^{\triangleleft}}{\Longrightarrow} & b \triangleleft f_2' \end{array} \qquad \text{or, alternatively,} \qquad \rho_0 \dashv^{\varphi_0}_{\varphi_0'} \rho_1 \dashv^{\varphi_1'}_{\varphi_1} \rho_2$$

We call ρ *cyclic*, if $\rho_0 = \rho_2$. In this case the *opposite Chu-morphism* $(\boldsymbol{\varphi}')^* \overset{\rho^*}{\Longrightarrow} \boldsymbol{\varphi}^*$ is given by $\langle \rho_1, \rho_0 = \rho_2, \rho_1 \rangle$.

Compared to the left presentation in Diagram (2-02) we have curried twice in order to display the Chu-spans as genuine spans with centers f_1 and f_1', respectively. Compared to span morphisms Chu-morphisms point in the opposite direction. This is necessary to obtain closed rather than coclosed Chu-bicategories and will later enable us to recover $\mathcal{B}$ rather than $\mathcal{B}^{co}$ inside those.

2.4. Example. The category ***set*** of sets and functions is symmetric monoidal with respect to the cartesian product $\times$. Recall that a span $a \xleftarrow{\varphi_0} f_1 \xrightarrow{\varphi_1} b$ in ***set*** may be thought of as a directed bi-partite (multi-)graph with node-set $a+b$, edge-set f_1, and domains and codomains given by φ_0 and φ_1, respectively.

Now we can interpret a Chu-span $\boldsymbol{\varphi} = \langle f_0, \varphi_0, f_1, \varphi_1, f_2 \rangle$ from a to b in ***set*** as a family of bipartite graphs with fixed node-set $a + b$ and edge-set f_1, where the domain- and codomain functions are *parameterized* by sets f_0 and f_2, respectively. In case of a cyclic Chu-span, the parameter sets coincide and we think of these functions as being jointly parameterized by f_0 rather than being separately parameterized by $f_0 \times f_0$.

For simple Chu-spans the graphs have just two nodes, hence are completely determined by the set of edges. The parameterization now only determines the size of the family, all members have the same trivial "shape".

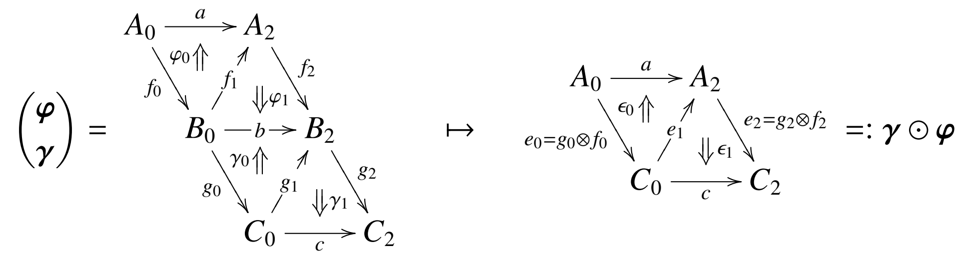
For a endo-Chu-spans the sets a and b coincide. Instead of bipartite graphs with node set $a+a$ we may therefore consider graphs just on a, where paths of length $\neq 1$ become available. In particular, this applies to classical Chu-spans, which by default are cyclic. Symmetry relating the domain and codomain functions forces every edge in the corresponding graph to be a loop.

Dualization of cyclic Chu-spans interchanges the roles of parameter set and edge set in the graph, besides reversing the arrows.

In Section 6 we will consider a different graph-theoretical interpretation of Chu-spans over ***set*** in the larger context of Chu-spans over the monoidal closed category $\langle \boldsymbol{rel}, \times, 1 \rangle$.

3. The composition of Chu-spans

In order to use Chu-spans as 1-cells in a new bicategory with the 1-cells of $\mathcal{B}$ as objects, we need to be able to compose them. Simplicity suggests composing the outer 1-cells (with even index) of matching Chu-spans just like 1-cells in $\mathcal{B}$:



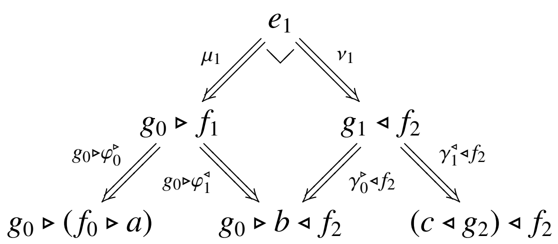
(3-03)

If $\boldsymbol{\varphi}$ and $\boldsymbol{\gamma}$ are trivial, the construction of e_1 reduces to the composition of ordinary spans in $\boldsymbol{B}\langle B_0, B_2\rangle$ by means of a pullback of φ_1 and γ_0 , leading to

3.1. Assumption. $\mathcal{B}$ locally has pullbacks.

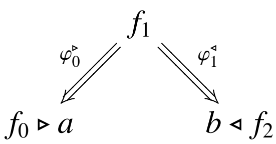
In the general case we still need to derive a 1-cell $C_0 \longrightarrow A_2$ from a pullback $B_0 \xrightarrow{p} B_2$ of φ_1 and γ_0 . Let us utilize the closedness of $\mathcal{B}$ (compare Assumption 2.2) to solve this problem. Recall that $(g_0 \triangleright p) \triangleleft f_2$ and $g_0 \triangleright (p \triangleleft f_2)$ have the same universal property, hence we may drop the parentheses. Since $g_0 \triangleright -$ and $- \triangleleft f_2$ are right-adjoint to $- \otimes g_0$ and $f_2 \otimes -$, respectively, we obtain the pullback $g_0 \triangleright p \triangleleft f_2$ as a first candidate for e_1 . The difficulty with this approach is to extract a Chu-span from a to c from this pullback. This would seem to require 2-cells $(f_2 \otimes f_1) \triangleleft f_2 \Longrightarrow f_1$ and $g_0 \triangleright (g_1 \otimes g_0) \Longrightarrow g_1$, for which there are no canonical candidates.

Instead, we could first curry and then obtain $e_1 \in \boldsymbol{B}\langle C_0, A_2\rangle$ as a pullback:

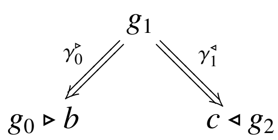


(3-04)

This construction links the non-matching transposes of the original Chu-spans $\boldsymbol{\varphi}$ and $\boldsymbol{\gamma}$

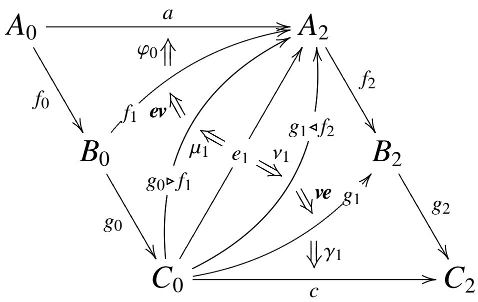


and



Now $g_0 \triangleright (f_0 \triangleright a) \cong (g_0 \otimes f_0) \triangleright a$ and $(c \triangleleft g_2) \triangleleft f_2 \cong c \triangleleft (g_2 \otimes f_2)$ allow us to extract a Chu-span from a to c from Diagram (3-04). Conceptually, e_1 may be thought of as the "object of formal adjunctions" with respect to φ_1 and γ_0 .

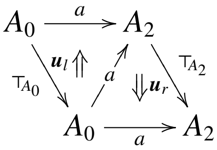
3.2. Definition. Given Chu-spans $\boldsymbol{\varphi} = \langle f_0, \varphi_0, f_1, \varphi_1, f_2 \rangle$ and $\boldsymbol{\gamma} = \langle g_0, \gamma_0, g_1, \gamma_1, g_2 \rangle$ from $A_0 \xrightarrow{a} A_2$ to $B_0 \xrightarrow{b} B_2$ to $C_0 \xrightarrow{c} C_2$, define their composite $\boldsymbol{\gamma} \odot \boldsymbol{\varphi}$ from a to c by



(3-05)

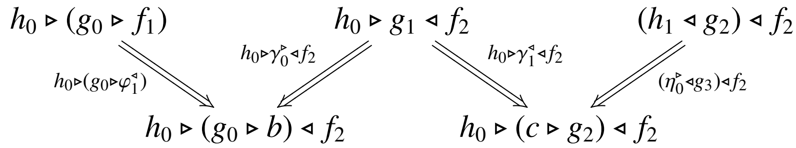
3.3. Theorem. *1-cells of* $\mathcal{B}$ *as objects, Chu-spans as 1-cells and Chu morphisms as 2-cells form a closed bicategory* $\boldsymbol{Chu_1}(\mathcal{B})$.

Proof. The identity Chu-span $\mathbf{1}_a$ on $A_0 \xrightarrow{a} A_2$ is given by the structural unit isomorphisms



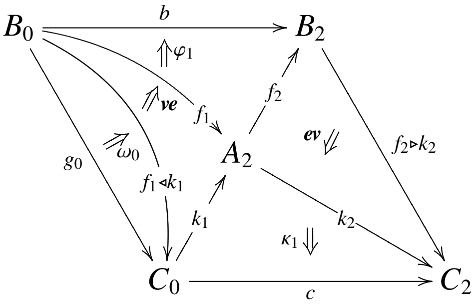
(3-06)

The essential associativity and the functoriality of the Chu-span composition follow, since the central 1-cell of both $(\boldsymbol{\eta} \otimes \boldsymbol{\gamma}) \otimes \boldsymbol{\varphi}$ and $\boldsymbol{\eta} \otimes (\boldsymbol{\gamma} \otimes \boldsymbol{\varphi})$ with $\boldsymbol{\eta} = \langle h_0, \eta_0, h_1, \eta_1, h_2 \rangle$ from c to $D_0 \xrightarrow{d_1} D_2$ derives from a limit (for example, via three pullbacks) of the diagram

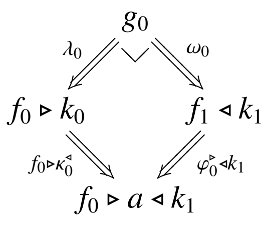


Note that the different variance in the even and odd components of a Chu-morphism is necessary for the functoriality of this composition.

Given $\boldsymbol{\varphi}$ and a Chu-span $\boldsymbol{\kappa} = \langle k_0, \kappa_0, k_1, \kappa_1, k_2 \rangle$ from a to c, define $\boldsymbol{\gamma} := \boldsymbol{\varphi} \triangleright \boldsymbol{\kappa}$ by



where



(3-07)

To show that this induces a right extension of $\boldsymbol{\kappa}$ along $\boldsymbol{\varphi}$, we need to construct an "evaluation" $\boldsymbol{E} := \boldsymbol{\gamma} \odot \boldsymbol{\varphi} \xRightarrow{\boldsymbol{Ev}} \boldsymbol{\kappa}$. For the outer components we use

$$g_0 \otimes f_0 \xRightarrow{\lambda_0 \otimes f_0} (f_0 \triangleright k_0) \otimes f_0 \xRightarrow{\boldsymbol{ev}} k_0 \qquad \text{and} \qquad (k_2 \otimes f_2) \otimes f_2 \xRightarrow{\boldsymbol{ev}} k_2\,.$$

Exponentially transposing $g_1 \otimes g_0 = f_2 \otimes k_1 \otimes g_0 \xRightarrow{\gamma_0} b$ above in different ways yields

$$\begin{array}{ccccc}
k_1 & \xRightarrow{\boldsymbol{ev}^{\triangleright}} & (f_1 \triangleleft k_1) \triangleright f_1 & \xRightarrow{\omega_0 \triangleright f_1} & g_0 \triangleright f_1 \\
{\scriptstyle \mathbf{1}^{\triangleleft}_{f_2 \otimes k_1}} \Downarrow & & & & \Downarrow {\scriptstyle g_0 \triangleright \varphi_1^{\triangleleft}} \\
(f_2 \otimes k_1) \triangleleft f_2 & & \xRightarrow[\gamma_0^{\triangleright} \triangleleft f_2]{} & & g_0 \triangleright b \triangleleft f_2
\end{array}$$

This induces $\boldsymbol{Ev}_1$ from k_1 into the pullback e_1 of $\gamma_0^{\triangleright} \triangleleft f_2$ and $g_0 \triangleright \varphi_1^{\triangleleft}$.

Now consider a Chu-span $b \xrightarrow{\gamma'} c$ and a Chu-morphism $\boldsymbol{E}' := \boldsymbol{\gamma}' \odot F \xRightarrow{\sigma} \boldsymbol{\kappa}$. To find a unique Chu-morphism $\boldsymbol{\gamma}' \xRightarrow{\sigma^{\triangleright}} \boldsymbol{\gamma} = \boldsymbol{\varphi} \triangleright \boldsymbol{\kappa}$ that satisfies $\boldsymbol{Ev} \circ \sigma^{\triangleright} \odot \boldsymbol{\varphi} = \sigma$, we set $(\sigma^{\triangleright})_2 = \sigma_2^{\triangleright}$. Exponentially transposing the property $\kappa_0 \circ k_1 \otimes \sigma_0 = \epsilon_0' \circ \sigma_1 \otimes e_0'$ of σ results in

$$\begin{array}{ccccc}
g_0' & \xRightarrow{\boldsymbol{ev}^{\triangleleft}} & f_1 \triangleleft (g_0' \triangleright f_1) & \xRightarrow{f_1 \triangleleft \sigma_1} & f_1 \triangleleft k_1 \\
{\scriptstyle \sigma_0^{\triangleright}} \Downarrow & & & & \Downarrow {\scriptstyle \varphi_0^{\triangleright} \triangleleft k_1} \\
f_0 \triangleright k_0 & & \xRightarrow[f_0 \triangleright \kappa_0^{\triangleleft}]{} & & f_0 \triangleright a \triangleleft k_1
\end{array}$$

The pullback g_0 of $f_0 \triangleright \kappa_0^{\triangleleft}$ and $\varphi_0^{\triangleright} \triangleleft k_1$ provides us with $g_0' \xRightarrow{(\sigma^{\triangleright})_0} g_0$. Finally, transposing $k_1 \xRightarrow{\sigma_1} e_1' \xRightarrow{\nu_1'} g_1' \triangleleft f_2$ exponentially yields the central component $g_1 = f_2 \otimes k_1 \xRightarrow{(\sigma^{\triangleright})_1} g_1'$. A straightforward computation establishes the desired property of $\sigma^{\triangleright}$.

The construction of right liftings is analogous. ■

3.4. Remarks.

(0) The construction of a right extension in Diagram (3-07) shows that the bicategories $\boldsymbol{spn}(\boldsymbol{B}\langle A_0, A_2\rangle)$ will in general not be left-closed with respect to span-composition: in the trivial case, for isomorphisms f_0 and f_2 as well as k_0 and k_2, the 2-cell $g_0 \xRightarrow{\omega_0} f_0 \triangleright k_0 \cong \mathbf{1}_a$ need not be an isomorphism. In other words, left extensions (and liftings) of ordinary spans will in general be proper Chu-spans.

(1) In case of two parallel Chu-spans $\boldsymbol{\varphi}$, $\boldsymbol{\varphi}'$ from a to b, the right extension $\boldsymbol{\gamma} := \boldsymbol{\varphi} \triangleright \boldsymbol{\varphi}'$ and the right lifting $\boldsymbol{\eta} := \boldsymbol{\varphi}' \triangleleft \boldsymbol{\varphi}$ may jointly be used to encode Chu-morphisms from $\boldsymbol{\varphi}$ to $\boldsymbol{\varphi}'$: combine 2-cells $B_0 \xRightarrow{\beta} g_0$ and $A_2 \xRightarrow{\alpha} e_2$ with the corresponding pullback projections and uncurry to obtain $f_0 \xRightarrow{\rho_0} f_0'$, $f_1' \xRightarrow{\rho_1, \sigma_1} f_1$ and $f_2 \xRightarrow{\sigma_2} f_2'$ with

$$\begin{array}{ccc}
 & f_1' \otimes f_0 & \\
{\scriptstyle f_1' \otimes \rho_0} \swarrow & & \searrow {\scriptstyle \rho_1 \otimes f_0} \\
f_1' \otimes f_0' & & f_1 \otimes f_0 \\
{\scriptstyle \varphi} \searrow & & \swarrow {\scriptstyle \varphi'} \\
 & a &
\end{array}
\quad \text{respectively} \quad
\begin{array}{ccc}
 & f_2' \otimes f_1 & \\
{\scriptstyle f_2' \otimes \sigma_1} \swarrow & & \searrow {\scriptstyle \sigma_2 \otimes f_1} \\
f_2' \otimes f_1' & & f_2 \otimes f_1 \\
{\scriptstyle \varphi} \searrow & & \swarrow {\scriptstyle \varphi'} \\
 & b &
\end{array}
\qquad (3\text{-}08)$$

compare Diagram (2-02). $\langle \beta, \alpha \rangle$ specifies a Chu-morphism iff $\rho_1 = \sigma_1$

3.5. Examples.

(0) Since ***set*** is cartesian closed, Chu-spans as in Example 2.4 can be composed as indicated above. The composition of ordinary spans may be visualized as the composition of bipartite graphs. This does carry over to the composition of Chu-spans: given $\boldsymbol{\varphi} = \langle f_0, \varphi_0, f_1, \varphi_1, f_2 \rangle$ and $\boldsymbol{\gamma} = \langle g_0, \gamma_0, g_1, \gamma_1, g_2 \rangle$ from a to b to c, the arrows of the composite Chu-span from a to c are pairs of functions $\langle g_0 \xrightarrow{\lambda} f_1, f_2 \xrightarrow{\rho} g_1 \rangle$ such that for all $\langle i, j \rangle \in g_0 \times f_2$ we have $\varphi_1 \langle \lambda(i), j \rangle = \gamma_0 \langle i, \rho(j) \rangle$, that is, λ and ρ are "formal adjoints" $\lambda \dashv^{\varphi_1}_{\gamma_0} \rho$. In case of $g_0 = 1 = f_2$ this reduces to the familiar notion of "matching" or "composable" arrows.

Similarly, if $\boldsymbol{\kappa} = \langle k_0, \kappa_0, k_1, \kappa_1, k_2 \rangle$ is another Chu-span from a to c, the arrow-set of the right-extension $\boldsymbol{\varphi} \triangleright \boldsymbol{\kappa}$ consists of the formal adjoints $\alpha \dashv^{\varphi_0}_{\kappa_0} \beta$.

(1) The bicategory ***rel*** with sets as objects, relations as 1-cells and inclusions as 2-cells is well-known to be closed with respect to 1-cell composition. Its hom-categories are power-sets and hence complete lattices. If we consider diagram (3-04) or (3-05) in ***rel***, the central relation $C_0 \xrightarrow{e_1} A_2$ is simply the intersection of the sets

$$g_0 \triangleright f_1 = \{ \langle z, x \rangle \in C_0 \times A_2 : \forall y \in B_0. \langle y, z \rangle \in g_0 \Longrightarrow \langle y, x \rangle \in f_1 \}$$
$$g_1 \triangleleft f_2 = \{ \langle z, x \rangle \in C_0 \times A_2 : \forall v \in B_2. \langle z, v \rangle \in g_1 \Longleftarrow \langle x, v \rangle \in f_2 \}$$

Once we restrict attention to cyclic and hence reversible Chu-spans, we would expect the resulting closed bicategory to be "$*$-autonomous" in a suitable sense. The following definition was introduced in [Kos01]. It minimizes coherence issues and implies closedness of the bicategory and the existence of "dualizing 1-cells" on every object.

3.6. Definition. A *cyclic $*$-autonomous* bicategory $\mathcal{B}$ is equipped with

(0) a "self-dual" family of equivalences

$$\boldsymbol{B}\langle A, B \rangle \xrightarrow{(-)^*} \boldsymbol{B}^{\mathbf{coop}}\langle A, B \rangle = \big(\boldsymbol{B}\langle B, A \rangle\big)^{\mathrm{op}}$$

for $\mathcal{B}_0$-objects A, B, that is,

$$\big(\boldsymbol{B}\langle A, B \rangle \xrightarrow{(-)^*} \boldsymbol{B}^{\mathbf{coop}}\langle A, B \rangle\big) \dashv \big(\boldsymbol{B}\langle B, A \rangle \xrightarrow{(-)^*} \boldsymbol{B}^{\mathbf{coop}}\langle B, A \rangle\big)^{\mathrm{op}}$$

(1) a natural family of 2-cells $r^* \otimes r \xRightarrow{\boldsymbol{ev}_r} (\top_A)^*$, $A \xrightarrow{r} B$ a 1-cell in $\mathcal{B}$, such that $\langle \boldsymbol{ev}_r, r^* \rangle$ is a right extension of $(\top_A)^*$ along r.

It is easy to see that $\langle r, \boldsymbol{ev}_r \rangle$ is a right lifting of $(\top_A)^*$ through r^*. Hence the "dualizing 1-cells" are given by the images of the identity 1-cells under $(-)^*$.

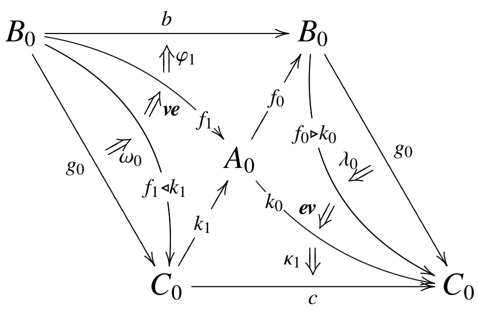
3.7. Remark. In a cyclic $*$-autonomous bicategory we may define a second tensor composition $\oplus$ ("par") by de Morgan duality, that is, $g \oplus f := (f^* \otimes g^*)^*$. This is essentially associative and has units of the form $\bot_A := (\top_A)^*$. In fact, we obtain a second bicategory structure on the objects, 1-cells and 2-cells of $\mathcal{B}$, which is linked with the original one via so-called "linear distributions" $(C \oplus B) \otimes A \xRightarrow{\delta_L} C \oplus (B \otimes A)$ and $C \otimes (B \oplus A) \xRightarrow{\delta_R} (C \otimes B) \oplus A$ subject to certain coherence requirements. "Linear bicategories" were introduced [CKS00] to study such related bicategory structures that do not necessarily arise via de Morgan duality.

3.8. Theorem. *Endo-1-cells of $\mathcal{B}$ as objects, cyclic Chu-spans as 1-cells and cyclic Chu-morphisms as 2-cells form a* cyclic $*$-autonomous *bicategory* $\boldsymbol{cChu_1}(\mathcal{B})$.

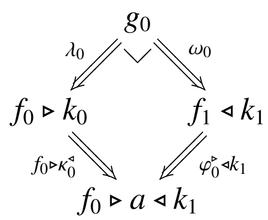
Proof. The dualization operation $(-)^*$ on cyclic Chu-spans and cyclic Chu-span morphisms provides the required family of equivalences.

If $\boldsymbol{\varphi}$ is cyclic from $A \xrightarrow{a} A$ to $B \xrightarrow{b} B$, the central component of $\boldsymbol{\varphi}^* \odot \boldsymbol{\varphi}$ is given by a pullback e_1 of the cospan $f_1 \triangleright f_1 \xLeftarrow{f_1 \triangleright \varphi_1^{\triangleleft}} f_1 \triangleright b \triangleleft f_0 \xRightarrow{\varphi_1^{\triangleright} \triangleleft f_0} f_0 \triangleleft f_0$. Exponential transposes of the identities on f_0 and f_1 now yield the central component $\top_A \Longrightarrow e_1$ of a Chu-morphism $\boldsymbol{\varphi}^* \odot \boldsymbol{\varphi} \xRightarrow{\boldsymbol{ev}_{\varphi}} (\mathbf{1}_a)^*$ with outer component φ_0. The right extension property and the naturality of the right extensions are routine verifications. ■

3.9. Remark. While the tensor-compositions in $\boldsymbol{cChu_1}(\mathcal{B})$ and $\boldsymbol{Chu_1}(\mathcal{B})$ coincide, right extensions in $\boldsymbol{Chu_1}(\mathcal{B})$ of cyclic Chu-spans $a \xrightarrow{\boldsymbol{\varphi}} b$ and $a \xrightarrow{\boldsymbol{\kappa}} c$ will in general not be cyclic. Instead, in $\boldsymbol{cChu_1}(\mathcal{B})$ the right extension of $\boldsymbol{\kappa}$ along $\boldsymbol{\varphi}$ is given by



where



(3-09)

Just as in ordinary $*$-autonomous categories, this may be expressed in terms of $\otimes$ and $(-)^*$ as $(\boldsymbol{\varphi} \otimes \boldsymbol{\kappa}^*)^*$. Right liftings behave dually.

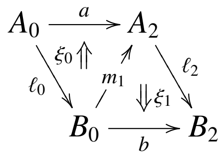
Recall that for a monoidal closed category $\mathcal{V}$ with finite limits the Chu-category $\boldsymbol{chu}\langle \mathcal{V}, t\rangle$ coincides with $\mathcal{V}^{\mathrm{op}} \times \mathcal{V}$. Hence the following result for Chu-bicategories, which extends the one for Chu-categories, is not surprising.

3.10. Proposition. *If $\mathcal{B}$ locally has $\mathcal{J}$-limits and $\mathcal{J}$-colimits, so do* $\boldsymbol{Chu_1}(\mathcal{B})$ *and* $\boldsymbol{cChu_1}(\mathcal{B})$.

Proof. For a functor $\mathcal{J} \xrightarrow{\boldsymbol{P}} (\boldsymbol{Chu_1}(\mathcal{B}))\langle a, b\rangle$ we obtain induced functors P_i, $i < 3$, from $\mathcal{D}$ into $\boldsymbol{B}\langle A_0, B_0\rangle$, $\boldsymbol{B}\langle B_0, A_2\rangle$ and $\boldsymbol{B}\langle A_2, B_2\rangle$, respectively. Consider limits $\langle \ell_0, \lambda_0\rangle$ and $\langle \ell_2, \lambda_2\rangle$ of P_0 and P_2, respectively, and a colimit $\langle \mu_1, m_1\rangle$ of P_1. For $j \in \mathcal{J}$ transposes of the 2-cell-components $P_1(j) \otimes P_0(j) \Longrightarrow a$ and $P_2(j) \otimes P_1(j) \Longrightarrow b$ of $\boldsymbol{P}(j)$ induce cocones

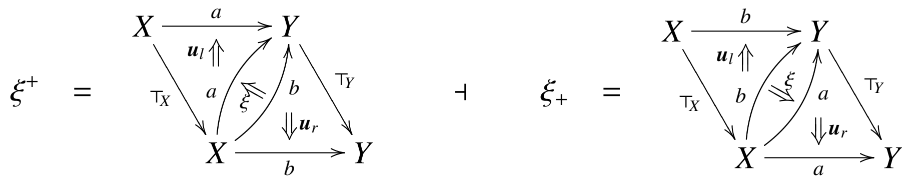
$$P_1(j) \Longrightarrow P_0(j) \triangleright a \xRightarrow{\lambda_{0\,j} \triangleright a} \ell_0 \triangleright a \quad \text{and} \quad P_1(j) \Longrightarrow b \triangleleft P_2(j) \xRightarrow{b \triangleleft \lambda_{2\,j}} b \triangleleft \ell_2$$

that both factor through μ_{1j}. Combination with the appropriate right extension $\boldsymbol{ev}$ and right lifting $\boldsymbol{ve}$ then yields a Chu-span



with projections $\langle \lambda_{0d}, \mu_{1d}, \lambda_{2d} \rangle$ into $\boldsymbol{P}(d)$ that clearly form a limit of $\boldsymbol{P}$. To obtain a colimit, start with a limit of P_1 and proceed dually, utilizing colimits of P_0 and P_2. ■

3.11. Remark. Since $\boldsymbol{Chu_1}(\mathcal{B})$ encompasses all bicategories $(\boldsymbol{spn}(\boldsymbol{B}\langle A_0, A_2\rangle))^{\mathrm{co}}$, all their maps (= right-adjoint spans) are preserved. In particular, every 2-cell $b \overset{\xi}{\Longrightarrow} a$ in $\boldsymbol{B}\langle X, Y\rangle$ induces a pair of adjoint Chu-spans



(3-10)

These admit particularly simple compositions with other Chu-spans that have domain, respectively, codomain b: just compose the appropriate 2-cell component with ξ.

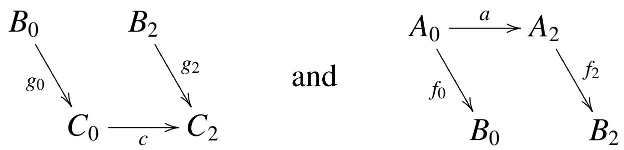
4. Relating $\mathcal{B}$ directly with the Chu-bicategories

In order to analyze how a closed bicategory $\mathcal{B}$ is related to $\boldsymbol{Chu_1}(\mathcal{B})$ and $\boldsymbol{cChu_1}(\mathcal{B})$, we first investigate, whether the latter form extensions of $\mathcal{B}$.

Then each object A of $\mathcal{B}$ ought to be mapped to a 1-cell of $\mathcal{B}$. The only canonical ones would seem to be $\top_A$, and perhaps the terminal 1-cell t_A, provided finite limits exist locally. In both cases $\boldsymbol{cChu_1}(\mathcal{B})$ appears to be a more natural candidate for an embedding of $\mathcal{B}$.

$\boldsymbol{Chu}(\mathcal{B})$, on the other hand, may be better suited as an extension of $\mathcal{B} \times \mathcal{B}$. Again the question arises, which canonical 1-cell $A_0 \longrightarrow A_2$ to choose. Unless the hom-categories of $\mathcal{B}$ have terminal objects, it is not clear which other suitable candidates to choose.

Turning the problem around, the (strict) domain/codomain functors $\boldsymbol{cChu_1}(\mathcal{B}) \longrightarrow \mathcal{B}$ and $\boldsymbol{Chu_1}(\mathcal{B}) \longrightarrow \mathcal{B} \times \mathcal{B}$ map a Chu-span $\boldsymbol{\varphi} = \langle f_0, \varphi_0, f_1, \varphi_1, f_2 \rangle$ to f_0 and $\langle f_0, f_2 \rangle$, respectively. For closed $\mathcal{B}$ with local pullbacks, we suspect that diagrams of the form

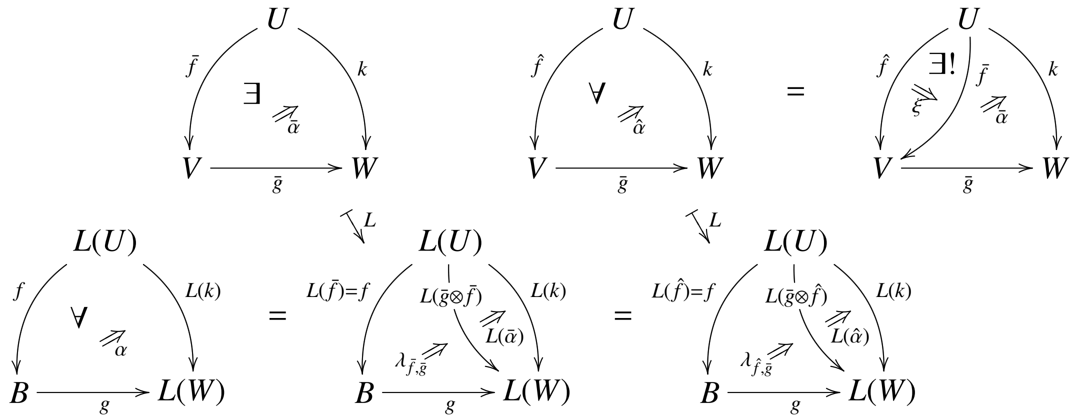


can be completed to Chu-spans into c, respectively, from a in some optimal fashion, where in the cyclic case we require $g_0 = g_2$, respectively $f_0 = f_2$. Technically this means that the above functors should be either lax or oplax counterparts of bifibrations, that is, admit the appropriately weakened form of "initial" and "terminal" lifts.

4.1. Definition. Given a lax functor $\mathcal{X} \xrightarrow{\langle L,\lambda\rangle} \mathcal{Y}$ and a 1-cell $B \xrightarrow{g} L(W)$ in $\mathcal{Y}$, we call a 1-cell $V \xrightarrow{\bar{g}} W$ in $\mathcal{X}$ a *lax initial lift* of g, provided that

- $L(\bar{g}) = g$, and
- for any 2-cell $g \otimes f \xRightarrow{\alpha} L(k)$ in $\mathcal{Y}$, there exists a 2-cell $\bar{g} \otimes \bar{f} \xRightarrow{\bar{\alpha}} k$ in $\mathcal{X}$ such that
  - $L(\bar{f}) = f$;
  - $\alpha = (L(\bar{\alpha})) \circ \lambda_{\bar{f},\bar{g}}$;
  - any other 2-cell $\bar{g} \otimes \hat{f} \xRightarrow{\hat{\alpha}} k$ in $\mathcal{X}$ with $L(\hat{f}) = f$ and $\alpha = (L(\hat{\alpha})) \circ \lambda_{\hat{f},\bar{g}}$ factors through $\bar{\alpha}$ by means of a unique 2-cell $\hat{f} \xRightarrow{\xi} \bar{f}$ in $\mathcal{X}$.

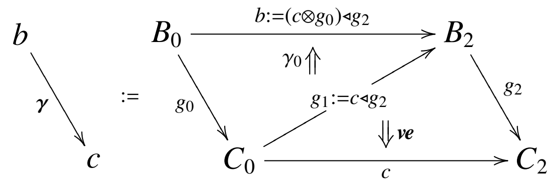
In form of pasting diagrams (starting in a zig-zag pattern at the lower left):



$\langle L, \lambda\rangle$ is a *lax fibration*, if each $B \xrightarrow{g} L(W)$ in $\mathcal{Y}$ has a lax initial lift. *Lax terminal lifts* and *lax opfibrations* are defined dually. A *lax bifibration* is both a lax fibration and a lax opfibration.

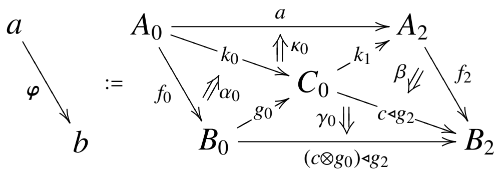
4.2. Theorem. *If $\mathcal{B}$ is closed and has local pullbacks, both functors $\boldsymbol{cChu_1}(\mathcal{B}) \longrightarrow \mathcal{B}$ and $\boldsymbol{Chu_1}(\mathcal{B}) \longrightarrow \mathcal{B} \times \mathcal{B}$ are lax bifibrations.*

Proof. In case of $\boldsymbol{Chu_1}(\mathcal{B}) \longrightarrow \mathcal{B} \times \mathcal{B}$ we need to construct a lax initial lift of $C_0 \xrightarrow{c} C_2$ along $\langle B_0 \xrightarrow{g_0} C_0, B_2 \xrightarrow{g_2} C_2\rangle$. Define a Chu-span into c by

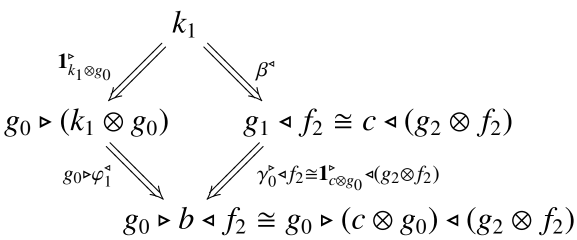


where γ_0 is a transpose of $\boldsymbol{ve} \otimes g_0$. For any Chu-span $\boldsymbol{\kappa} = \langle k_0, \kappa_0, k_1, \kappa_1, k_2\rangle$ from $A_0 \xrightarrow{a} A_2$ to c, and all 2-cells $g_i \otimes f_i \xRightarrow{\alpha_i} k_i$, $i \in \{0, 2\}$, in $\mathcal{B}$, we need to find an essentially unique Chu-morphism $\boldsymbol{\gamma} \odot \boldsymbol{\varphi} \xRightarrow{\alpha} \boldsymbol{\kappa}$ with outer components α_0 and α_2 such that every other Chu-morphism

$\boldsymbol{\gamma} \odot \hat{\boldsymbol{\varphi}} \overset{\alpha'}{\Longrightarrow} \boldsymbol{\kappa}$ with the same outer components factors as $\alpha' = \alpha \circ (\boldsymbol{\gamma} \odot \xi)$ by means of a unique Chu-morphism $\hat{\boldsymbol{\varphi}} \overset{\xi}{\Longrightarrow} \boldsymbol{\varphi}$. Concretely, with $\beta := (\kappa_1 \circ (\alpha_2 \otimes k_1))^{\triangleleft}$ we set



For $\varphi_1 = \gamma_0 \circ (\beta \otimes g_0)$ we then compute $(\varphi_1^{\triangleright})^{\triangleleft} = (\varphi_1^{\triangleleft})^{\triangleright}$ from k_1 to $g_0 \triangleright b \triangleleft f_2$:



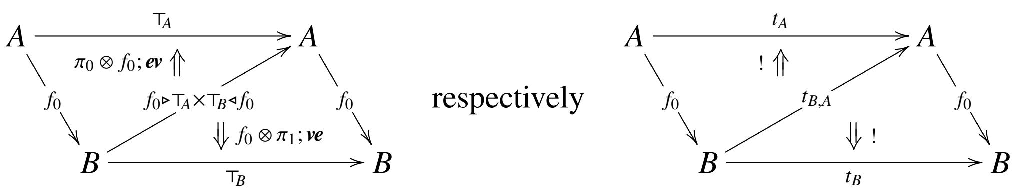
Here we utilized $(x \triangleleft g_2) \triangleleft f_2 \cong x \triangleleft (g_2 \otimes f_2)$ for $x = c$ and for $x = c \otimes g_0$. The central 1-cell e_1 of $\boldsymbol{\epsilon} := \boldsymbol{\gamma} \odot \boldsymbol{\varphi}$ being a pullback of the lower cospan induces the required 2-cell $k_1 \overset{\alpha_1}{\Longrightarrow} e_1$. A straightforward calculation establishes $\boldsymbol{\alpha} = \langle \alpha_0, \alpha_1, \alpha_2 \rangle$ as a Chu-morphism from $\boldsymbol{\gamma} \odot \boldsymbol{\varphi}$ to $\boldsymbol{\kappa}$.

Notice that the transpose $\boldsymbol{\varphi} \overset{\alpha^{\triangleright}}{\Longrightarrow} \boldsymbol{\kappa} \triangleright \boldsymbol{\gamma}$ has the identity on $k_1 \otimes g_0$ as central 1-cell. Hence if $\boldsymbol{\gamma} \odot \hat{\boldsymbol{\varphi}} \overset{\hat{\alpha}}{\Longrightarrow} \boldsymbol{\kappa}$ also has the outer components α_0 and α_2, the central 1-cell of its transpose $\hat{\boldsymbol{\varphi}} \overset{\hat{\alpha}^{\triangleright}}{\Longrightarrow} \boldsymbol{\kappa} \triangleright \boldsymbol{\gamma}$ provides us with a candidate for the central 1-cell of the desired $\boldsymbol{\varphi} \overset{\xi}{\Longrightarrow} \boldsymbol{\varphi}$, the outer 1-cells being fixed as identities. A simple calculation then shows $\hat{\alpha} = \alpha \circ (\boldsymbol{\gamma} \odot \xi)$.

If $f_0 = f_2$, the Chu-span $\boldsymbol{\gamma}$ is cyclic, hence the same construction applies.

"Lax terminal lifts" for both $\boldsymbol{Chu_1}(\mathcal{B}) \longrightarrow \mathcal{B} \times \mathcal{B}$ and $\boldsymbol{cChu_1}(\mathcal{B}) \longrightarrow \mathcal{B}$ are computed dually with the help of right extensions. ∎

In order to embed $\mathcal{B}$ into $\boldsymbol{cChu_1}(\mathcal{B})$ we need to choose endo-1-cells for each $\mathcal{B}$-object and Chu-spans for each 1-cell of $\mathcal{B}$. With identities and terminals, respectively, we can build Chu-spans that are closed under composition:

 (4-11)

In fact, $t_{B,A} = f_0 \triangleright t_a \times t_B \triangleleft f_0$, so both constructions follow the pattern of forming limits in $\boldsymbol{cChu_1}(\mathcal{B})$, see Proposition 3.10. However, the second choice is both "lax initial" and "lax terminal", while the first one has neither of these properties. This partially explains the usefulness of simple Chu-spans in [Bar96]. The second type of Chu-span also generalizes to the non-cyclic case. This yields an embedding of $\mathcal{B} \times \mathcal{B}$ into $\boldsymbol{Chu_1}(\mathcal{B})$, provided local terminals exist.

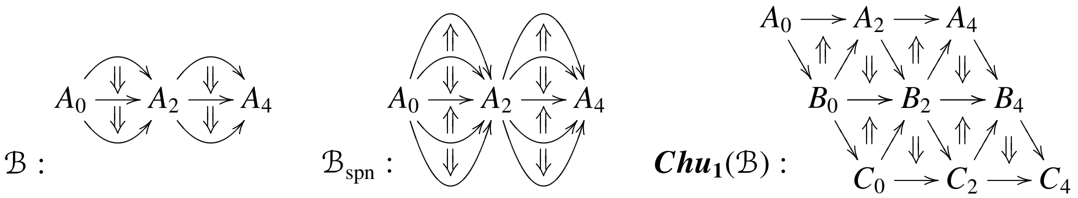
However, the cyclic Chu-spans of Diagram (4-11) are *not* closed under dualization $(-)^*$. So in case that $\mathcal{B}$ happens to be cyclic $*$-autonomous in the sense of Definition 3.6, these embeddings are not very useful. As Example 5.2 below will show, additional structure on $\mathcal{B}$ provides other possibilities for embeddings.

In view of these shortcomings a more conceptual comparison of $\mathcal{B}$ with the Chu-categories is needed.

A right-closed bicategory $\mathcal{B}$ may be specified by its hom-categories $\boldsymbol{B}\langle X,Y\rangle$ (where 2-cells are composed "vertically"), the composition functors $\boldsymbol{B}\langle X,Y\rangle \times \boldsymbol{B}\langle Y,Z\rangle \xrightarrow{\otimes} \boldsymbol{B}\langle X,Z\rangle$ (responsible for the "horizontal" composition of 1- and 2-cells), and the adjunctions $-\otimes r \dashv r \triangleright -$, respectively, $s\otimes - \dashv - \triangleleft s$ between appropriate hom-categories (expressing closedness).

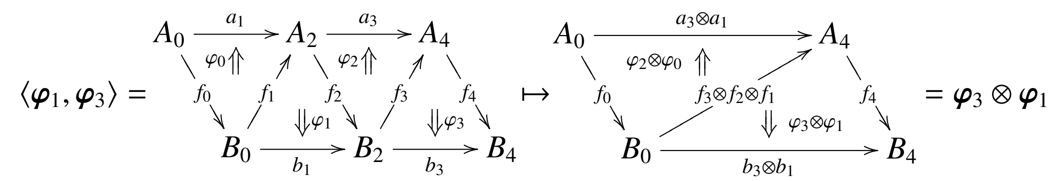
Provided $\mathcal{B}$ locally has pullbacks, we replacing its hom-categories by their bicategories of spans (with 2-cells reversed) yields a 3-dimensional structure. Since 1-cell composition in $\mathcal{B}$ is left rather than right adjoint, local limits need not be preserved. Hence the composition functors of $\mathcal{B}$ only extend to normal lax functors, and instead of an interchange law for vertical and horizontal span compositions, we only have a 3-cell in one direction.

Generalizing from ordinary spans to Chu-spans further reduces the possibilities of meaningful horizontal composition, but instead provides us with an extended vertical composition that even admits right extensions and right liftings – which could be seen as a trade-off. This justifies drawing Chu-spans vertically rather than horizontally: the composition of Chu-spans generalizes the vertical composition of $\mathcal{B}$, *not* the horizontal one. Schematically we have



(4-12)

To see concretely, why horizontal composition of Chu-spans is unlikely to work, consider



(4-13)

While for another pair of Chu-spans $\langle b_1, b_3\rangle \xrightarrow{\langle\gamma_1,\gamma_3\rangle} \langle c_1, c_3\rangle$ we can always construct a Chu-morphism $(\boldsymbol{\gamma}_3 \odot \boldsymbol{\varphi}_3)\otimes(\boldsymbol{\gamma}_1 \odot \boldsymbol{\varphi}_1) \Longrightarrow (\boldsymbol{\gamma}_3 \otimes \boldsymbol{\gamma}_1)\odot(\boldsymbol{\varphi}_3 \otimes \boldsymbol{\varphi}_1)$ with trivial outer components, there are serious problems with this approach.

- In general, there are no identities for this operation. In fact, f_2 admits a horizontal right or left identity iff f_2 is an isomorphism.

- This proposed horizontal composition of Chu-spans does not extend to Chu-morphisms; the different variance in the even and odd 1-cell components prevents us from finding a 2-cell from $f_3' \otimes f_2' \otimes f_1'$ to $f_3 \otimes f_2 \otimes f_1$, as required. In fact, it appears impossible to combine f_1, f_2 and f_3 into a 1-cell $B_0 \longrightarrow A_4$ that avoids this problem.

5. Bicategories of Chu-chains

While the idea of composing Chu-spans horizontally seems to hold little promise, the left side of Diagram (4-13) indicates the possibility of extending the notion of Chu-span to link *typed paths* of 1-cells in $\mathcal{B}$ by *chaining* matching Chu-spans together horizontally.

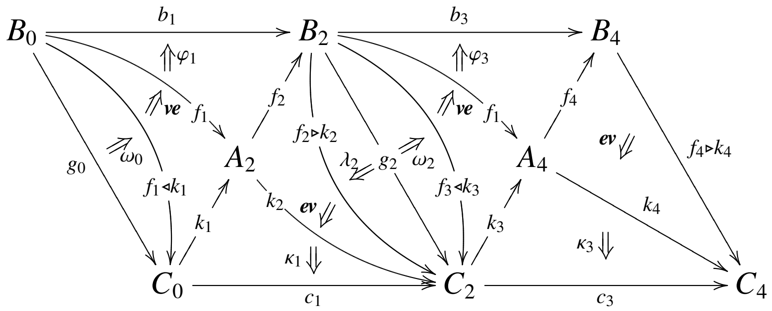
Since paths of length 0 are just objects, we expect to find $\mathcal{B}$ at the bottom of a hierarchy of right-closed bicategories $\boldsymbol{Chu_n}(\mathcal{B})$, $n \in \mathbb{N}$. Moreover, infinite paths can be considered as well.

Our indexing scheme for the 1-cell-paths is intended to avoid the need for index-tuples or double indices, which would clutter up the notation even further.

5.1. Theorem. *Let $\mathcal{B}$ be right-closed with local pullbacks, and $n \in \mathbb{N}$. Typed 1-cell-paths $\boldsymbol{a} = \langle A_{2i} \xrightarrow{a_{2i+1}} A_{2i+2} : i < n \rangle$ of length n as objects, typed Chu-span paths as 1-cells and the evident $(2n+1)$-sequences of 2-cells in $\mathcal{B}$ of alternating variance as new 2-cells form a right-closed bicategory $\boldsymbol{Chu_n}(\mathcal{B})$.*

Similarly we obtain closed bicategories $\boldsymbol{Chu}_{\mathbb{N}}(\mathcal{B})$, $\boldsymbol{Chu}_{\mathbb{Z}}(\mathcal{B})$ and $\boldsymbol{Chu}_{\mathbb{Z}\setminus\mathbb{N}}(\mathcal{B})$, where the 1-cells $A_{2i} \xrightarrow{a_{2i+1}} A_{2i+2}$ in the objects are indexed by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Z} \setminus \mathbb{N}$, respectively.

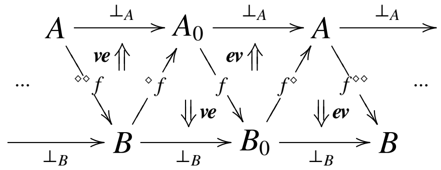
Proof. For $n \in \{0, 1\}$ we recover $\mathcal{B}$ and $\boldsymbol{Chu_1}(\mathcal{B})$, respectively. For $n > 1$ the operations in $\boldsymbol{Chu_n}(\mathcal{B})$ work component-wise, except the formation of right extensions and right liftings. These take the neighboring component into account, and only the rightmost component of a right extension has the shape of Diagram (3-07), while the other components have the shape of Diagram (3-09). For example, $\langle a_1, a_3 \rangle \xrightarrow{\langle \varphi_1, \varphi_3 \rangle} \langle b_1, b_3 \rangle$ and $\langle a_1, a_3 \rangle \xrightarrow{\langle \kappa_1, \kappa_3 \rangle} \langle c_1, c_3 \rangle$ have a right extension $\langle \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_3 \rangle \triangleright \langle \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_3 \rangle$ given by



(5-14)

The same phenomenon occurs for $\mathbb{Z} \setminus \mathbb{N}$-indexed paths. For left extensions, the leftmost component of finite or $\mathbb{N}$-indexed paths for lack of a left neighbor will display an exceptional shape, whereas in the $\mathbb{Z}$-indexed case no exceptional shapes occur. ∎

5.2. Example. We call a linear bicategory $\mathcal{B}$ in the sense of Remark 3.7 *right-closed*, if any $A \xrightarrow{f} B$ has so-called left and right "linear adjoints" $f^{\diamond} := f \triangleright \bot_A$ and ${}^{\diamond}f =: \bot_B \triangleleft f$, where $\bot$ picks out the units for the second tensor $\oplus$. With $A_i = A$ and $B_i = B$ for all $i \in 2\mathbb{Z}$ we may consider a $\mathbb{Z}$-path of left and right "linear adjoints" of f

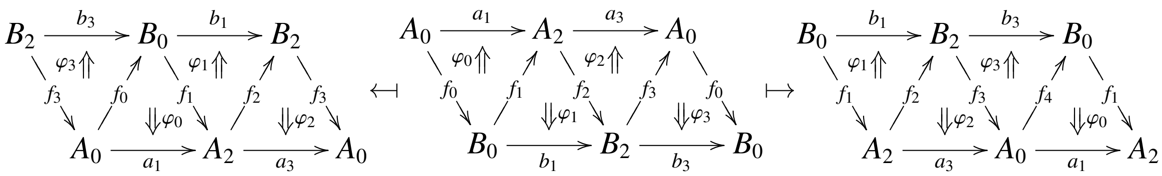


The 2-cell $f^{\diamond} \otimes f \overset{ev}{\Longrightarrow} \bot_A$ should be viewed as the counit of the linear adjunction $f^{\diamond} \dashv\!\!\mid f$; its unit is given by $(f^{\diamond\diamond} \otimes f^{\diamond} \overset{ev}{\Longrightarrow} \bot_B)^{\diamond} = (\top_B \overset{ev^{\diamond}}{\Longrightarrow} f \oplus f^{\diamond})$.

If $\mathcal{B}$ happens to be cyclic $*$-autonomous (compare Definition 3.6), ${}^{\diamond}f$ and $f^{\diamond}$ are coherently isomorphic to f^{*}. Then $\mathbb{Z}$-paths may be "curled up" into cyclic paths of lengths 1. Thus we obtain a $(-)^{*}$-preserving embedding of $\mathcal{B}$ into $\boldsymbol{cChu_1}(\mathcal{B})$.

Let us now turn to the general cyclic case. What is $\boldsymbol{cChu_n}(\mathcal{B})$ supposed to be for $n \neq 1$? Recall that the cyclic $*$-autonomous bicategory $\boldsymbol{cChu_1}(\mathcal{B})$ sits inside $\boldsymbol{Chu_1}(\mathcal{B})$ as a non-1-full sub-bicategory containing all cyclic and hence reversible 1-cells. For $n > 1$ one can consider the corresponding sub-bicategory of $\boldsymbol{Chu_n}(\mathcal{B})$. However, cyclicity of Chu-span-paths in general will no longer insure unique reversibility as for $n = 1$.

5.3. Example. For $n = 2$ a cyclic pair of Chu-spans $\langle a_1, a_3\rangle \xrightarrow{\langle\varphi_1,\varphi_3\rangle} \langle b_1, b_3\rangle$ has two different possibilities for reversal



which might be denoted by ${}^{\diamond}\langle\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_3\rangle$ and $\langle\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_3\rangle^{\diamond}$, respectively. The operations ${}^{\diamond}(-)$ and $(-)^{\diamond}$ are clearly inverses, cyclically shifting the rightmost, respectively, leftmost 2-cell to the other side, which after four iterations reproduces $\langle\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_3\rangle$. In general, ${}^{\diamond\diamond}\langle\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_3\rangle = \langle\boldsymbol{\varphi}_3, \boldsymbol{\varphi}_1\rangle = \langle\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_3\rangle^{\diamond\diamond}$ differs from $\langle\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_3\rangle$.

We now obtain an analogon of a linear adjunction, compare Example 5.2: The same computation we utilized in the proof of Theorem 3.8 generates canonical 2-cells

$$\langle\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_3\rangle \otimes {}^{\diamond}\langle\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_3\rangle \Longrightarrow {}^{\diamond}\langle\mathbf{1}_{b_1}, \mathbf{1}_{b_3}\rangle \quad \text{and} \quad \langle\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_3\rangle^{\diamond} \otimes \langle\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_3\rangle \Longrightarrow \langle\mathbf{1}_{a_1}, \mathbf{1}_{a_3}\rangle^{\diamond}$$

However, for $n > 1$ it does not seem possible to combine $\otimes$, ${}^{\diamond}(-)$ and $(-)^{\diamond}$ into a sensible $\oplus$-operation, and thus to obtain a linear structure, on the sub-bicategory of $\boldsymbol{Chu_n}(\mathcal{B})$ generated by the cyclic Chu-spans.

Turning to infinite paths, every 1-cell $\boldsymbol{a} \xrightarrow{\boldsymbol{\varphi}} \boldsymbol{b}$ of $\boldsymbol{Chu}_{\mathbb{Z}}(\mathcal{B})$ is cyclic, but unless $\boldsymbol{\varphi}$ is periodic, all the 1-cells ${}^{\diamond^{2n}}\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}^{\diamond^{2n}}$, $n \in \mathbb{N}$, will be distinct. Again, we obtain the counterpart to a linear adjunction, namely canonical 2-cells $\boldsymbol{\varphi} \otimes {}^{\diamond}\boldsymbol{\varphi} \Longrightarrow {}^{\diamond}(\mathbf{1}_{\boldsymbol{b}})$ and $\boldsymbol{\varphi}^{\diamond} \otimes \boldsymbol{\varphi} \Longrightarrow (\mathbf{1}_{\boldsymbol{a}})^{\diamond}$, but no $\oplus$-operation.

Finally, even though none of the 1-cells in $\boldsymbol{Chu}_{\mathbb{N}}(\mathcal{B})$ is cyclic, we can still define an operation $(-)^{\diamond}$ on all 1-cells $\boldsymbol{a} \xrightarrow{\boldsymbol{\varphi}} \boldsymbol{b}$, which simply removes the leftmost 2-cell $f_1 \otimes f_0 \overset{\varphi_0}{\Longrightarrow} a_1$. Now there is only one canonical 2-cell $\boldsymbol{\varphi}^{\diamond} \otimes \boldsymbol{\varphi} \Longrightarrow (\mathbf{1}_{\boldsymbol{a}})^{\diamond}$ without a counterpart: the absence of rightmost 2-cells in $\mathbb{N}$-indexed Chu-span paths prevents us from defining a ${}^{\diamond}(-)$-operation.

While the sub-bicategories of $\boldsymbol{Chu_n}(\mathcal{B})$, $n > 1$, described in the preceding example may be of some independent interest, they, as well as $\boldsymbol{Chu}_{\mathbb{Z}}(\mathcal{B})$, fail to be cyclic $*$-autonomous, apparently due to a shortage of 1-cells. To fix this, we propose a slight shift in perspective.

5.4. Definition. We write $\boldsymbol{n}_-$ and $\boldsymbol{n}_\circ$ for the graphs with node-set $n+1 = \{ i \in \mathbb{N} : i \leq n \}$ and the successor relation, respectively, node-set n and the successor relation modulo n. Similarly, the graphs $\mathbb{N}_-$ and $\mathbb{Z}_\circ$ have node-sets $\mathbb{N}$, respectively, $\mathbb{Z}$ and the successor relation.

For simplicity, we will refer to graph morphisms from $\boldsymbol{n}_\circ$ or from $\mathbb{Z}_\circ$ into $\mathcal{B}$ as $\mathcal{B}$-*loops* and to 1-cells between these as *Chu-loops*.

An object $\boldsymbol{a}$ of $\boldsymbol{Chu_n}(\mathcal{B})$ can be viewed as a graph morphism from $\boldsymbol{n}_-$ into $\mathcal{B}$, mapping i to A_{2i} and $\langle i, i+1 \rangle$ to a_{i+1}. If the endpoints A_0 and A_{2n} accidentally coincide, $\boldsymbol{a}$ may be used as domain or codomain for cyclic Chu-span paths as well as non-cyclic ones. In Example 5.3 the paths

$$A_0 \xrightarrow{a_1} A_2 \xrightarrow{a_3} A_0 \quad \text{and} \quad A_2 \xrightarrow{a_3} A_0 \xrightarrow{a_1} A_2$$

are different images of the graph $2_- = (0 \longrightarrow 1 \longrightarrow 2)$, whereas they can be viewed as different presentations of a single image of $2_\circ = (0 \rightleftarrows 1)$. Clearly, it is preferable to consider graph-morphisms from $\boldsymbol{n}_\circ$ as objects, when we are interested just in cyclic $\mathcal{B}$-loops. Then Chu-loops take the shape of "triangulated cylinders", which can readily be turned upside down. This operation ought to be an involution.

Observe that the graphs $\boldsymbol{n}_\circ$ and $\mathbb{Z}_\circ$ admit non-trivial endomorphisms, which are in fact automorphisms, in contrast to the "rigid" graphs $\boldsymbol{n}_-$ and $\mathbb{N}_-$. While any linear Chu-span path $\boldsymbol{a} \xrightarrow{\varphi} \boldsymbol{b}$ has to start with a 1-cell $A_0 \xrightarrow{f_0} B_0$, in case of a Chu-loop we can relax this requirement of *matching base points* and allow the codomain of the 1-cell originating at A_0, say f_0, to be B_{2k} for some $k \in \mathbb{Z}$; in case of loops of length n all subscripts are to be understood modulo $2n$. The value k identifies an automorphism to be performed on the target $\mathcal{B}$-loop before building a Chu-loop. The latter will also contain 1-cells $A_{-2k} \longrightarrow B_0$ and $B_0 \longrightarrow A_{2-2k}$. By making k part of the 1-cells, we obtain $\langle k : \boldsymbol{\varphi} \rangle^* = \langle 1-k : \boldsymbol{\varphi}^* \rangle$.

5.5. Definition. For $n > 0$ the bicategory $\boldsymbol{cChu_n}(\mathcal{B})$ has graph-morphisms $\boldsymbol{n}_\circ \longrightarrow \mathcal{B}$ as objects. Its 1-cells $\boldsymbol{a} \xrightarrow{\langle k:\boldsymbol{\varphi} \rangle} \boldsymbol{b}$ consist of an automorphism $\boldsymbol{n}_\circ \xrightarrow{k} \boldsymbol{n}_\circ$ and a Chu-loop $\boldsymbol{a} \xrightarrow{\boldsymbol{\varphi}} \boldsymbol{b} \circ k$, while 2-cells $\langle k : \boldsymbol{\varphi} \rangle \xRightarrow{\rho} \langle k' : \boldsymbol{\varphi}' \rangle$ only exist for $k = k'$ and consist of 2-cells $f_{2i} \xRightarrow{\rho_i} f'_{2i}$ and $f'_{2i+1} \xRightarrow{\rho_{2i+1}} f_{2i+1}$, $i < n$, subject to the evident axioms. $\boldsymbol{cChu}_{\mathbb{Z}}(\mathcal{B})$ is defined analogously.

5.6. Proposition. *The bicategories* $\boldsymbol{cChu_n}(\mathcal{B})$, $n > 0$, *and* $\boldsymbol{cChu}_{\mathbb{Z}}(\mathcal{B})$ *are cyclic* $*$-*autonomous, and the canonical functors into the subcategories of graphs with single object* $\boldsymbol{n}_\circ$, *respectively,* $\mathbb{Z}_\circ$ *are bifibrations.*

Proof. Cyclic $*$-autonomy is an immediate consequence of our considerations above. For example, the right extension of $\boldsymbol{a} \xrightarrow{\langle p:\boldsymbol{\kappa} \rangle} \boldsymbol{c}$ along $\boldsymbol{a} \xrightarrow{\langle i:\boldsymbol{\varphi} \rangle} \boldsymbol{b}$ is given by $\boldsymbol{b} \xrightarrow{\langle p-i:\boldsymbol{\varphi} \triangleright \boldsymbol{\kappa} \rangle} \boldsymbol{c}$. All morphisms in the base being automorphisms, the existence of initial and terminal lifts is trivial. ■

Each of these rather simple bifibrations has a single fibre corresponding to the subcategory of $\boldsymbol{Chu_n}(\mathcal{B})$ described in Example 5.3, respectively, $\boldsymbol{Chu}_{\mathbb{Z}}(\mathcal{B})$.

For $n > 0$ the bicategories $\boldsymbol{cChu_n}(\mathcal{B})$ clearly can be recovered within $\boldsymbol{cChu}_{\mathbb{Z}}(\mathcal{B})$ as non-1-full sub-bicategories of n-periodic $\mathcal{B}$-loops and n-periodic Chu-loops; this embedding preserves right extensions and right liftings. Thus in a sense $\boldsymbol{cChu}_{\mathbb{Z}}(\mathcal{B})$ addresses most aspects of cyclic $*$-autonomy related to $\mathcal{B}$. Of course, this rather satisfactory situation does not carry over to non-cyclic paths.

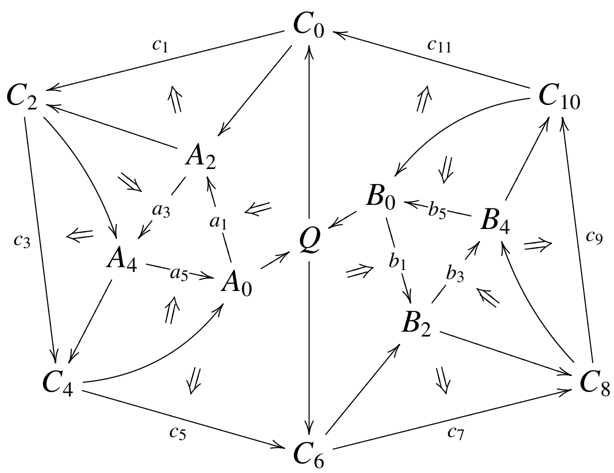
Our observations in the cyclic case so far raise at least two issues beyond the scope of the current paper: what is $\boldsymbol{cChu_0}(\mathcal{B})$ supposed to be, and how can we combine all these single fibers into a more meaningful bifibration over some subcategory of $\boldsymbol{grph}$, such that the total bicategory remains cyclic $*$-autonomous?

Clearly, the objects of a hypothetical bicategory $\boldsymbol{cChu_0}(\mathcal{B})$ ought to be just $\mathcal{B}$-objects, respectively identity 1-cells. In order to guarantee uniquely reversible 1-cells, we may either consider isomorphisms, which yields a non-1-full sub-bicategory of $\boldsymbol{Chu_0}(\mathcal{B}) = \mathcal{B}$, or adjoint equivalences, which yields a non-1-full sub-bicategory of $\boldsymbol{cChu_1}(\mathcal{B})$. We tend to prefer the second choice, but both allow us to at least partially address the second question.

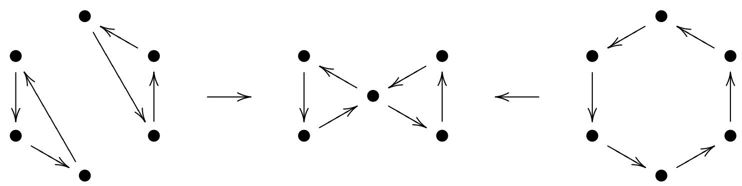
Recall the geometric interpretation of 1-cells in the Chu-bicategories between $\mathcal{B}$-loops of the same size as triangulated cylinders. Those triangles, which carry an identity 2-cell, or at least the counit of an adjoint equivalence, should essentially be collapsible. In other words, if we extend two $\mathcal{B}$-loops $\boldsymbol{a}$ and $\boldsymbol{b}$ of potentially different size by inserting identity 1-cells until the resulting $\mathcal{B}$-loops $\boldsymbol{a'}$ and $\boldsymbol{b'}$ have the same length, then any Chu-loop $\boldsymbol{a'} \xrightarrow{\varphi} \boldsymbol{b'} \circ k$ whose 2-cell components into the newly inserted 1-cells are identities, or counits of adjoint equivalences, ought to qualify as a Chu-loop from $\boldsymbol{a}$ to $\boldsymbol{b}$. This suggests some form of bisimulation between graphs as a candidate for the morphisms in the base of our hypothetical bifibration.

Of course, this also works for non-cyclic $\mathcal{B}$-paths, hence the bicategories $\boldsymbol{Chu_n}(\mathcal{B})$, $n \in \mathbb{N}$, should form the fibers of a larger entity as well.

Taking the speculation a bit further, we might even hope to mimic some form of cobordism in this context. For example, two disjoint $\mathcal{B}$-loops $\boldsymbol{a}$ and $\boldsymbol{b}$ could be linked to a single $\mathcal{B}$-loop $\boldsymbol{c}$ in the following fashion



Here the 1-cells $A_0 \longrightarrow C_0$, $A_0 \longrightarrow C_6$ as well as $B_0 \longrightarrow C_0$ and $B_0 \longrightarrow C_6$ factor through a common object Q. The underlying $\boldsymbol{grph}$-morphisms form a cospan



6. Symmetry and the game product

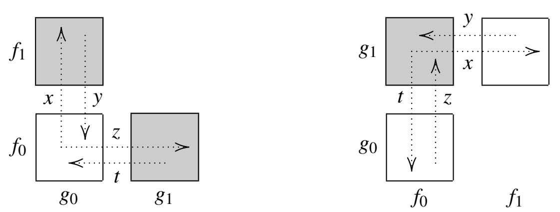
Interactions and games have lately been studied to find models for certain fragments of linear logic. They have mostly been represented in terms of trees (compare for example, [Bla92] [Abr97], [AJ94], [HO93], and [Hyl97]), but here we wish to use labeled transition systems (LTSs) instead. These can be viewed as graph morphisms from a small graph into $\boldsymbol{rel}$, or even into $\boldsymbol{spn}$, if one wants to allow repetition of labels along parallel arrows.

In order to model interaction, we wish to use *bipartite* graphs with two nodes, 0 and 1, for *Opponent* and *Player*, respectively. If the interaction is to be strictly alternating, the hom-sets $[0,0]$ and $[1,1]$ will be empty, that is, there are no internal transitions. Call the other hom-sets $a := [0,1]$ and $b := [1,0]$. A graph morphism into $\boldsymbol{rel}$ now specifies state sets f_0 (for Opponent) and f_1 (for Player) together with a function $a \longrightarrow \boldsymbol{rel}\langle f_0, f_1\rangle = \mathcal{P}(f_0 \times f_1)$ assigning to each element $x \in a$ the set of x-labeled *moves* from states in f_0 to states in f_1, and a similar function $b \longrightarrow \boldsymbol{rel}\langle f_1, f_0\rangle = \mathcal{P}(f_1 \times f_0)$. Of course, these functions are equivalent to relations $f_0 \times f_1 \xrightarrow{\varphi_0} a$ and $f_1 \times f_0 \xrightarrow{\varphi_1} b$.

Observe that $\boldsymbol{rel}$ is monoidal closed with respect to the cartesian product $\times$: exponentiation is also given by $\times$, but now interpreted as a functor $\boldsymbol{rel}^{\mathrm{op}} \times \boldsymbol{rel} \longrightarrow \boldsymbol{rel}$. Consequently, the evaluation relations $a \times (a \times b) \xrightarrow{\boldsymbol{ev}} b$ and $(a \times b) \times b \xrightarrow{\boldsymbol{ve}} a$ as transposes of the diagonal on $a \times b$ satisfy $\langle w, \langle x, y\rangle, z\rangle \in \boldsymbol{ev}$, respectively, $\langle\langle x, y\rangle, z, w\rangle \in \boldsymbol{ve}$ iff $x = w$ and $y = z$. Hence a bipartite LST may be identified with a cyclic Chu-span in the symmetric monoidal closed category $\langle \boldsymbol{rel}, \times, 1\rangle$, compare Diagram (2-01).

Similarly, non-cyclic Chu-spans may be thought of as "tripartite" LTSs. If in the cyclic case a set $f_0^0 \subseteq f_0$ of initial states for Opponent is specified, we can define the subsets $f_1^{2k+1} \subseteq f_1$ and $f_0^{2k+2} \subseteq f_0$ reachable after $2k+1$ and $2k+2$ steps, respectively. The corresponding finite or $\mathbb{N}$-indexed path of Chu-spans can be thought of as the *unfolding* or *trellis* of the interaction.

If $\boldsymbol{\gamma} = \langle g_0, \gamma_0, g_1, \gamma_1, g_0\rangle$ is another bipartite LTS with move sets c for Opponent and d for Player, a standard game-theoretic operation is to interleave $\boldsymbol{\gamma}$ with $\boldsymbol{\varphi}$. Of particular interest are those cases, where only one participant is allowed to switch games. We distinguish $\boldsymbol{\omega} := \boldsymbol{\varphi} \boxtimes \boldsymbol{\gamma}$ (only Opponent can switch) and $\boldsymbol{\pi} := \boldsymbol{\varphi} \multimap \boldsymbol{\gamma}$ (only Player can switch). The moves keep one state-component fixed, as indicated in the following schematic state transition diagram with $x \in a$, $y \in b$, $z \in c$ and $t \in d$:



$$w_0 = g_0 \times f_0$$
$$w_1 = g_0 \times f_1 + g_1 \times f_0$$
$$p_0 = f_0 \times g_0 + f_1 \times g_1$$
$$p_1 = f_0 \times g_1$$

$$a + c \xrightarrow{\varphi \boxtimes \gamma} b + d \qquad c + b \xrightarrow{\varphi \multimap \gamma} a + d$$

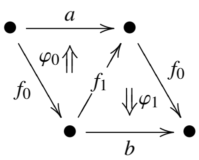
Switching between games is indicated by a change of the transition's direction. In case of unrestricted interleaving, Opponent's state set is $f_0 \times g_0 + f_1 \times g_1$, while Player's state set is $g_0 \times f_1 + g_1 \times f_0$. Hence allowing only Opponent or Player to switch games effectively renders parts of their state sets unreachable.

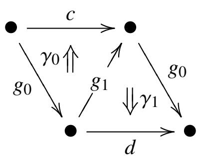
Clearly, the operation $\boxtimes$ is symmetric and we have

$$\boldsymbol{\varphi} \multimap \boldsymbol{\gamma} = (\boldsymbol{\varphi} \boxtimes \boldsymbol{\gamma}^*)^* \quad \text{and} \quad (\boldsymbol{\gamma} \boxtimes \boldsymbol{\varphi}) \multimap \boldsymbol{\delta} = \boldsymbol{\gamma} \multimap (\boldsymbol{\varphi} \multimap \boldsymbol{\delta})$$

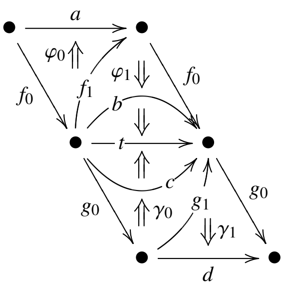
We emphasize again that the domains and codomains of the cyclic Chu-spans do not need to match for these operations to make sense.

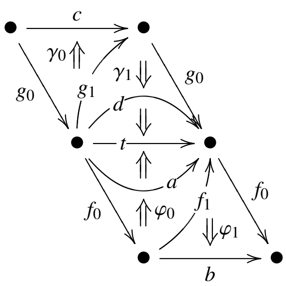
Recall that disjoint union $+$ in $\boldsymbol{rel}$ provides the categorical product, in particular $\emptyset$ is the terminal object. Since the constructions above do *not* use 2-cells of $\boldsymbol{rel}$, they must be available in any symmetric monoidal closed category $\mathcal{V}$ with finite products. We denote the identity for $\otimes$ by $\top$, the terminal object by t, and write τ for the terminal projections. While two arbitrary cyclic Chu-spans

$\boldsymbol{\varphi} =$ 

and $\boldsymbol{\gamma} =$ 

in general cannot be composed, we can compose appropriate partial "simplifications":



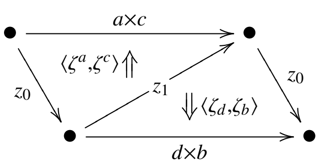
$\mapsto \boldsymbol{\gamma}^t \odot \boldsymbol{\varphi}_t$ and 

$\mapsto \boldsymbol{\varphi}^t \odot \boldsymbol{\gamma}_t$

where $\boldsymbol{\varphi}_t := \tau_+ \odot \boldsymbol{\varphi}$ and $\boldsymbol{\varphi}^t := \boldsymbol{\varphi} \odot \tau^+$, compare Remark 3.11. By the symmetry of $\otimes$, not only do the outer 1-cells $z_0 := g_0 \otimes f_0$ and $f_0 \otimes g_0$ agree, but also both central pullbacks coincide, which because of the terminal t reduce to products:

$$z_1 := g_0 \triangleright f_1 \times g_1 \triangleleft f_0 \cong f_0 \triangleright g_1 \times f_1 \triangleleft g_0$$

6.1. Definition. If $\boldsymbol{\gamma}^t \odot \boldsymbol{\varphi}_t = \langle z_0, \zeta^a, z_1, \zeta_d, z_0 \rangle$ and $\boldsymbol{\varphi}_t \odot \boldsymbol{\gamma}_t = \langle z_0, \zeta^c, z_1, \zeta_b, z_0 \rangle$, combining both composites yields

$\boldsymbol{\gamma} \boxtimes \boldsymbol{\varphi} \quad :=$ 

Even though the motivating example $\langle \boldsymbol{rel}, \times, 1 \rangle$ fails to have all pullbacks and hence does not allow the formation of a Chu-bicategory, let us collect the results above.

6.2. Proposition. *For any symmetric monoidal closed category $\langle \mathcal{V}, \otimes, \top \rangle$ with finite products, up to natural isomorphism the operation $\boxtimes$ on cyclic Chu-spans is symmetric, associative and has the identity Chu-span $\mathbf{1}_t$ as a unit. Moreover, it extends to cyclic Chu-morphisms. Defining $\boldsymbol{\varphi} \multimap \boldsymbol{\gamma} := (\boldsymbol{\gamma}^* \boxtimes \boldsymbol{\varphi})^*$ from $b \times c$ to $d \times a$ yields an operation $\multimap$ that satisfies*

$$(\boldsymbol{\gamma} \boxtimes \boldsymbol{\varphi}) \multimap \boldsymbol{\delta} = (\boldsymbol{\gamma} \boxtimes \boldsymbol{\varphi} \boxtimes \boldsymbol{\delta}^*)^* = \boldsymbol{\gamma} \multimap (\boldsymbol{\varphi} \boxtimes \boldsymbol{\delta}^*)^* = \boldsymbol{\gamma} \multimap (\boldsymbol{\varphi} \multimap \boldsymbol{\delta})$$

and extends to cyclic Chu-morphisms as well. ∎

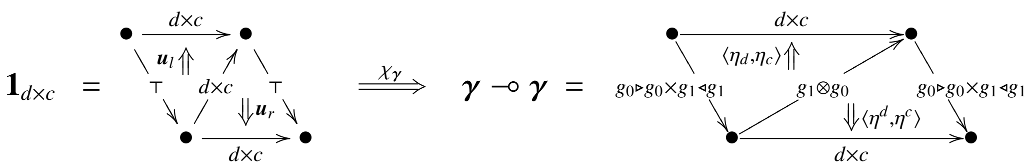
In the presence of pullbacks in $\mathcal{V}$ we obtain

6.3. Theorem. *If $\langle \mathcal{V}, \otimes, \top \rangle$ is symmetric monoidal closed and has finite limits, then the cyclic $*$-autonomous bicategory $\boldsymbol{cChu_1}(\mathcal{V})$ is symmetric monoidal with respect to $\boxtimes$, which on objects coincides with $\times$.*

6.4. Remark. Even though the operation $\boxtimes$ of $\boldsymbol{cChu_1}(\mathcal{V})$ coincides with the categorical product on objects, which thus trivially carry a cocommutative comonoid structure, in general $\boldsymbol{cChu_1}(\mathcal{V})$ will *not* be a cartesian bicategory in the sense of Carboni and Walters [CW87]. They in addition required every 1-cell to be a lax comonoid homomorphism. In our context for a Chu-span $a \xrightarrow{\boldsymbol{\varphi}} b$ this amounts to requiring Chu-morphisms between $(\iota(b))^+ \odot \boldsymbol{\varphi}$ and $(\iota(a))^+$, respectively, between $(\delta(b))^+ \odot \boldsymbol{\varphi}$ and $(\boldsymbol{\varphi} \boxtimes \boldsymbol{\varphi}) \odot (\delta(a))^+$, subject to certain axioms (where $a \xRightarrow{\iota(a)} \top$ and $a \xRightarrow{\delta(a)} a \times a$ constitute the product-induced comonoid structure on a in $\mathcal{V}$). But without canonical $\mathcal{V}$-morphisms between $\top$ and f, respectively, $f \otimes f$ and f for every $\mathcal{V}$-object f, such Chu-morphisms need not exist.

Now the obvious question arises, how to interpret cyclic Chu-spans in $\mathcal{V}$ as objects of a symmetric $*$-autonomous category with $\boxtimes$ as tensor, $(-)^*$ as dualization and $\multimap$ as internal hom. First we formulate the copy-cat strategy in terms of a Chu-morphism.

6.5. Definition. Given a cyclic Chu-span $c \xrightarrow{\boldsymbol{\gamma}} d$, let η^d and η_d be the 2-cell components of $\boldsymbol{\gamma}^t \odot \boldsymbol{\gamma}^*_t$, and let η^c and η_c be the 2-cell components of $(\boldsymbol{\gamma}^*)^t \odot \boldsymbol{\gamma}_t$. Setting $(\chi_{\boldsymbol{\gamma}})_0 = \langle \boldsymbol{id}^{\triangleright}_{g_0}, \boldsymbol{id}^{\triangleleft}_{g_1} \rangle$ and $(\chi_{\boldsymbol{\gamma}})_1 = \langle \gamma_0, \gamma_1 \rangle$ specifies a cyclic Chu-morphism



This suggests the possibility of encoding arrows (that is, strategies for games) from $a \xrightarrow{\boldsymbol{\varphi}} b$ to $c \xrightarrow{\boldsymbol{\gamma}} d$ by means of cyclic Chu-morphisms into $\boldsymbol{\varphi} \multimap \boldsymbol{\gamma}$. Hence given another cyclic Chu-span $e \xrightarrow{\boldsymbol{\delta}} f$, we wish to combine cyclic Chu-morphisms

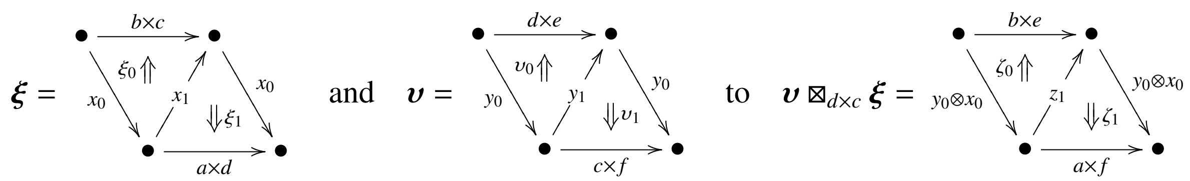
$$b \times c \underset{\boldsymbol{\varphi} \multimap \boldsymbol{\gamma}}{\overset{\boldsymbol{\xi}}{\rightrightarrows}} a \times d \; (\Downarrow \rho) \qquad \text{and} \qquad d \times e \underset{\boldsymbol{\gamma} \multimap \boldsymbol{\delta}}{\overset{\boldsymbol{\vartheta}}{\rightrightarrows}} c \times f \; (\Downarrow \sigma) \tag{6-15}$$

in such a way that allows us to obtain a Chu-span morphism into $b \times e \xrightarrow{\varphi \multimap \delta} a \times f$. But rather than simply forming

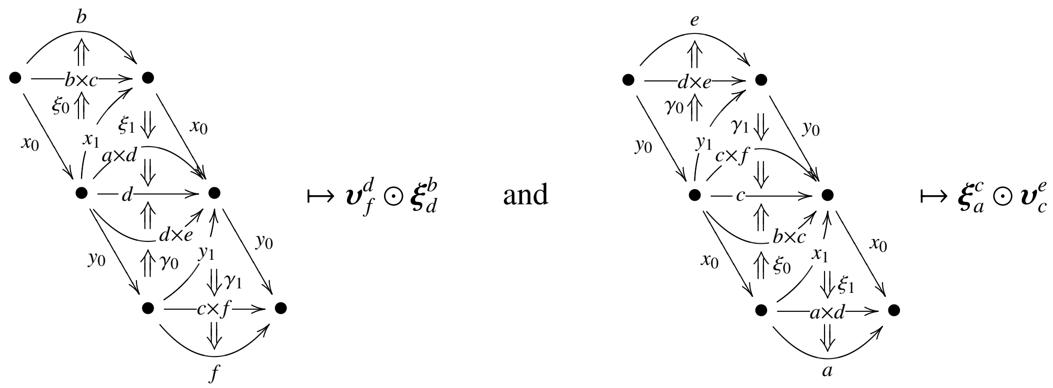
$$d \times e \times b \times c \underset{(\delta \multimap \gamma) \boxtimes (\varphi \multimap \delta)}{\overset{\vartheta \boxtimes \xi}{\underset{\Downarrow \sigma \boxtimes \rho}{\rightrightarrows}}} c \times f \times a \times d$$

and projecting out the required domain and codomain, we wish to take the partial matches between the domains and codomains into account, in other words, we wish to form a *trace*. So far the existence of finite products was sufficient to perform the Chu-span compositions we needed. However, we now will be concerned with non-trivial composites, which depend on the existence of pullbacks. Although ***rel*** in general lacks pullbacks, the following construction is of course inspired by the idea of hiding appropriate pairs of moves as indicated above.

6.6. Definition. In the spirit of Definition 6.1 we compose cyclic Chu-spans



by combining



where the unlabeled 2-cells denote projections compare Proposition 3.11. In this case z_1 is obtained by pulling back a transpose $y_o \triangleright x_1 \Longrightarrow y_0 \triangleright (c \times d) \triangleleft x_0 \Longleftarrow y_1 \triangleleft x_0$ of the induced cospan

$$x_1 \otimes x_0 \xRightarrow{\langle \xi_0;\pi_c,\xi_1;\pi_d \rangle} c \times d \xLeftarrow{\langle \upsilon_1;\pi_c,\upsilon_0;\pi_d \rangle} y_0 \otimes y_1$$

6.7. Theorem. *For every symmetric monoidal closed category $\mathcal{V}$, cyclic Chu-spans as objects and cyclic Chu-span morphisms into $\boldsymbol{\varphi} \multimap \boldsymbol{\gamma}$ as morphisms from $\boldsymbol{\varphi}$ to $\boldsymbol{\gamma}$ form a symmetric $*$-autonomous category $[\boldsymbol{cChu_1}(\mathcal{B})]^{\boxtimes}$ with tensor $\boxtimes$.*

Proof. Setting $\boldsymbol{\lambda} := \boldsymbol{\varphi} \multimap \boldsymbol{\gamma}$ and $\boldsymbol{\mu} := \boldsymbol{\gamma} \multimap \boldsymbol{\delta}$ and, we only need to construct a Chu-span morphism from $\boldsymbol{\omega} := \boldsymbol{\lambda} \boxtimes_{d \times c} \boldsymbol{\mu}$ to $\boldsymbol{\nu} := \boldsymbol{\varphi} \multimap \boldsymbol{\delta}$.

Although $\otimes$ need not preserve products, we certainly obtain 2-cells from

$$w_0 = m_0 \otimes \ell_0 = (g_0 \triangleright d_0 \times g_1 \triangleleft d_1) \otimes (f_0 \triangleright g_0 \times f_1 \triangleleft g_1)$$

into $(f_0 \triangleright g_0 \otimes g_0 \triangleright d_0) \times (f_1 \triangleleft g_1 \otimes g_1 \triangleleft d_1)$ and from there into $(f_0 \triangleright d_0) \times (f_1 \triangleleft d_1) = n_0$. On the other hand, transposing

$$f_0 \otimes d_1 \otimes (g_1 \triangleleft d_1) \xrightarrow{id \otimes ve} f_0 \otimes g_1 \qquad \text{and} \qquad (f_0 \triangleright g_0) \otimes f_0 \otimes d_1 \xrightarrow{ev \otimes id} g_0 \otimes d_1$$

induces 2-cells from $n_1 = f_0 \otimes d_1$ to $(g_1 \triangleleft d_1) \triangleright (g_1 \otimes f_0)$ and to $(d_1 \otimes g_0) \triangleleft (f_0 \triangleright g_0)$, and from there via symmetry and projections in the exponent to $m_0 \triangleright \ell_1$ and $m_1 \triangleleft \ell_0$. The universal property of the pullback w_1 now yields the desired 2-cell $n_1 \Longrightarrow w_1$.

The functoriality of $\boxtimes_{d\times c}$ is obvious. Given cyclic Chu-span morphisms as in Diagram (6-15), we define the composition of $\sigma \boxtimes_{d\times c} \rho$ with the Chu-span morphism constructed above to be the composite arrow $\sigma \circ \rho$ from $\boldsymbol{\varphi}$ to $\boldsymbol{\delta}$. The remaining verifications of the associativity and of the fact that the copy-cat Chu-spans are units for this composition are lengthy but straightforward. ∎

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