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BIPRODUCTS AND COMMUTATORS FOR NOETHERIAN FORMS

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ABSTRACT. Noetherian forms provide an abstract self-dual context in which one can establish homomorphism theorems (Noether isomorphism theorems and homological diagram lemmas) for groups, rings, modules and other group-like structures. In fact, any semi-abelian category in the sense of G. Janelidze, L. Márki and W. Tholen, as well as any exact category in the sense of M. Grandis (and hence, in particular, any abelian category), can be seen as an example of a noetherian form. In this paper we generalize the notion of a biproduct of objects in an abelian category to a noetherian form and apply it do develop commutator theory in noetherian forms. In the case of semi-abelian categories, biproducts give usual products of objects and our commutators coincide with the so-called Huq commutators (which in the case of groups are the usual commutators of subgroups). This paper thus shows that the structure of a noetherian form allows for a self-dual approach to products and commutators in semi-abelian categories, similarly as has been known for homomorphism theorems.

1. Introduction

A "noetherian form" is a mathematical structure that allows to recapture the noetherian isomorphism theorems for groups (and other group-like structures) in a general self-dual setting. This notion evolved through the following papers: [10], [11], [12], [13], [4]. A *noetherian form* can be defined as a category $\mathbb{C}$ for which

- every object A is equipped with a bounded lattice $\mathsf{sub}\, A$, called the lattice of *(formal) subobjects*,
- and every morphism $f\colon A \to B$ induces two maps

$$f_*\colon \mathsf{sub}\, A \to \mathsf{sub}\, B \quad \text{and} \quad f^*\colon \mathsf{sub}\, B \to \mathsf{sub}\, A$$

  called the *direct* and *inverse image* maps of f respectively.

The *image* of a morphism f is defined to be $\mathsf{Im}\, f = f_*1$, and the *kernel* is defined to be $\mathsf{Ker}\, f = f^*0$ (where 1 and 0 denote top and bottom elements of the subobject lattices). Further, a subobject is called *normal* if it occurs as a kernel of some morphism, and it is called *conormal* if it occurs as an image of some morphism. The category with this added

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structure must also satisfy five axioms listed at the start of Section 2. The definition of a noetherian form is self-dual in the following sense, where we extend categorical duality to:

- f_* is dual to f^*, for any morphism f;
- $X \leq Y$ is dual to $Y \leq X$, for any two subobjects X and Y of some object A.

It follows, in particular, that 'kernel' is dual to 'image', 'normal subobject' is dual to 'conormal subobject'.

Noetherian forms in which every subobject is both normal and conormal, and in which both products and coproducts exist, turn out to be precisely the abelian categories. In an abelian category products are the same as 'biproducts' — these can be seen as simultaneous products and coproducts. In this paper (building on [17]) we generalise the notion of a biproduct to noetherian forms, and as an application, develop commutator theory in the context of a noetherian form (having biproducts). For any two objects A and B, the *biproduct* of A and B is an object G equipped with four maps

$$A \underset{p_1}{\overset{e_1}{\rightleftarrows}} G \underset{p_2}{\overset{e_2}{\leftrightarrows}} B$$

such that $p_1e_1 = 1_A$, $p_2e_2 = 1_B$, $\mathsf{Im}\, e_1 = \mathsf{Ker}\, p_2$ and $\mathsf{Im}\, e_2 = \mathsf{Ker}\, p_1$, which satisfies two further, more technical conditions. This definition is self-dual in the sense that was previously described. The definition of biproducts was inspired by one of the characterizations of categorical biproducts in an abelian category found in [3]. The two further conditions were added to deduce that biproducts are unique up to isomorphism, but interestingly it turns out that under the assumption that biproducts exist, to check whether something is a biproduct, it is sufficient to only check the four explicitly stated equalities above.

These biproducts and (categorical) products share some common properties, like both give rise to a monoidal structure on a category. In fact, there are even relations between them, for example, in any noetherian form with products, coproducts and biproducts (thus necessarily pointed), the canonical morphism from the coproduct to the product factorizes through the biproduct. Another relation is that biproducts and products coincide exactly when the category is Barr-exact [1] and protomodular (in the sense of Bourn [2]). Consequently, biproducts and products coincide in any semi-abelian category [8] (seen as a noetherian form).

In this paper we explore the extent to which products in the context of a semi-abelian category have a self-dual treatment as biproducts in the more general context of a noetherian form. In particular, we show that this self-dual treatment is sufficient for recovering the theory of (Huq) commutators.

Defining commutators from biproducts is analogous as to how Huq commutators (see [7]) are defined from products. In the context of semi-abelian categories, where biproduct are exactly products, these commutators coincide with the Huq commutators.

For commutators, we mostly prove that results known for Huq commutators in semi-abelian categories hold more generally for our commutators. Using commutators we also introduce *commutative* objects in our context and prove that the collection of commutative objects of a noetherian form (with biproducts) constitutes a full reflective subcategory whose opposite category is Barr-exact and protomodular. Further, this subcategory is isomorphic to the category of internal monoids with respect to biproducts. Dually, the subcategory of cocommutative objects is Barr-exact and protomodular. This category falls short of being semi-abelian due to the absence of coproducts.

Any semi-abelian category determines a noetherian form, where the subobjects of an object are the categorical subobjects (equivalence classes of monomorphisms into the object). And as noted before, biproducts are exactly products and the commutators are Huq commutators. In a semi-abelian the subcategory of cocommutative objects is the entire category, while the subcategory of commutative objects is the subcategory of abelian objects. Another source of examples of noetherian forms is given by Grandis exact categories [5, 6]. Since existence of biproducts forces pointedness, Grandis exact categories having biproducts are forced to be Puppe-Mitchell exact [15, 16]. Biproducts in these categories (when they exist) are the same as split products in the sense of Grandis [5, 6] (which in general are not quite the same as split products, as defined in the present paper). In particular, in the category of sets and partial bijections, biproducts are given by disjoint union, while in the category of distributive lattices and modular connections (see [5, 6]), biproducts are the same as cartesian products of distributive lattice. In the first example, the commutator of two subsets is their intersection, while the cocommutator is their union. In the second example, where 'subobjects' of a distributive lattice are elements of the lattice, commutator is given by meet and cocommutator by join.

In fact, more generally, we will prove that in any noetherian form within which all conormal subobjects are normal and direct images preserve meets, commutators are given by meets of subobjects. Both this condition and its dual hold in the last two examples, this explains why commutators are meets and cocommutators are joins in those examples.

2. Preliminary results on noetherian forms

All the results in this section were established in [4].

We begin by listing the axioms that a noetherian form must satisfy:

(1) For any two composable morphisms f and g, $(fg)_* = f_*g_*$ and $(fg)^* = g^*f^*$, and for any identity morphism $(1_A)_* = 1_{\mathsf{sub}\,A} = 1_A^*$. Further for any morphism f, f_* and f^* forms a monotone Galois connection: for any subobject X of the domain and subobject Y of the codomain of f,

$$f_*X \leq Y \quad \Leftrightarrow \quad X \leq f^*Y.$$

(2) For any morphism f and subobject X of the domain and subobject Y of the codomain,

$$f_*f^*Y = Y \wedge \mathsf{Im}\, f \quad \text{and} \quad f^*f_*X = X \vee \mathsf{Ker}\, f.$$

(3) For any conormal $X \in \mathsf{sub}\, A$, there is a morphism $\iota_X \colon X_O \to A$, called the *embedding* of X, such that $(\iota_X)_* 1 \leq X$ and for any f into A with $f_* 1 \leq X$, there is a unique h such that $f = \iota_X h$. Also, for any normal $Y \in \mathsf{sub}\, A$, there is a morphism $\pi_Y \colon A \to A/Y$, called the *projection* of Y, such that $\pi_Y^* 0 \geq Y$, and for any g from A with $g^* 0 \geq Y$ there is a unique morphism k such that $k\pi_Y = g$.

(4) Any morphism f factorizes as $f = \iota_{\mathsf{Im}\, f} h \pi_{\mathsf{Ker}\, f}$ for some isomorphism h.

(5) the join of normal subobjects is normal, and dually the meet of conormal subobjects is conormal.

Note that, for a conormal subobject X of object A, by the notation X_O we mean that there is an embedding of X with domain X_O.

We will, for convenience, mostly denote $f_* X$ by fX, and $f^* Y$ by $f^{-1} Y$.

2.1. Remark. *In* [17]*, a stronger version of axiom (3) was used where existence of embeddings/projections was required for all subobjects and not only the conormal/normal ones.*

Here is a list of useful basic properties of a noetherian form, all of which follows directly from the axioms.

- $f0 = 0$ and $f^{-1}1 = 1$ for any morphisms f;
- direct images preserves joins, and inverse images preserves meets;
- for any conormal subobject C, $\mathsf{Im}\, \iota_C = C$, and dually for any normal subobject N, $\mathsf{Ker}\, \pi_N = N$;
- any morphism f is a projection (of some subobject) if and only if $\mathsf{Im}\, f = 1$, and dually it is an embedding if and only if $\mathsf{Ker}\, f = 0$;
- any split epi is a projection, and dually any split mono is an embedding;
- any projection is an epimorphism, and dually any embedding is a monomorphism;
- any morphism f is an isomorphism if and only if $\mathsf{Ker}\, f = 0$ and $\mathsf{Im}\, f = 1$;
- for any normal subobject N and projection p, pN is normal, and dually for any conormal subobject C and embedding m, $m^{-1}C$ is conormal.

Although this result wasn't explicitly stated in [4], the proof of this result was given in their proof of the Restricted Modular Law (the proposition hereafter). Since this result has some usefulness on its own, we make it explicit.

2.2. Proposition. *For a morphism f and a subobject X below the image of f and a normal subobject N of the codomain of f, we have*

$$f^{-1}(X \vee N) = f^{-1}X \vee f^{-1}N.$$

PROOF. Suppose $N = g^{-1}0$, for some morphism g. We then have

$$\begin{aligned} f^{-1}X \vee f^{-1}N &= f^{-1}X \vee f^{-1}g^{-1}0 \\ &= f^{-1}X \vee (gf)^{-1}0 \\ &= (gf)^{-1}(gf)f^{-1}X \\ &= f^{-1}g^{-1}gff^{-1}X \\ &= f^{-1}g^{-1}gX \\ &= f^{-1}(X \vee g^{-1}0) \\ &= f^{-1}(X \vee N). \end{aligned}$$

■

2.3. PROPOSITION. [Restricted Modular Law] *For any three subobjects X, Y, and Z of a object G, if Y is normal and Z is conormal (or if Y is conormal and X is normal), then*

$$X \leq Z \quad \Rightarrow \quad X \vee (Y \wedge Z) = (X \vee Y) \wedge Z.$$

PROOF. Suppose $Y = g^{-1}0$ and $Z = f1$ for some morphisms g and f, and suppose $X \leq Z$. We have

$$\begin{aligned} X \vee (Y \wedge Z) &= X \vee (g^{-1}0 \wedge f1) \\ &= ff^{-1}X \vee ff^{-1}g^{-1}0 \\ &= f(f^{-1}X \vee f^{-1}g^{-1}0) \\ &= ff^{-1}(X \vee g^{-1}0) \\ &= (X \vee g^{-1}0) \wedge f1 \\ &= (X \vee Y) \wedge Z. \end{aligned}$$

■

2.4. SEMI-ABELIAN AND RELATED CATEGORIES. From results in [10] (see also [13]), it follows that with ordinary categorical subobjects a semi-abelian category constitutes a noetherian form. Moreover, from [13] we know that any noetherian form in which any subobject is conormal is a semi-abelian category. Here we are going to prove a stronger result that any noetherian form in which inverse images of conormal subobjects are conormal, is semi-abelian, provided that it is pointed and has products and coproducts.

An example of a noetherian form, where there are more formal subobjects than categorical subobjects, is the form over the category of groups, where formal subobjects of a group G are pairs (X, Y) of subgroups of G. Direct and inverse images, and meets and joins are defined component-wise. The conormal subobjects in this form are those pairs of subgroups where the two subgroups are equal, thus this form cannot be (isomorphic to) the form of categorical subobjects. But here the inverse images of conormal subobjects are conormal.

2.5. Lemma. *In a noetherian form with a zero object, in which all bottom subobjects* 0 *are conormal (or dually, all top subobjects* 1 *are normal), the zero objects are exactly those objects* T *for which* $\mathsf{sub}\,T$ *has exactly one element. The zero morphisms are exactly those morphisms with kernel being* 1*, or equivalently, with image being* 0*.*

Proof. Suppose T is the zero object. Let $\iota_0\colon 0 \to T$ be the embedding of the conormal subobject 0 of T. Since T is in particular an initial object, there exists an $h\colon T \to 0$ and $\iota_0 h = 1_T$. Consequently ι_0 is an isomorphism. So $0 = \mathsf{Im}\,\iota_0 = 1$, and thus $\mathsf{sub}\,T$ only has one element. If $\mathsf{sub}\,T'$ has one element for any object T', then the unique morphism $t\colon T \to T'$ is necessarily an isomorphism.

If $f\colon G \to H$ is a zero morphism, then it factors through the zero object T, and consequently $f1 = 0$ and $f^{-1}0 = 1$. If f has image 0, then it has to factor through the embedding ι_0 of the bottom subobject of its codomain. The domain of ι_0 is a zero object. Thus f is a zero morphism. ■

2.6. Corollary. *In a noetherian form with zero object, all bottom subobjects are conormal if and only if all top subobjects are conormal.*

Proof. If either condition is satisfied then for any object there is at least one zero morphism to it and one from it. Consequently the top subobject is normal and the bottom subobject is conormal. ■

2.7. Lemma. *In a noetherian form, the diagram, where* m *is an embedding,*

$$\begin{array}{ccc} P & \overset{n}{\dashrightarrow} & A \\ {\scriptstyle g}\,\vdots & & \big\downarrow{\scriptstyle f} \\ C & \xrightarrow{m} & B \end{array}$$

has a pullback if and only if there is a largest conormal subobject $\underline{f^{-1}m1}$ *contained in* $f^{-1}m1$*. The pullback is given by* (g, n) *where* n *is the embedding of* $\underline{f^{-1}m1}$*.*

Proof. Suppose the pullback of m and f exists, that it is the diagram

$$\begin{array}{ccc} P & \xrightarrow{n} & A \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ C & \xrightarrow{m} & B \end{array}$$

We have

$$n1 \leq f^{-1}fn1 = f^{-1}mg1 \leq f^{-1}m1.$$

So $n1$ is a conormal subobject contained in $f^{-1}m1$. Suppose D is a conormal subobject contained in $f^{-1}m1$. Let ι_D be the embedding of D. Then

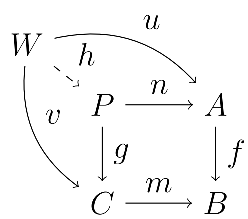
$$f\iota_D 1 \leq ff^{-1}m1 \leq m1.$$

Since m is an embedding, there is a unique u such that $mu = f\iota_D$. Since g and n is the pullback, there is in particular an h such that $nh = \iota_D$. Consequently $n1$ is the largest conormal subobject contained in $f^{-1}m1$.

For the converse, suppose there is a largest conormal subobject $\underline{f^{-1}m1}$ contained in $f^{-1}m1$. Let n be the embedding of it. We have

$$fn1 = f\underline{f^{-1}m1} \leq ff^{-1}m1 \leq m1.$$

Thus there is a unique g such that $mg = fn$. Consider the diagram



where $fu = mv$. We have

$$u1 \leq f^{-1}fu1 = f^{-1}mv1 \leq f^{-1}m1.$$

Thus $u1 \leq \underline{f^{-1}m1}$. Since n is the embedding of $\underline{f^{-1}m1}$, there is a unique h such that $nh = u$. We have

$$mgh = fnh = fu = mv.$$

And since m is a monomorphism, $gh = v$. And so g and n is the pullback of m and f. ∎

2.8. Corollary. *In a noetherian form with a zero object, where all the bottom subobjects are conormal, f has a categorical kernel if and only if there is a largest conormal subobject $\underline{f^{-1}0}$ contained in $f^{-1}0$. In either case, the embedding of $\underline{f^{-1}0}$ is the categorical kernel of f.*

2.9. Lemma. *Any noetherian form with pullbacks along embeddings and with finite products is finitely complete.*

Proof. Since any split monomorphism is an embedding, the lemma follows from the fact that pullbacks along split monomorphisms and finite products implies finitely complete. ∎

2.10. Lemma. *In a noetherian form with zero object, in which normal subobjects are conormal, the normal epimorphisms are exactly the projections. Further, the embeddings are exactly the monomorphisms.*

Proof. Suppose p is a normal epi. Then $p = me$ for some projection e and embedding m. Suppose p is the cokernel of f. Then $mef = 0 = m0$, which implies $ef = 0$. Thus there is an d such that $e = dp$. Thus $p = me = mdp$, and so $md = 1$. Consequently m is an isomorphism, and thus p a projection.

Conversely, take any projection p. Since $p^{-1}0$ is conormal, by Corollary 2.8, p has a categorical kernel k, which is the embedding of $p^{-1}0$. By the dual of Corollary 2.8, p is then the cokernel of k ($k1$ is normal, p is the projection of $k1$, and all the top subobjects are normal).

For the second part, we already have that any embedding is a monomorphism. Take any monomorphism $m\colon A \to B$. By Corollary 2.8, m has a kernel $k\colon K \to A$. So $mk = 0 = m0$, and so $k = 0$. So $m^{-1}0 = k0 = 0$, thus m is an embedding. ∎

The following lemma is a version of the short five lemma in a noetherian form.

2.11. Lemma. *In a noetherian form, consider the following commutative diagram*

$$\begin{array}{ccccc} K & \xrightarrow{k} & A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle v} \\ K' & \xrightarrow{k'} & A' & \xrightarrow{f'} & B' \end{array}$$

where k and k' are embeddings, and f and f' are projections, and $k1 = f^{-1}0$ and $k'1 = f'^{-1}0$. If u and v are projections, then so is w. Dually, if u and v are embeddings, then so is w.

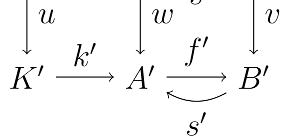
Proof. Suppose u and v are projections. We have

$$\begin{aligned} & f'w1 = vf1 = 1 \\ \Rightarrow\ & f'^{-1}0 \vee w1 = 1 \\ \Rightarrow\ & k'1 \vee w1 = 1 \\ \Rightarrow\ & k'u1 \vee w1 = 1 \\ \Rightarrow\ & wk1 \vee w1 = 1 \\ \Rightarrow\ & w1 = 1. \end{aligned}$$

∎

2.12. Theorem. *Any noetherian form with zero object, in which all normal subobjects are conormal, is protomodular.*

Proof. Consider the commutative diagram



where k is the kernel of f, and k' is the kernel of f', and $fs = 1$ and $f's' = 1$. So k and k' are readily embeddings, and f and f' are projections. By Corollary 2.8, $k1 = f^{-1}0$ and $k'0 = f'^{-1}0$. Then by Lemma 2.11, it follows that if u and v are isomorphisms, so is w. Thus the category is protomodular. ■

2.13. LEMMA. *In a pointed category with products, for any product $A \times B$, $(1, 0)$ is the kernel of π_2 and $(0, 1)$ is the kernel of π_1.*

PROOF. Suppose $\pi_2 k = 0$. Then $k = (\pi_1 k, 0) = (1, 0)(\pi_1 k)$. Since $(1, 0)$ is a mono, $\pi_1 k$ is the unique morphism such that $k = (1, 0)\pi_1 k$. Since also $\pi_2(1, 0) = 0$, $(1, 0)$ is the kernel of π_2. Similarly, $(0, 1)$ is the kernel of π_1. ■

2.14. LEMMA. *In any noetherian form with a zero object and with products, in which all normal subobjects are conormal, for any binary product, we have*

- $\mathsf{Im}\,(1, 0) = \mathsf{Ker}\,\pi_2$ *and* $\mathsf{Im}\,(0, 1) = \mathsf{Ker}\,\pi_1$.
- $\mathsf{Im}\,(1, 0) \vee \mathsf{Im}\,(0, 1) = 1$ *and* $\mathsf{Ker}\,\pi_1 \wedge \mathsf{Ker}\,\pi_2 = 0$.
- *For any f and g,*

$$\mathsf{Ker}\,f \times g = \pi_1^{-1}\mathsf{Ker}\,f \wedge \pi_2^{-1}\mathsf{Ker}\,g \quad \textit{and} \quad \mathsf{Im}\,f \times g = (1, 0)\mathsf{Im}\,f \vee (0, 1)\mathsf{Im}\,g.$$

PROOF. The first part of proof follows from the above lemma and Corollary 2.8. For the second part, we have

$$(1, 0)1 \vee (0, 1)1 = (1, 0)1 \vee \pi_1^{-1}0 = \pi_1^{-1}\pi_1(1, 0)1 = \pi_1^{-1}1 = 1,$$

$$\pi_1^{-1}0 \wedge \pi_2^{-1}0 = \pi_1^{-1}0 \wedge (1, 0)1 = (1, 0)(1, 0)^{-1}\pi_1^{-1}0 = (1, 0)0 = 0.$$

For the last part:

$$\begin{aligned}(f \times g)^{-1}0 = (f \times g)^{-1}(\pi_1^{-1}0 \wedge \pi_2^{-1}0) =& (f \times g)^{-1}\pi_1^{-1}0 \wedge (f \times g)^{-1}\pi_2^{-1}0 \\ =& \pi_1^{-1}f^{-1}0 \wedge \pi_2^{-1}g^{-1}0.\end{aligned}$$

For the other one, first notice that $(f \times g)(1, 0) = (1, 0)f$ and $(f \times g)(0, 1) = (0, 1)g$. This can be demonstrated by composing with π_1 and π_2. Then

$$\begin{aligned}(f \times g)1 = (f \times g)((1, 0)1 \vee (0, 1)1) &= (f \times g)(1, 0)1 \vee (f \times g)(0, 1)1 \\ &= (1, 0)f1 \vee (0, 1)g1.\end{aligned}$$

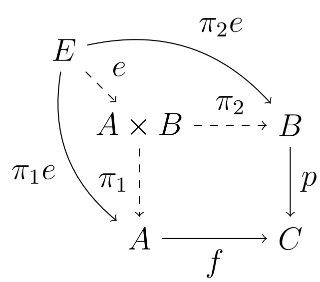
■

2.15. Theorem. *Any noetherian form with a zero object and with products, in which inverse images of conormal subobjects are conormal and all the bottom subobjects are conormal, is a regular category.*

Proof. By Lemma 2.9, it is finitely complete. Further, regular epimorphisms are exactly normal epimorphisms (since by Theorem 2.12, the category is protomodular, and is also pointed). So by Lemma 2.10, the regular epimorphisms are exactly the projections.

Take the kernel pair (K, k_1, k_2) of morphism $f\colon A \to B$. Consider the factorization $f = ep$ where p is a projection and e is an embedding. Then $pk_1 = pk_2$ as well. If $pu = pv$, then $fu = fv$, so there is a unique h then such that $k_1h = u$ and $k_2h = v$. Thus (K, k_1, k_2) is also the kernel pair of projection p. Since p is a regular epi, it is the coequalizer of its kernel pair. Thus coequalizers of kernel pairs exists.

To show that regular epimorphisms are pullback stable, take any regular epimorphism $p\colon B \to C$ and morphism $f\colon A \to C$. Their pullback can be constructed as follows:



where e is the equalizer of $f\pi_1$ and $p\pi_2$. So we need to prove that $\pi_1 e1 = 1$. We have that e is the embedding of $(f\pi_1, p\pi_2)^{-1}(1,1)1 = (f \times p)^{-1}(1,1)1$, since e can be constructed as the horizontal part of the pullback of $(f\pi_1, p\pi_2)$ and $(1_C, 1_C)$. We have, using Proposition 2.2,

$$\begin{aligned} f(\pi_1 e1) &= f\pi_1(f \times p)^{-1}(1,1)1 \\ &= \pi_1(f \times p)(f \times p)^{-1}(1,1)1 \\ &= \pi_1((f \times p)1 \wedge (1,1)1) \\ &= \pi_1(((1,0)f1 \vee (0,1)p1) \wedge (1,1)1) \\ &= \pi_1(((1,0)f1 \vee (0,1)1) \wedge (1,1)1) \\ &= \pi_1(((1,0)f1 \vee \pi_1^{-1}0) \wedge (1,1)1) \\ &= \pi_1((1,0)f1 \vee \pi_1^{-1}0) \wedge \pi_1(1,1)1) \\ &= \pi_1((1,0)f1 = f1 \end{aligned}$$

We also have, making again use of Proposition 2.2,

$$\begin{aligned}
\pi_1 e1 &= \pi_1(f\times p)^{-1}(1,1)1\\
&\geq \pi_1(f\times p)^{-1}0\\
&= \pi_1(\pi_1^{-1}f^{-1}0\wedge\pi_2^{-1}g^{-1}0)\\
&= \pi_1(\pi_1^{-1}f^{-1}0)\wedge\pi_1\pi_2^{-1}g^{-1}0\\
&= f^{-1}0\wedge\pi_1\pi_2^{-1}g^{-1}0\\
&\geq f^{-1}0\wedge\pi_1\pi_2^{-1}0\\
&= f^{-1}0\wedge\pi_1(1,0)1\\
&= f^{-1}0
\end{aligned}$$

Putting these two calculations together, we get $\pi_1 e1 = 1$. ■

2.16. Theorem. *In any noetherian form with a zero object and products, in which all normal subobjects are conormal, any reflexive relation (R, d_1, d_2, s) on any object X is effective.*

Proof. Since $d_1 s = 1 = d_2 s$, both d_1 and d_2 are projections. Since $\overline{d_1 d_2^{-1}0} = d_1 d_2^{-1}0$, the pushout

$$\begin{array}{ccc}
R & \xrightarrow{d_2} & X\\
{\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle f'}\\
X & \xrightarrow[f]{} & Y
\end{array}$$

exists by the dual of Lemma 2.7, where f is the projection of $d_1 d_2^{-1}0$. Notice that $f = fd_1 s = f'd_2 s = f'$. Also notice that

$$\begin{aligned}
d_2 d_1^{-1}0 = d_2(d_1^{-1}0\vee d_2^{-1}0) = d_2(d_1^{-1}d_1 d_2^{-1}0) &= d_2 d_1^{-1}f^{-1}0\\
&= d_2 d_2^{-1}f^{-1}0 = f^{-1}0 = d_1 d_2^{-1}0.
\end{aligned}$$

Further notice that $\pi_1(d_1, d_2)d_1^{-1}0 = 0$, thus $\pi_1^{-1}0 \geq (d_1, d_2)d_1^{-1}0$ and similarly $\pi_2^{-1}0 \geq (d_1, d_2)d_2^{-1}0$. From all this, we have, together with the restricted modular law,

$$\begin{aligned}
(f\times f)^{-1}0 &= \pi_1^{-1}f^{-1}0\wedge\pi_2^{-1}f^{-1}0\\
&= \pi_1^{-1}d_1 d_2^{-1}0\wedge\pi_2^{-1}d_2 d_1^{-1}0\\
&= \pi_1^{-1}\pi_1(d_1,d_2)d_2^{-1}0\wedge\pi_2^{-1}\pi_2(d_1,d_2)d_1^{-1}\\
&= (\pi_1^{-1}0\vee(d_1,d_2)d_2^{-1}0)\wedge(\pi_2^{-1}0\vee(d_1,d_2)d_1^{-1}0)\\
&= ((\pi_1^{-1}0\vee(d_1,d_2)d_2^{-1}0)\wedge\pi_2^{-1}0)\vee(d_1,d_2)d_1^{-1}0\\
&= ((\pi_1^{-1}0\wedge\pi_2^{-1}0)\vee(d_1,d_2)d_2^{-1}0\vee(d_1,d_2)d_1^{-1}0\\
&= (d_1,d_2)d_1^{-1}0\vee(d_1,d_2)d_2^{-1}0\\
&\leq (d_1,d_2)1.
\end{aligned}$$

Further have,

$$\begin{aligned} & 1 = fd_1 1 \\ \Rightarrow & (1,1)1 = (1,1)fd_1 1 = (fd_1, fd_1)1 = (fd_1, fd_2)1 = (f \times f)(d_1, d_2)1 \\ \Rightarrow & (f \times f)^{-1}(1,1)1 = (d_1, d_2)1 \vee (f \times f)^{-1}0 = (d_1, d_2)1. \end{aligned}$$

Since R is a relation, (d_1, d_2) is a monomorphism. Since the category is pointed and normal subobjects are conormal, (d_1, d_2) is an embedding. From the above calculations, (d_1, d_2) is the embedding of $(f \times f)^{-1}(1,1)1$, thus (d_1, d_2) is the equalizer of $f\pi_1$ and $f\pi_2$. Thus $(R, \pi_1(d_1, d_2), \pi_2(d_1, d_2))$ is the pullback of f and f, that is (R, d_1, d_2) is the kernel pair of f. ■

Putting everything together, we get

2.17. Theorem. *Any noetherian form with zero object, products and coproducts, in which inverse images of conormal subobjects are conormal and* 0 *is conormal, is semi-abelian.*

3. Biproducts

Throughout this section, we assume that we are working in a noetherian form.

3.1. Introduction to biproducts.

3.2. Definition. *A* split product *of* A *and* B *is an object* G *equipped with four maps*

$$A \underset{p_1}{\overset{e_1}{\rightleftarrows}} G \underset{p_2}{\overset{e_2}{\leftrightarrows}} B$$

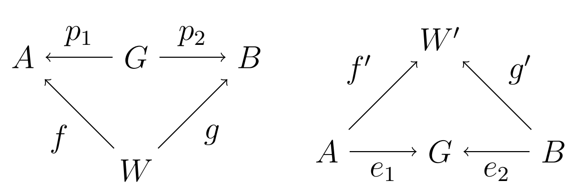
such that

$$\begin{aligned} \mathsf{Ker} p_1 = \mathsf{Im} e_2, \quad p_1 e_1 = 1 \\ \mathsf{Ker} p_2 = \mathsf{Im} e_1, \quad p_2 e_2 = 1 \end{aligned}$$

Sometimes we will just refer to G as a split product of A and B, and assume that their respective embeddings are given by e_1 and e_2, and their respective projections are given by p_1 and p_2.

The two additional conditions on a split product in the following definition were suggested to the author by Zurab Janelidze.

3.3. Definition. *A* biproduct *of* A *and* B *is a split product* G *of* A *and* B *such that for the following diagrams*



the left one has a limit for any f and g, and the right one has a colimit for any f' and g'.

To make it easier to refer to those diagrams in the definition, the left one will be denoted by $\mathsf{L}_G(f,g)$ and the right one by $\mathsf{C}_G(f',g')$. The subscript G may be dropped when it is clear to which biproduct we are referring to.

Note that both the notions of a split- and biproduct are self-dual.

Some trivial properties of split products:

3.4. Proposition. *If G is a split product of A and B, then we have*

(1) $e_1 1 \vee e_2 1 = 1 = p_1^{-1} 0 \vee p_2^{-1} 0$;

(2) $e_1 1 \wedge e_2 1 = 0 = p_1^{-1} 0 \wedge p_2^{-1} 0$;

Proof. (1) Since $p_2 e_2 1 = 1$, we have

$$e_1 1 \vee e_2 1 = p_2^{-1} 0 \vee e_2 1 = p_2^{-1} p_2 e_2 1 = p_2^{-1} 1 = 1.$$

(2) is the dual of (1). ∎

3.5. Corollary. *If the split product of any two object exists, then the top subobject 1 is normal and the bottom subobject 0 is conormal for any object.*

Proof. By (1) of the previous proposition, 1 in $A \oplus A$ is normal, since it is the join of normal subobjects. Since $p_1 \colon A \oplus A \to A$ is a projection, $1 = p_1 1$ is normal. Dually 0 is a conormal subobject of A. ∎

Having biproducts forces pointedness:

3.6. Theorem. *If the biproduct of any two objects exists in a non-empty noetherian form, then the category is pointed.*

Proof. Take any object G. By Corollary 3.5, $1 = G$ is a normal subobject of G. Let $T = G/G$. We have

$$1^T = \pi_G \pi_G^{-1} 1 = \pi_G 1 = 0^T.$$

Thus $\mathsf{sub}\, T$ has exactly one element. Let B be a biproduct of T and T. Then

$$1 = e_1 1 \vee e_2 1 = e_1 0 \vee e_2 0 = 0.$$

Thus $\mathsf{sub}\, B$ also has one element. From this in particular follows that both e_1 and e_2 are isomorphisms. For any object A, there is at least a morphism from T to A, for example compose the embedding from T to a biproduct of T and A with the projection from the same biproduct to A. We would like to show that there is at most one morphism from $T \to A$. Consider any $f, g \colon T \to A$. Let $(C, e \colon A \to C, m \colon B \to C)$ be a colimit (or even a cocone) of $\mathsf{C}_B(f,g)$. Since $\mathsf{sub}\,(T)$ only has one element, both f and g are embeddings of 0 in A. Thus there exists an isomorphism $h \colon T \to T$ such that $fh = g$. We have

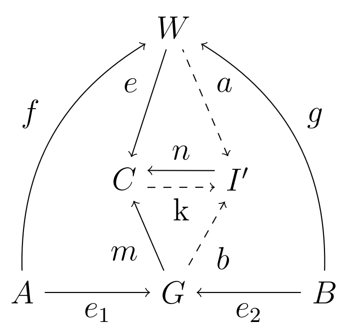
$$me_2 = eg = efh = me_1 h,$$

and since m is (trivially) an embedding, $e_2 = e_1 h$. Consequently $fe_1^{-1} = ge_2^{-1}$ for any $f, g\colon T \to A$. In the case when $f = g = 1_T$, $e_1 = e_2$. So in the general case, $f, g\colon T \to A$ implies $f = g$. So T is an initial object. With dual arguments, one can show that T is also a terminal object. Thus T is a zero object. ∎

The remaining results of this section are some general properties of biproducts.

3.7. Proposition. *Suppose G is a biproduct of A and B. For any $f\colon A \to W$ and $g\colon B \to W$, if $(C, e\colon W \to C, m\colon G \to C)$ is a colimit of $\mathsf{C}(f, g)$, then e is a projection.*

Proof. Suppose (C, e, m) is a colimit of $\mathsf{C}(f, g)$. Let $I = \mathsf{Im}\, e$, and let $n\colon I' \to C$ be an embedding corresponding to I. We have the following diagram



Morphism a exists such that $na = e$, since n is an embedding and $\mathsf{Im}\, e \leq \mathsf{Im}\, n$. We (always) have $\mathsf{Im}\, m \leq I$, since

$$m1 = m(e_1 1 \vee e_2 1) = me_1 1 \vee me_2 1 = ef1 \vee eg1 \leq \mathsf{Im}\, e = I.$$

Thus morphism b exists such that $nb = m$.
We have

$$nbe_1 = me_1 = ef = naf$$

Which implies $be_1 = af$, since n is a monomorphism. Similarly $be_2 = ag$. Thus (I', a, b) is a cocone of $\mathsf{C}(f, g)$. Since C is a colimit, there exists a morphism $k\colon C \to I'$ such that $ke = a$ and $km = b$. Composing k and n, we get a morphism $nk\colon C \to C$ such that $(nk)e = e$ and $(nk)m = m$. But $1_C\colon C \to C$ is the unique such morphism. Thus $nk = 1_C$, and thus n is a projection. Consequently e is a projection. ∎

3.8. Proposition. *If G is a biproduct of A and B, then e_1 and e_2 are jointly epi, and dually p_1 and p_2 are jointly mono.*

Proof. Suppose $(C, e\colon G \to C, m\colon G \to C)$ is a colimit of $\mathsf{C}_G(e_1, e_2)$. Since $(G, 1_G, 1_G)$ is a cocone, there is an $h\colon C \to G$ such that $he = 1_G$ and $hm = 1_G$, consequently e is an embedding. By Proposition 3.7, e is also a projection, thus e is an isomorphism. Consequently h is also an isomorphism, and $(G, 1_G, 1_G)$ is also a colimit of $\mathsf{C}(e_1, e_2)$. Suppose for some $u, v\colon G \to W$, $ue_1 = ve_1$ and $ue_2 = ve_2$. Then (W, u, v) forms a cocone of $\mathsf{C}(e_1, e_2)$. Thus there is an $h\colon G \to W$ such that $u = h1_G$ and $v = h1_G$, from which we get $u = v$. ∎

3.9. Proposition. *Suppose G is a biproduct of A and B. For any $f\colon A \to W$ and $g\colon B \to W$, any cocone $(C, e\colon W \to C, m\colon G \to C)$ of $\mathsf{C}(f, g)$ is a colimit if and only if*

- *e is a projection;*
- *for any cocone $(D, d\colon W \to D, n\colon G \to D)$ we have $\mathsf{Ker}e \leq \mathsf{Ker}d$.*

Proof. Suppose (C, e, m) is a colimit. Then by Proposition 3.7, e is a projection. For the second point, since C is a colimit, there is an h such that $he = d$ from which it follows.

Conversely suppose there is such a cocone (C, e, m) with those properties. Let

$$(D, d\colon W \to D, n\colon G \to D)$$

be another cocone. Since e is a projection and $\mathsf{Ker}e \leq \mathsf{Ker}d$, there is a unique $h\colon C \to D$ such that $he = d$. We further have

$$hme_1 = hef = df = ne_1,$$

and similarly $hme_2 = ne_2$. By Proposition 3.8, e_1 and e_2 are jointly epi, and thus $hm = n$. Thus (C, e, m) is indeed a colimit of $\mathsf{C}(f, g)$. ■

3.10. Biproducts of morphisms. The following theorem is the main theorem of this subsection.

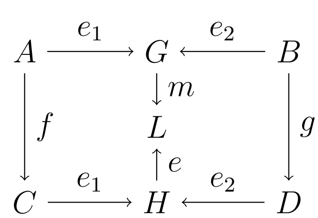
3.11. Theorem. *Suppose G is a biproduct of A and B, and H is a split product of C and D. For any pair of morphisms $f\colon A \to C$ and $g\colon B \to D$, there is a unique morphism $h\colon G \to H$ such that*

$$\begin{aligned} he_1 &= e_1 f, \quad p_1 h = f p_1 \\ he_2 &= e_2 g, \quad p_2 h = g p_2. \end{aligned}$$

Furthermore

$$\begin{aligned} &\mathsf{Im}h = e_1\mathsf{Im}f \vee e_2\mathsf{Im}g, \\ &\mathsf{Ker}h = p_1^{-1}\mathsf{Ker}f \wedge p_2^{-1}\mathsf{Ker}g. \end{aligned}$$

Proof. Let $(L, e\colon H \to L, m\colon G \to L)$ be a colimit of $\mathsf{C}_G(e_1 f, e_2 g)$. The following commutative diagram just makes it easier to follow:



The aim will be to deduce that e is an isomorphism, then $h = e^{-1}m$ is our desired morphism. Both $(C, p_1\colon H \to C, fp_1\colon G \to C)$ and $(D, p_2\colon H \to Dfp_2\colon G \to D)$ are cocones of $\mathsf{C}(e_1 f, e_2 g)$. So $\mathsf{Ker} e \leq \mathsf{Ker} p_1, \mathsf{Ker} p_2$, thus $\mathsf{Ker}\, e = 0$. By Proposition 3.7, e is a projection and thus an isomorphism. Let $h = e^{-1}m$. By choice of h, $he_1 = e_1 f$ and $he_2 = e_2 g$. We further have

$$p_1 he_1 = p_1 e_1 f = f = f p_1 e_1 \quad \text{and} \quad p_1 he_2 = p_1 e_2 g = 0 = f p_1 e_2.$$

Since e_1 and e_2 are jointly epic by Proposition 3.8, $p_1 h = f p_1$. Similarly $p_2 h = g p_2$. To compute the image of h, we have

$$\mathsf{Im}\, h = h1 = h(e_1 1 \vee e_2 1) = he_1 1 \vee he_2 1 = e_1 f 1 \vee e_2 g 1 = e_1 \mathsf{Im}\, f \vee e_2 \mathsf{Im}\, g.$$

By a dual argument, we get the formula for the kernel of h. ∎

3.12. Corollary. *Biproducts are unique up to isomorphisms.*

Proof. Let both G and H be biproducts of A and B. Take $f = 1_A$ and $g = 1_B$. Then the induced morphism h in the proposition above, is an isomorphism which commutes with the projections and with the embeddings. ∎

3.13. Notation. *The biproduct of A and B will be denoted by $A \oplus B$. For $f\colon A \to C$ and $g\colon B \to D$, the unique $h\colon A \oplus B \to C \oplus D$ in the statement of Theorem 3.11 will be denoted by $f \oplus g$.*

The following follows immediately from Theorem 3.11:

3.14. Corollary. *The following holds for any morphisms:*

- $1_A \oplus 1_B = 1_{A \oplus B}$;
- $(f \oplus g)(u \oplus v) = fu \oplus gv$, *whenever the compositions are defined.*

Further basic results that follows from Theorem 3.11:

3.15. Proposition. *In any biproduct $A \oplus B$, we have:*

(1) For normal subobjects N of A and M of B, $p_1^{-1}N \wedge p_2^{-1}M$ is a normal subobject of $A \oplus B$.

(2) For conormal subobjects C of A and D of B, $e_1 C \vee e_2 D$ is a conormal subobject of $A \oplus B$.

(3) if N is a normal subobject of A, then $e_1 N$ is a normal subobject of $A \oplus B$;

(4) if C is a conormal subobject of $A \oplus B$, then $p_1^{-1}C$ is a conormal subobject of A;

(5) For any $X \in \mathsf{sub}\, A$ and $Y \in \mathsf{sub}\, B$, if X is a normal or conormal subobject, or Y is a normal or conormal subobject, then

$$e_1 X \vee e_2 Y = p_1^{-1}X \wedge p_2^{-1}Y.$$

Proof. (1) The subobject $p_1^{-1}N \wedge p_2^{-1}M$ is normal, since it is the kernel of $\pi_N \oplus \pi_M$.

(2) is the dual of (1).

(3) Since $p_1 e_1 = 1$, we get $N = 1^{-1}N = e_1^{-1}p_1^{-1}N$. Then we have

$$e_1 N = e_1 e_1^{-1} p_1^{-1} N = e_1 1 \wedge p_1^{-1} N = p_1^{-1} N \wedge p_2^{-1} 0,$$

thus $e_1 N$ is normal by (1).

(4) is the dual of (3).

(5) As already noticed in the proof of (3), $e_1 X = e_1 1 \wedge p_1^{-1} X$ and similarly $e_2 Y = e_2 1 \wedge p_2^{-1} Y$. These are in particular just special cases of (5) for $Y = 0$ or $X = 0$. Suppose X is normal. Then we have, making use of the restricted modular law twice,

$$\begin{aligned} & e_1 X \vee e_2 Y \\ = & (e_1 1 \wedge p_1^{-1} X) \vee (e_2 1 \wedge p_2^{-1} Y) \\ = & ((e_1 1 \wedge p_1^{-1} X) \vee e_2 1) \wedge p_2^{-1} Y \\ = & (p_1^{-1} X \wedge (e_1 1 \vee e_2 1)) \wedge p_2^{-1} Y \\ = & p_1^{-1} X \wedge p_2^{-1} Y. \end{aligned}$$

The second equality follows, since $p_2^{-1}Y \geq p_2^{-1}0 = e_1 1 \geq e_1 1 \wedge p_1^{-1} X$, and $e_1 1 \wedge p_1^{-1} X = e_1 X$ is normal by (3) and $e_2 1$ is conormal. The third follows, since $p_1^{-1} X \geq p_1^{-1} 0 = e_2 1$, and $e_2 1$ is normal and $e_1 1$ is conormal.

So for X normal the result is true. By duality it is also true if X is conormal. The case for when Y is normal or conormal is similar. ■

The remaining results of this subsection, is about when an object is a biproduct of two other objects. All this still relies on Theorem 3.11.

3.16. Theorem. *For any A and B, any split product of A and B is a biproduct, assuming the biproduct of A and B exists.*

Proof. Let G be a split product of A and B. Theorem 3.11 guarantees an isomorphism h (taking $f = 1_A$ and $g = 1_B$) between G and $A \oplus B$ which commutes with the projections and with the embeddings. From this, it can be readily checked that the split product G will satisfy the remaining biproduct conditions. ■

3.17. Corollary. *If G has two subobjects A and B such that*

- *A and B are both normal and conormal;*
- *$A \vee B = 1$ and $A \wedge B = 0$,*

then, $G \cong A_O \oplus B_O$, assuming $A_O \oplus B_O$ exists, where A_O and B_O are the domains of embeddings of A and B respectively.

Proof. Let $e_1 = \iota_A \colon A_O \to G$ and $e_2 = \iota_B \colon B_O \to G$. Notice that

$$\pi_B \iota_A 1 = \pi_B A = \pi_B (A \vee B) = \pi_B 1 = 1,$$

and

$$\iota_A^{-1} \pi_B^{-1} 0 = \iota_A^{-1} B = \iota_A^{-1}(A \wedge B) = \iota_A^{-1} 0 = 0.$$

Thus $\pi_B \iota_A$ is an isomorphism. Denote the inverse by h_1. Similarly $\pi_A \iota_B$ is an isomorphism. Denote the inverse by h_2. Define $p_1 = h_1 \pi_B$ and $p_2 = h_2 \pi_A$. A straightforward verification shows that G together with e_1, e_2, p_1, and p_2 forms a splitproduct of A_O and B_O, thus a biproduct of A_O and B_O by the above theorem. ■

3.18. Corollary. *If $f \colon B \to A$ has a right inverse $s \colon A \to B$, and $\mathsf{Im} s$ is normal and $\mathsf{Ker} f$ is conormal, then $B \cong (\mathsf{Im} s)_O \oplus (\mathsf{Ker} f)_O$, assuming that biproduct exists.*

Proof. Both $\mathsf{Im}\, s$ and $\mathsf{Ker}\, f$ is both normal and conormal. We have

$$f^{-1}0 \vee s1 = f^{-1} f s 1 = f^{-1} 1 = 1 \quad \text{and} \quad f^{-1} 0 \wedge s1 = s s^{-1} f^{-1} 0 = s0 = 0.$$

Thus by Corollary 3.17 the result follows. ■

3.19. Comparison with categorical products and coproducts.

3.20. Proposition. *Consider any noetherian form that has biproducts, (categorical) products and coproducts. For any pair of objects A and B, the canonical morphism $I \colon A + B \to A \times B$ (that is $\pi_i I \iota_j = \delta_{i,j}$) factors as*

$$A + B \xrightarrow[e]{} A \oplus B \xrightarrow[m]{} A \times B \qquad (I \colon A + B \to A \times B)$$

where e is a projection and m is an embedding, such that $e\iota_1 = e_1$, $e\iota_2 = e_2$, $p_1 = \pi_1 m$, and $p_2 = \pi_2 m$.

Proof. Let e be the unique morphism $A + B \to A \oplus B$ such that $e\iota_i = e_i$ for $i = 1, 2$. Then

$$1 = e_1 1 \vee e_2 1 = e\iota_1 1 \vee e\iota_2 1 = e(\iota_1 1 \vee \iota_2 1) \leq e1 \leq 1.$$

Thus e is a projection.

Dually, the unique morphism $m \colon A \oplus B \to A \times B$ such that $\pi_i m = p_i$ for $i = 1, 2$, is an embedding. Furthermore, we have

$$\pi_i m e \iota_j = p_i e_j = \delta_{i,j}.$$

Thus me is the canonical morphism. ■

3.21. THEOREM. *For any noetherian form $\mathbb{C}$ with biproducts and (categorical) products, the following are equivalent*

(1) For any two objects, their biproduct and product coincide.

(2) Inverse images of conormal subobjects are conormal.

(3) Normal subobjects are conormal.

(4) For any product, $(1,0)$ and $(0,1)$ are jointly extremal epimorphic

Whenever any of the above holds, $\mathbb{C}$ is protomodular and Barr exact.

PROOF. (1) ⇒ (2): Take a morphism $f\colon A \to B$ and a conormal subobject X of B. Consider the biproduct $A \oplus B$. Since it is a product, there is a morphism $h\colon A \to A \oplus B$ such that $p_1 h = 1$ and $p_2 h = f$. By Proposition 3.15(4), $p_2^{-1}X$ is conormal. Since h is a split mono, it is an embedding, from which it follows that

$$f^{-1}X = h^{-1}p_2^{-1}X$$

is conormal.

(2) ⇒ (3): Since biproducts exist, all bottom subobjects 0 are conormal, from which it follows that all normal subobjects are conormal.

(3) ⇒ (4): Consider any product $A \times B$. Take any monomorphism $f\colon W \to A \times B$ such that $(1,0)$ and $(0,1)$ factor through f, that is $(1,0) = fa$ and $(0,1) = fb$, for some morphisms a and b. By Lemma 2.10, f is an embedding. Further by Lemma 2.14, we have

$$f1 \geq fa1 \vee fb1 = (1,0)1 \vee (0,1)1 = 1.$$

Thus f is an isomorphism.

(4) ⇒ (1): Consider the morphism $m = (p_1, p_2)\colon A \oplus B \to A \times B$. Since $me_1 = (1,0)$ and $me_2 = (0,1)$, and m is a monomorphism, m is an isomorphism.

For the last part, if either of the above points hold, then by Theorems 2.12, 2.15, and 2.16, $\mathbb{C}$ is protomodular and Barr exact. ■

3.22. COROLLARY. *Any noetherian form with a zero object, biproducts, products and coproducts is semi-abelian if and only if the inverse image of any conormal subobject is conormal.*

3.23. MONOIDALITY OF BIPRODUCTS. Throughout this subsection, we are working in a noetherian form.

3.24. NOTATION. *For $a\colon A \to C$ and $b\colon A \to D$, if there is a morphism $h\colon A \to C \oplus D$ such that $p_1 h = a$ and $p_2 h = b$, then it is unique by Proposition 3.8, and h will be denoted by (a, b).*

Notice that any morphism $h\colon A \to C \oplus D$ can be written as $h = (p_1 h, p_2 h)$.

Some basic properties:

3.25. Proposition. *For any morphisms a, b, f, and g, we have, whenever the composites are defined:*

(1) $(a, b)f = (af, bf)$*;*

(2) $f \oplus g = (fp_1, gp_2)$*;*

(3) $(f \oplus g)(a, b) = (fa, gb)$*.*

(4) $(p_1, p_2) = 1$ *for the projections of any biproduct.*

Proof. Since p_1 and p_2 are jointly monic, we only need to verify that both sides are equal when composing with p_1 and p_2 on both sides.

(1) $p_1(a, b)f = af = p_1(af, bf)$ and $p_2(a, b)f = p_2(af, bf)$.

(2) $p_1(f \oplus g) = fp_1 = p_1(fp_1, gp_2)$ and $p_2(f \oplus g) = p_2(fp_1, gp_2)$.

(3) $(f \oplus g)(a, b) = (fp_1, gp_2)(a, b) = (fp_1(a, b), gp_2(a, b)) = (fa, gb)$.

(4) $p_1 = p_1(p_1, p_2)$ and $p_2 = p_2(p_1, p_2)$.

■

The dual of the above will be:

3.26. Notation. *For $a\colon A \to C$ and $b\colon B \to C$, if there is a morphism $h\colon A \oplus B \to C$ such that $he_1 = a$ and $he_2 = b$, it is unique, and h will be denoted by $[a, b]$.*

We also have dual properties, which are true by duality:

3.27. Proposition. *For any morphisms a, b, f, and g, we have, whenever the composites are defined:*

(1) $f[a, b] = [fa, fb]$*;*

(2) $f \oplus g = [e_1 f, e_2 g]$*;*

(3) $[a, b](f \oplus g) = [af, bg]$*;*

(4) $[e_1, e_2] = 1$ *for the embeddings of any biproduct.*

3.28. Lemma. *For any objects A, B, and C the morphism*

$$\alpha = ((p_1, p_1p_2), p_2p_2) = [e_1e_1, [e_1e_2, e_2]]\colon A \oplus (B \oplus C) \to (A \oplus B) \oplus C$$

exists. Moreover, α is a natural isomorphism.

Proof. Consider the following diagram:

$$A \oplus B \underset{1 \oplus p_1}{\overset{1 \oplus e_1}{\rightleftarrows}} A \oplus (B \oplus C) \underset{p_2 p_2}{\overset{e_2 e_2}{\leftrightarrows}} C$$

We have $(1 \oplus p_1)(1 \oplus e_1) = 1_A \oplus 1_B = 1_{A \oplus B}$ and $p_2 p_2 e_2 e_2 = 1_C$. Further

$$\begin{aligned} &\mathsf{Ker}\,(1 \oplus p_1) \\ &= p_1^{-1}0 \wedge p_2^{-1}p_1^{-1}0 \\ &= e_2 1 \wedge p_2^{-1}p_1^{-1}0 \\ &= e_2 e_2^{-1} p_2^{-1} p_1^{-1} 0 \\ &= e_2 p_1^{-1} 0 \\ &= e_2 e_2 1 = \mathsf{Im}\,(e_2 e_2), \end{aligned} \qquad \begin{aligned} &\mathsf{Im}\,(1 \oplus e_1) \\ &= e_1 1 \vee e_2 e_1 1 \\ &= p_2^{-1} 0 \vee e_2 e_1 1 \\ &= p_2^{-1} p_2 e_2 e_1 1 \\ &= p_2^{-1} e_1 1 \\ &= p_2^{-1} p_2^{-1} 0 = \mathsf{Ker}\,(p_2 p_2). \end{aligned}$$

Thus the above diagram is a split product, thus a biproduct. By Theorem 3.11 (selecting $f = 1_{A \oplus B}$ and $g = 1_C$) there is morphism $\alpha\colon A \oplus (B \oplus C) \to (A \oplus B) \oplus C$ such that

$$p_1 \alpha = (1 \oplus p_1) \quad \text{and} \quad p_2 \alpha = p_2 p_2$$

By the same theorem, α is furthermore an isomorphism. We have

$$\alpha = (p_1 \alpha, p_2 \alpha) = (1 \oplus p_1, p_2 p_2) = ((p_1, p_1 p_2), p_2 p_2).$$

We also have

$$p_1 \alpha e_1 = (1 \oplus p_1) e_1 = e_1 = p_1 e_1 e_1 \quad \text{and} \quad p_2 \alpha e_1 = p_2 p_2 e_1 = 0 = p_2 e_1 e_1.$$

Thus $\alpha e_1 = e_1 e_1$. And also

$$p_1 \alpha e_2 = (1 \oplus p_1) e_2 = e_2 p_1 = p_1 (e_2 \oplus 1) \quad \text{and} \quad p_2 \alpha e_2 = p_2 p_2 e_2 = p_2 = p_2 (e_2 \oplus 1).$$

Thus $\alpha e_2 = e_2 \oplus 1$. Consequently

$$\alpha = [e_1 e_1, e_2 \oplus 1] = [e_1 e_1, [e_1 e_2, e_2]].$$

To verify naturality, we must show that the following diagram commutes for any f, g, and h:

$$\begin{array}{ccc} A \oplus (B \oplus C) & \xrightarrow{\ \alpha\ } & (A \oplus B) \oplus C \\ \big\downarrow {\scriptstyle f \oplus (g \oplus h)} & & \big\downarrow {\scriptstyle (f \oplus g) \oplus h} \\ X \oplus (Y \oplus Z) & \xrightarrow{\ \alpha\ } & (X \oplus Y) \oplus Z \end{array}$$

It does indeed commute:

$$\begin{aligned}
&\alpha(f \oplus (g \oplus h)) \\
&= ((p_1, p_1p_2), p_2p_2)(f \oplus (g \oplus h)) \\
&= ((p_1(f \oplus (g \oplus h)), p_1p_2(f \oplus (g \oplus h))), p_2p_2(f \oplus (g \oplus h))) \\
&= ((fp_1, p_1(g \oplus h)p_2), p_2(g \oplus h)p_2) \\
&= ((fp_1, gp_1p_2), hp_2p_2) \\
&= ((f \oplus g)(p_1, p_1p_2), hp_2p_2) \\
&= ((f \oplus g) \oplus h)((p_1, p_1p_2), p_2p_2) \\
&= ((f \oplus g) \oplus h)\alpha.
\end{aligned}$$

■

3.29. Theorem. *Any noetherian form $\mathbb{C}$ with biproducts forms a monoidal category*

$$\langle \mathbb{C}, \oplus, 0, \alpha, p_2^{0\oplus -}, p_1^{-\oplus 0} \rangle.$$

Proof. Corollary 3.14 shows that $\oplus$ forms a functor from $\mathbb{C} \times \mathbb{C}$ to $\mathbb{C}$.

We already know that having biproducts forces pointedness, thus the zero object 0 exists.

For any object A, $p_2^{0\oplus A} \colon 0 \oplus A \to A$ is a natural isomorphism: It is a projection, since it is a split epi. Also

$$p_2^{-1}0 = e_1 1 = e_1 0 = 0.$$

Thus it is also an embedding, thus an isomorphism. For naturality, take any $f \colon A \to X$. We must show that the diagram

$$\begin{array}{ccc}
0 \oplus A & \xrightarrow{\;p_2\;} & A \\
\big\downarrow{\scriptstyle 1 \oplus f} & & \big\downarrow{\scriptstyle f} \\
0 \oplus X & \xrightarrow{\;p_2\;} & X
\end{array}$$

commutes. Indeed it does by the definition of $1 \oplus f$. Similarly, $p_1^{A\oplus 0} \colon A \oplus 0 \to A$ is a natural isomorphism for any object A.

What is still left, is to prove that two certain diagrams commute. The first one, for any objects A and C, the following diagram commutes:

$$\begin{array}{ccc}
A \oplus (0 \oplus C) & \xrightarrow{\;\alpha\;} & (A \oplus 0) \oplus C \\
\big\downarrow{\scriptstyle 1 \oplus p_2} & & \big\downarrow{\scriptstyle p_1 \oplus 1} \\
A \oplus C & \xrightarrow{\;1\;} & A \oplus C
\end{array}$$

Indeed:

$$\begin{aligned}(p_1 \oplus 1)\alpha &= (p_1 \oplus 1)((p_1, p_1p_2), p_2p_2) = (p_1(p_1, p_1p_2), 1p_2p_2)\\ &= (p_1, p_2p_2) = (1 \oplus p_2)(p_1, p_2) = 1 \oplus p_2.\end{aligned}$$

The second diagram, for any A, B, C, and D, the following diagram commutes:

$$\begin{array}{ccc} A \oplus (B \oplus (C \oplus D)) \xrightarrow{\alpha} (A \oplus B) \oplus (C \oplus D) & \xrightarrow{\alpha} & ((A \oplus B) \oplus C) \oplus D \\ \downarrow 1 \oplus \alpha & & \uparrow \alpha \oplus 1 \\ A \oplus ((B \oplus C) \oplus D) & \xrightarrow{\alpha} & (A \oplus (B \oplus C)) \oplus D \end{array}$$

Indeed:

$$\begin{aligned}&(\alpha \oplus 1)\alpha(1 \oplus \alpha)\\ =&(\alpha p_1, p_2)\alpha(1 \oplus \alpha)\\ =&(((p_1, p_1p_2), p_2p_2)p_1, p_2)\alpha(1 \oplus \alpha)\\ =&(((p_1p_1, p_1p_2p_1), p_2p_2p_1), p_2)\alpha(1 \oplus \alpha)\\ =&(((p_1p_1\alpha, p_1p_2p_1\alpha), p_2p_2p_1\alpha), p_2\alpha)(1 \oplus \alpha)\\ =&(((p_1, p_1p_1p_2), p_2p_1p_2), p_2p_2)(1 \oplus \alpha)\\ =&(((p_1(1 \oplus \alpha), p_1p_1p_2(1 \oplus \alpha)), p_2p_1p_2(1 \oplus \alpha)), p_2p_2(1 \oplus \alpha))\\ =&(((p_1, p_1p_1\alpha p_2), p_2p_1\alpha p_2), p_2\alpha p_2)\\ =&(((p_1, p_1p_2), p_1p_2p_2), p_2p_2p_2)\\ =&((p_1\alpha, p_1p_2\alpha), p_2p_2\alpha)\\ =&((p_1, p_1p_2), p_2p_2)\alpha\\ =&\alpha\alpha\end{aligned}$$

■

3.30. Remark. *The monoidal structure given by biproducts is in fact a 'monoidal sum structure' in the sense of* [9]. *This follows from Theorem 3.29 and Proposition 3.8.*

4. Commutators

Throughout this section we are working in a noetherian form with biproducts.

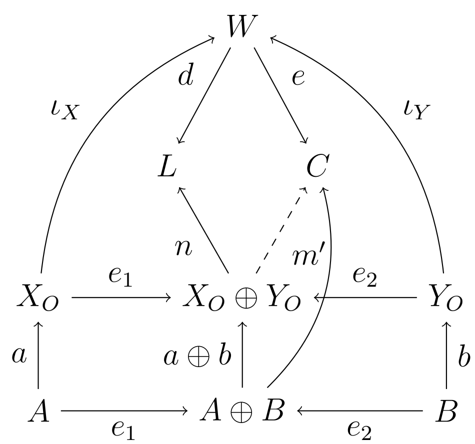
4.1. The general theory.

4.2. Definition. *For a object G and conormal subobjects X and Y, the* commutator $[X, Y]_G$ *is defined as follows: If $(C, e: G \to C, m: X_O \oplus Y_O \to C)$ is a colimit of $\mathsf{C}_{X_O \oplus Y_O}(\iota_X, \iota_Y)$, then $[X, Y]_G = \mathsf{Ker}\, e$.*

The commutator $[1, 1]_G$ will be denoted by $[G, G]_G$ instead.

4.3. Proposition. *For any* $f\colon A \to W$ *and* $g\colon B \to W$, *if* $(C, e\colon W \to C, m\colon A \oplus B \to C)$ *is a colimit of* $\mathsf{C}_{A\oplus B}(f,g)$, *then* $\mathsf{Ker}\, e = [\mathsf{Im}\, f, \mathsf{Im}\, g]_W$.

Proof. Let $\mathsf{Im}\, f = X$ and $\mathsf{Im}\, g = Y$. Factorize $f = \iota_X a$ and $g = \iota_Y b$, where a and b are projections of the kernels of f and g respectively. Consider the following diagram, where (L, d, n) is a colimit of $\mathsf{C}_{X_O \oplus Y_O}(\iota_X, \iota_Y)$, and (C, e, m) is a colimit of $\mathsf{C}_{A\oplus B}(f, g)$



The unlabelled arrow $A \oplus B \to C$ is m.

Since $(L, d, n(a \oplus b))$ forms a cocone of $\mathsf{C}_{A\oplus B}(f, g)$, $\mathsf{Ker}\, e \le \mathsf{Ker}\, d$. Notice, by Proposition 3.15 (5),

$$\mathsf{Ker}\,(a \oplus b) = p_1^{-1}\mathsf{Ker}\, a \wedge p_2^{-1}\mathsf{Ker}\, b = e_1\mathsf{Ker}\, a \vee e_2\mathsf{Ker}\, b.$$

Using this, we have

$$\begin{aligned} m(a \oplus b)^{-1}0 = m(e_1 a^{-1}0 \vee e_2 b^{-1}0) &= m e_1 a^{-1}0 \vee m e_2 b^{-1} \\ &= e f a^{-1}0 \vee e g b^{-1}0 = 0. \end{aligned}$$

So $\mathsf{Ker}\, m \ge \mathsf{Ker}\, a \oplus b$. Since $a \oplus b$ is a projection, there is a unique $m'\colon X_O \oplus Y_O \to C$ such that $m'(a \oplus b) = m$. Then we have

$$m' e_1 a = m'(a \oplus b) e_1 = m e_1 = e f = e \iota_X a.$$

Since a is a projection, it is an epi, thus $m' e_1 = e\iota_X$. Similarly $m' e_2 = e \iota_Y$. Thus (C, e, m') is a cocone of $\mathsf{C}_{X_O \oplus Y_O}(\iota_X, \iota_Y)$. Consequently $\mathsf{Ker}\, d \le \mathsf{Ker}\, e$, and thus $\mathsf{Ker}\, e = \mathsf{Ker}\, d = [X, Y]_W$. ■

We establish some basic properties.

4.4. Proposition. *If* A, B, X, *and* Y *are conormal subobjects of* G, *with* $X \le A$ *and* $Y \le B$, *then*

$$[X, Y] \le [A, B].$$

Proof. Since $X \le A$ there is a unique k such that $\iota_A k = \iota_X$. Similarly, there is a unique l such that $\iota_B l = \iota_Y$. Suppose $(C, e\colon G \to C, m\colon A_O \oplus B_O \to C)$ is a colimit of $\mathsf{C}_{A_O \oplus B_O}(\iota_A, \iota_B)$. Then $(C, e\colon G \to C, m(k \oplus l)\colon X_O \oplus Y_O \to C)$ is a cocone of $\mathsf{C}_{X_O \oplus Y_O}(\iota_X, \iota_Y)$. Thus $[X, Y] \le \mathsf{Ker}\, e = [A, B]$. ■

4.5. Proposition. *For any conormal subobjects X and Y of G, we have $[X,Y] \leq N$ where N is a normal subobject containing X. In particular if there is a smallest normal subobject $\overline{X}$ containing X, then $[X,Y] \leq \overline{X}$.*

Proof. We have that $(G/N, \pi_N \colon G \to G/N, \pi_N \iota_Y p_2 \colon X_O \oplus Y_O \to G/N)$ is a cocone of $\mathsf{C}_{X_O \oplus Y_O}(\iota_X, \iota_Y)$. Consequently $N = \mathsf{Ker}\, \pi_N \geq [X,Y]$. ∎

4.6. Proposition. *For any morphism $f \colon G \to H$ and conormal subobjects X and Y of G, we have*

(1) $f[X,Y] \leq [fX, fY]$;

(2) $[fX, fY] \leq N$ for any normal subobject $N \geq f[X,Y]$.

Proof. (1) Suppose $(C, e \colon H \to C, m \colon X_O \oplus Y_O \to C)$ is a colimit of $\mathsf{C}(f\iota_X, f\iota_Y)$. Then (C, ef, m) is a cocone of $\mathsf{C}(\iota_X, \iota_Y)$. Consequently

$$[X,Y] \leq f^{-1}e^{-1}0 = f^{-1}[fX, fY] \quad \Rightarrow \quad f[X,Y] \leq [fX, fY].$$

(2) Suppose $(C, e \colon G \to C, m \colon X_O \oplus Y_O \to C)$ is a colimit of $\mathsf{C}(\iota_X, \iota_Y)$. Since $f[X,Y] \leq N$, we have $[X,Y] \leq f^{-1}N$, and so $\mathsf{Ker}\, e \leq \mathsf{Ker}\, \pi_N f$. Thus there is an h such that $he = \pi_N f$. One can then readily observe that $(H/N, \pi_N, hm)$ is a cocone of $\mathsf{C}(f\iota_X, f\iota_Y)$. From this it follows that $[fX, fY] \leq N$. ∎

4.7. Corollary. *We have the following immediate consequences*

(1) if there is a smallest normal subobject $\overline{f[X,Y]}$ containing $f[X,Y]$, then $\overline{f[X,Y]} = [fX, fY]$;

(2) if f is a projection, then $f[X,Y] = [fX, fY]$.

Proof. (1) Since $[fX, fY]$ is normal which contains $f[X,Y]$, we have

$$[fX, fY] \leq \overline{f[X,Y]} \leq [fX, fY],$$

from which equality follows.

(2) If f is a projection, then $f[X,Y]$ is normal. Then from (1) the result follows. ∎

Here is another corollary.

4.8. Corollary. *If any commutator of any two conormal subobjects is their meet, then any direct image preserves meets of conormal subobjects and any conormal subobject is normal.*

Proof. The last part is clear: take any conormal subobject X, then $[X, X] = X \wedge X = X$, and consequently X is normal.

Take any two conormal subobjects X and Y of the same object A. It is sufficient to prove that their meet is preserved under embeddings and projections. Let $d\colon A \to B$ be an embedding, when we have

$$d(\iota_X 1 \wedge \iota_Y 1) = d(\iota_X \iota_X^{-1} \iota_Y 1) = d\iota_X \iota_X^{-1} d^{-1} d\iota_Y 1 = d\iota_X 1 \wedge d\iota_Y 1.$$

If $d\colon A \to B$ is a projection, we use the corollary above:

$$d(X \wedge Y) = d[X, Y] = [dX, dY] = dX \wedge dY.$$

■

4.9. Proposition. *If direct images preserve meets of conormal subobjects, then the commutator of any two conormal subobjects contains their meet.*

Proof. For conormal subobjects X and Y of A, let $(C, e\colon A \to C, m\colon X_O \oplus Y_O \to C)$ be the colimit of $\mathsf{C}(\iota_X, \iota_Y)$. We have

$$e(X \wedge Y) = eX \wedge eY = e\iota_X 1 \wedge e\iota_Y 1 = me_1 1 \wedge me_2 1 = m(e_1 1 \wedge e_2 1) = m0 = 0.$$

Thus $X \wedge Y \leq \mathsf{Ker}\, e = [X, Y]$. ■

4.10. Proposition. *The commutator of any conormal subobjects X and Y is $X \wedge Y$ if and only if the direct images preserves the meet of conormal subobjects and all conormal subobjects are normal.*

Proof. The one direction is given by the corollary above.

For the converse, by Proposition 4.5, the commutator of X and Y is contained in $X \wedge Y$. And also by the proposition above, $X \wedge Y$ is contained in the commutator. Thus the commutator is $X \wedge Y$. ■

Commutators in biproducts can be computed component-wise as in:

4.11. Proposition. *If A, C are conormal subobjects of G, and B, D are conormal subobjects of H, then*

$$\begin{aligned}[e_1 A \vee e_2 B, e_1 C \vee e_2 D]_{G \oplus H} &= [e_1 A, e_1 C]_{G \oplus H} \vee [e_2 B, e_2 D]_{G \oplus H} \\ &= e_1 [A, C]_G \vee e_2 [B, D]_H.\end{aligned}$$

Proof. Since $[A, C]_G$ is normal, so is $e_1[A, C]_G$ by Proposition 3.15 (3), thus by Corollary 4.7 (1) we have $e_1[A, C]_G = [e_1 A, e_1 C]_{G \oplus H}$. Similarly $e_2[B, D]_H = [e_2 B, e_2 D]_{G \oplus H}$. By Proposition 3.15 (2) both $e_1 A \vee e_2 B$ and $e_1 C \vee e_2 D$ are conormal. Further, from Proposition 4.4 it follows that

$$[e_1 A \vee e_2 B, e_1 C \vee e_2 D]_{G \oplus H} \geq [e_1 A, e_1 C]_{G \oplus H} \vee [e_2 B, e_2 D]_{G \oplus H}.$$

Notice that

$$A_O \oplus B_O \underset{p_1 \oplus p_1}{\overset{e_1 \oplus e_1}{\rightleftarrows}} (A_O \oplus C_O) \oplus (B_O \oplus D_O) \underset{p_2 \oplus p_2}{\overset{e_2 \oplus e_2}{\leftrightarrows}} C_O \oplus D_O$$

Is a split product by Theorem 3.11 and Proposition 3.15(5), thus a biproduct of $A_O \oplus B_O$ and $C_O \oplus D_O$. Suppose $(L_1, d_1 \colon G \to L_1, n_1 \colon A_O \oplus C_O \to L_1)$ is a colimit of $\mathsf{C}(\iota_A, \iota_C)$, and $(L_2, d_2 \colon H \to L_2, n_2 \colon B_O \oplus D_O \to L_2)$ is a colimit of $\mathsf{C}(\iota_B, \iota_D)$. Then $(L_1 \oplus L_2, d_1 \oplus d_2, n_1 \oplus n_2)$ is a cocone of $\mathsf{C}(\iota_A \oplus \iota_B, \iota_C \oplus \iota_D)$. Consequently

$$\begin{aligned}[e_1 A \vee e_2 B, e_1 C \vee e_2 D] &= [\mathsf{Im}\,(\iota_A \oplus \iota_B), \mathsf{Im}\,(\iota_C \oplus \iota_D)] \\ &\leq \mathsf{Ker}\,(d_1 \oplus d_2) \\ &= e_1 d_1^{-1} 0 \vee e_2 d_2^{-1} 0 = e_1[A, C] \vee e_2[B, D].\end{aligned}$$

Thus the result is true. ■

4.12. Corollary. *For objects A and B, we have*

$$[A \oplus B, A \oplus B]_{A \oplus B} = e_1[A, A]_A \vee e_2[B, B]_B.$$

Proof. We have

$$[A \oplus B, A \oplus B] = [e_1 A \vee e_2 B, e_1 A \vee e_2 B] = e_1[A, A] \vee e_2[B, B].$$

■

4.13. Trivial commutators.

4.14. Lemma. *For any pair of morphisms $f \colon A \to W$ and $g \colon B \to W$, if $[\mathsf{Im}\, f, \mathsf{Im}\, g] = 0$, then there exists a unique $h \colon A \oplus B \to W$ such that $he_1 = f$ and $he_2 = g$; that is $[f, g]$ exists. Conversely, if $[f, g]$ exists, then $[\mathsf{Im}\, f, \mathsf{Im}\, g] = 0$.*

Proof. Suppose $(C, e \colon W \to C, m \colon A \oplus B \to C)$ is a colimit of $\mathsf{C}(f, g)$. Suppose $[\mathsf{Im}\, f, \mathsf{Im}\, g] = 0$. Then $\mathsf{Ker} e = 0$ by Proposition 4.3, and since e is a projection by Proposition 3.7, e is an isomorphism. Then the required h is $e^{-1}m$. The converse also follows easily from Proposition 4.3. ■

4.15. Proposition. *For any object G and conormal subobjects X and Y such that $X \vee Y = 1$, if $[X, Y] = 0$, then both X and Y are normal subobjects of G.*

Proof. Since $[\mathsf{Im}\, \iota_X, \mathsf{Im}\, \iota_Y] = [X, Y] = 0$, by Lemma 4.14 there is a morphism

$$h \colon X_O \oplus Y_O \to G$$

such that $he_1 = \iota_X$ and $he_2 = \iota_Y$. Notice that h is a projection, since

$$h1 = h(e_1 1 \vee e_2 1) = he_1 1 \vee he_2 1 = X \vee Y = 1.$$

Since $e_1 1$ and $e_2 1$ are normal subobjects of $X_O \oplus Y_O$, $he_1 1$ and $he_2 1$ are normal subobjects of G, that is, X and Y are normal subobjects of G. ■

Usually in group theory, a normal subgroup X of G is defined to be a subgroup such that for any $g \in G$, $gXg^{-1} \subseteq X$, or equivalently $[X, G] \leq X$. The above result allows to prove the same here:

4.16. Corollary. *For any object G and conormal subobject X, X is a normal subobject if and only if $[X, G] \leq X$.*

Proof. By Proposition 4.5, if X is normal then $[X, G] \leq X$.

For the converse, consider the projection $p = \pi_{[X,G]}$. We have $[pX, pG] = p[X, G] = 0$, and $pX \vee pG = 1$, and pX and pG are conormal subobjects. Thus by Proposition 4.15, pX is a normal subobject of $G/[X, G]$. Since X contains the kernel of p, $X = p^{-1}pX$. Consequently X is normal as well. ∎

4.17. Remark. *If we apply this corollary to semi-abelian categories seen as noetherian forms, we recover the main result Theorem 6.3 of [14].*

One can readily observe that $[e_1 1, e_2 1] = 0$ for the embeddings of any biproduct.

Another way to recognize whether an object is a biproduct of two subobjects:

4.18. Theorem. *If object G has conormal subobjects A and B such that*

- $A \vee B = 1$,
- $A \wedge B = 0$,
- $[A, B]_G = 0$,

then $G \cong A_O \oplus B_O$.

Proof. By Proposition 4.15, the first and the last points implies that both A and B are normal subobjects. Then the result follows from Corollary 3.17. ∎

4.19. Commutative objects.

4.20. Definition. *An object A is called* commutative *when $[A, A]_A = 0$.*

From previous results on commutators, we have the following list of basic properties of commutative objects:

- for projection $p\colon A \to B$, if A is commutative, then so is B;
- for embedding $e\colon B \to A$, if A is commutative, then so is B;
- any conormal subobject of a commutative object is normal;
- $A \oplus B$ is commutative if and only if both A and B are commutative.

4.21. Theorem. *For any noetherian form $\mathbb{C}$ with biproducts, the full subcategory $\mathbb{A}$ of all commutative objects is a reflective subcategory.*

PROOF. Take any object G in $\mathbb{C}$. We have

$$[G/[G,G], G/[G,G]] = [\pi_{[G,G]}G, \pi_{[G,G]}G] = \pi_{[G,G]}[G,G] = 0.$$

Thus $G/[G,G]$ is commutative. Take any morphism $f\colon G \to A$, where A is commutative. Then we have

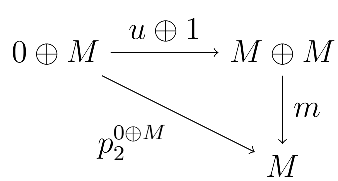
$$f[G,G] \leq [fG, fG] = 0.$$

Thus $\mathsf{Ker}\, f \geq [G,G]$, and thus there is a unique $h\colon G/[G,G] \to A$ such that $f = h\pi_{[G,G]}$. Consequently, $\mathbb{A}$ is a full reflective subcategory of $\mathbb{C}$. ∎

There is another way of getting this full subcategory of commutative objects:

4.22. THEOREM. *The internal monoids (with respect to $\oplus$) are exactly the commutative objects. The internal monoid structure is uniquely determined on every commutative object. Further any morphism between commutative objects preserves the monoid structures.*

PROOF. Suppose $(M, m\colon M \times M \to M, u\colon 0 \to M)$ is an internal monoid. Then in particular the following diagram commutes

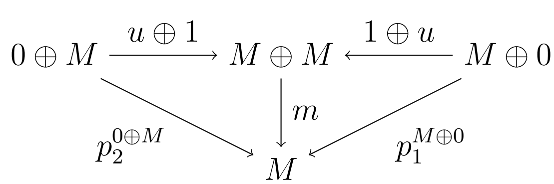


We have

$$me_2^{M\oplus M} = m(u \oplus 1)e_2^{0\oplus M} = p_2^{0\oplus M}e_2^{0\oplus M} = 1_M.$$

Similarly we get that $me_1^{M\oplus M} = 1_M$. Thus $(M, 1_M\colon M \to M, m\colon M\oplus M \to M)$ is a cocone of $\mathsf{C}_{M\oplus M}(1_M, 1_M)$. Since it is a cocone with least possible kernel, by Proposition 3.9 it is a colimit, and thus $[M,M] = [\mathsf{Im}\, 1_M, \mathsf{Im}\, 1_M] \leq \mathsf{Ker} 1_M = 0$. So M is commutative. Notice that we were forced to have $m = [1,1]$, so if an object is an internal monoid, it is uniquely so.

Conversely, take any commutative object M. By Lemma 4.14 there is a unique morphism $m = [1,1]\colon M \oplus M \to M$ such that $me_1 = 1 = me_2$. There is also a unique morphism $u\colon 0 \to M$. The following diagram commutes:



since

$$m(u \oplus 1) = [1,1](u \oplus 1) = [u,1] = p_2^{0\oplus M}.$$

and similarly the other triangle commutes. Further, the following diagram also commutes:

$$\begin{array}{ccccc} M \oplus (M \oplus M) & \xrightarrow{\alpha} & (M \oplus M) \oplus M & \xrightarrow{m \oplus 1} & M \oplus M \\ \downarrow{\scriptstyle 1 \oplus m} & & & & \downarrow{\scriptstyle m} \\ M \oplus M & & \xrightarrow{\qquad\quad m \qquad\quad} & & M \end{array}$$

Recalling from Section 3.23, $\alpha = [e_1 e_1, [e_1 e_2, e_2]]$. We have

$$\begin{aligned} & m(m \oplus 1)\alpha \\ =& m(m \oplus 1)[e_1 e_1, [e_1 e_2, e_2]] \\ =& m[(m \oplus 1)e_1 e_1, [(m \oplus 1)e_1 e_2, (m \oplus 1)e_2]] \\ =& m[e_1 m e_1, [e_1 m e_2, e_2]] \\ =& m[e_1, [e_1, e_2]] \\ =& m[e_1, 1] \\ =& [m e_1, m] \\ =& [m e_1, m e_2 m] \\ =& m[e_1, e_2 m] \\ =& m(1 \oplus m). \end{aligned}$$

For the morphism part: take any $f \colon A \to B$ between two commutative objects A and B. As demonstrated before $(A, [1_A, 1_A], u_A)$ and $(B, [1_B, 1_B], u_B)$ are the unique internal monoid structures on A and B respectively. Trivially $f u_A = u_B$, since 0 is an initial object. Also

$$[1_B, 1_B](f \oplus f) = [f, f] = f[1_A, 1_A].$$

Thus f is an internal monoid morphism from A to B. ■

4.23. Remark. *Commutative objects defined above are precisely the indiscrete objects in the sense of Definition 2.8.1 of* [9]. *Although we have included the proof of Theorem 4.22, it is actually a simple corollary of Theorem 2.8.2 of* [9]. *Furthermore, it is easy to show that biproducts give a symmetric monoidal structure and hence, by Theorem 2.8.3 of* [9], *the unique internal monoid structure on each commutative object is an internal commutative monoid structure, which answers the question posed to the author by Tim Van der Linden.*

The result below generalizes the fact that split extensions of abelian groups is the product.

4.24. Proposition. *For any commutative objects A, X, and B, if*

$$A \xrightarrow{\ g\ } X \underset{s}{\overset{f}{\rightleftarrows}} B$$

satisfies $\mathsf{Ker} f = \mathsf{Im} g$, *$g$ is an embedding and $fs = 1_B$, then $X \cong A \oplus B$.*

PROOF. The image of s is normal, since the image of s is conormal and X is commutative. Further, the kernel of f is conormal, since it is the image of g. Thus, by Corollary 3.18, we have

$$X \cong (\mathsf{Ker}\, f)_O \oplus (\mathsf{Im}\, s)_O = A \oplus B.$$

■

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