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PUSHFORWARDS AND GAUGE TRANSFORMATIONS FOR CATEGORICAL CONNECTIONS

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Abstract. We construct and establish results for a categorical counterpart of pushforwards of connections on principal bundles. This categorical pushforward takes as input a categorical connection $\mathbb{A}_{\mathbf{P}}$ on a categorical bundle $\mathbf{P}$ and an appropriate functor $\mathbb{S} : \mathbf{P} \to \mathbf{Q}$ and outputs a categorical connection $\mathbb{S}_*\mathbb{A}_{\mathbf{P}}$ on the categorical bundle $\mathbf{Q}$. Applying this construction to the case of categorical bundles arising from decorated path spaces in principal bundles, we obtain a transformation of classical connections that combines the traditional gauge transformation with an affine translation.

1. Introduction

Categorical bundle theory provides a rich framework within which geometric notions such as connections and parallel transport can be formulated and studied at multiple levels using the language and techniques of category theory. There are distinct formalisms for categorical bundle theory; in this paper we follow the categorical framework for connections over path spaces developed in [11, 12]; for ease of reference, section 4 includes a largely self-contained description of the framework. Very briefly put, a categorical principal bundle is a categorical counterpart of a classical principal bundle; it is given by a functor $\pi : \mathbf{P} \to \mathbf{M}$, where $\mathbf{P}$, the 'bundle category', is a category on which a given categorical group $\mathbf{G}$ acts and $\mathbf{M}$ is the 'base category', just as in the traditional case a principal G-bundle is given by means of a surjective map $\pi : P \to M$ between manifolds, along with an action of the Lie group G on P. Categorical groups (defined in section 2) are essential to our whole framework in the same way that Lie groups are to traditional principal bundle theory. Just as a connection on $\pi : P \to M$ can be specified through all horizontal lifts of paths on M to paths on P, a categorical connection is defined as a specification of 'horizontal lifts' of functors in the base category $\mathbf{M}$ to functors in the bundle category $\mathbf{P}$, satisfying certain geometrically-motivated conditions.

For a good enough bundle morphism $f : P \to Q$ between classical principal bundles there is the traditional notion of pushforward for any connection on P to a connection on Q (horizontal paths in Q would be f-images of horizontal paths in P). A major goal of this paper is to *introduce and study a categorical counterpart of the traditional pushforward*

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for categorical connections on categorical bundles. The definition of pushforward of a categorical connection is given in section 6.2 (equations (6.3)).

Using the new notion of pushforwards of categorical connections we study a concrete example arising from categories associated to path spaces. This leads us, in section 8, to an extended form of the traditional gauge transformation of connection forms.

The main objectives of this paper are to:

1. construct and study the notion of a *pushforward* for connections on categorical bundles;

2. use the pushforward to construct an extension of the notion of the traditional gauge transformation to include affine translates $A \mapsto \phi^* A + \Lambda$, where ϕ is a traditional gauge transformation and Λ is an appropriate type of 1-form.

A principal G-bundle is a smooth submersion $\pi : P \to M$ of manifolds, along with a smooth free right action $(p, g) \mapsto pg$ of a Lie group G on P, preserving the fibers of the projection π; there is also a local triviality property (see Kobayashi and Nomizu [28] for the theory). A connection A on this bundle is a 1-form on P with values in the Lie algebra $L(G)$ with certain properties; the geometric significance of A is that it leads to a way of lifting a path γ on M to a path $\tilde{\gamma}_p$, initiating at p, on P, with $\pi \circ \tilde{\gamma}_p = \gamma$.

To review briefly, there are different counterparts of the classical theory of bundles in the categorical framework. In the approach we follow, a categorical principal bundle is given by a functor $\pi_{\mathbf{P}} : \mathbf{P} \to \mathbf{M}$, and there is an action $\mathbf{P} \times \mathbf{G} \to \mathbf{M}$, where $\mathbf{G}$ is a categorical group; we will explain these concepts in section 4. A categorical connection is a prescription to lift morphisms γ of the base category $\mathbf{M}$ to morphisms $\overline{\gamma}_p$ in $\mathbf{P}$. A traditional gauge transformation is a smooth mapping $P \to P$ that preserves fibers and the action of G, and is specified by a smooth function $\theta : P \to G$ which has an equivariance property. We will show (in section 7.4) that the categorical counterpart of this, a *categorical gauge transformation*, is specified by both *the function* θ *and a* 1*-form on* $\mathrm{Mor}(\mathbf{P})$ *that takes values in the Lie algebra of* a subgroup of $\mathrm{Mor}(\mathbf{G})$.

1.1. Technical description. In classical bundle theory a connection on a principal bundle can be pushed forward to produce a connection on a different bundle. In more detail, suppose $\pi_P : P \to M$ and $\pi_Q : Q \to M$ are principal G- and $K-$bundles, where G and K are Lie groups. Suppose $s : G \to K$ is a Lie group homomorphism and

$$\begin{array}{ccc} P & \xrightarrow{\quad S \quad} & Q \\ & {\scriptstyle \pi_P}\searrow \quad \swarrow{\scriptstyle \pi_Q} & \\ & M & \end{array} \tag{1.1}$$

a commutative diagram, with S a smooth map that satisfies $S(pg) = S(p)s(g)$ for all $p \in P$ and $g \in G$. Then a connection A_P on P produces a connection $S_* A_P$ on Q essentially by declaring that S map A_P-horizontal paths on P to $S_* A_P$-horizontal paths on Q. (For details see [28, Proposition 6.2].)

A basic example is the case where M is a Riemannian manifold of dimension n, K is the orthogonal group $O(n)$, with s being the inclusion into the general linear group $GL(n)$, P is the bundle of orthonormal frames for the tangent bundle of M, and S is the inclusion map into the bundle Q of all frames over M.

In section 4 we give a self-contained description of the mathematical formalism developed in our earlier works, but in a modified form that brings out some features more clearly. We review the notion of a categorical group $\mathbf{G}$, which involves two groups G and H intertwined in a special structure called a crossed module; when $\mathbf{G}$ is a categorical Lie group, the associated groups G and H are Lie groups. We also describe the notions of categorical path spaces and a categorical principal bundle $\pi_{\mathbf{P}} : \mathbf{P} \to \mathbf{M}$. Briefly, $\mathbf{M}$ is a category whose objects are points of a manifold and whose morphisms correspond to paths on the manifold. A central example of interest for categorical bundles is that of a *decorated* bundle $\mathbf{P}^{A,\mathrm{dec}} \to \mathbf{M}$, which arises from a classical principal G-bundle $\pi : P \to M$, a connection A on this bundle, and an additional new structure group H as mentioned above. Then the objects of $\mathbf{P}^{A,\mathrm{dec}}$ are just the points of P, while morphisms are of the form $(\tilde{\gamma}^A, h)$, where $\tilde{\gamma}^A$ is any A-horizontal path on P and $h \in H$ is a decoration of that path.

In section 6 we construct and study the categorical counterpart of the pushforward (1.1). Briefly, if

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\ \mathbb{S}\ } & \mathbf{Q} \\ & {\scriptstyle \pi_{\mathbf{P}}} \searrow \quad \swarrow {\scriptstyle \pi_{\mathbf{Q}}} & \\ & \mathbf{M} & \end{array} \tag{1.2}$$

is a categorical counterpart to the diagram (1.1) then, by pushing forward horizontal lifts, we obtain a categorical connection $\mathbb{S}_*\mathbf{A}_{\mathbf{P}}$ on $\mathbf{Q}$ from a given categorical connection $\mathbf{A}_{\mathbf{P}}$ on $\mathbf{P}$.

In our presentation we build up to this general notion of pushforward by *first studying examples* of interest in sections 4.4 and 5.7.

In section 5.5 we use a process that is a kind of inverse of the pushforward that works only in the context we need. Briefly, if $\mathbb{S} : \mathbf{P} \to \mathbf{Q}$ is a functor between categorical principal bundles, preserving all relevant structures, then a categorical connection $\mathbf{A}_{\mathbf{Q}}$ can be "lifted" to a categorical connection $\mathbf{A}_{\mathbf{P}}$ on $\mathbf{P}$ such that the pushforward of $\mathbf{A}_{\mathbf{P}}$ to $\mathbf{Q}$ is the original connection $\mathbf{A}_{\mathbf{Q}}$.

We use pushforwards to construct an extension of the notion of gauge transformation of connections. In classical bundle theory, a global gauge transformation is specified by a smooth map $\theta : P \to G$ that is equivariant in the sense that

$$\theta_{pg} = g^{-1}\theta_p g, \tag{1.3}$$

for all $p \in P$ and $g \in G$. In Theorem 7.5 we show that a categorical gauge transformation on $\mathbf{P}^{A,\mathrm{dec}}$ is specified by a pair (θ, Λ^H), where $\theta : P \to G$ is as above, and Λ^H is a smooth 1-form on P with values in the Lie algebra $L(H)$ (recall $\mathbf{G}$ is associated with two Lie groups, G and H) that satisfies the equivariance condition

$$\Lambda^H_{pg}(vg) = \alpha(g^{-1})\Lambda^H_p(v), \tag{1.4}$$

for all $p \in P$, $v \in T_pP$, and $g \in P$.

A *categorical bundle morphism* $\Theta : \mathbf{P}^{A,\mathrm{dec}} \to \mathbf{P}^{A,\mathrm{dec}}$ is a functor that preserves the categorical bundle structure. In Theorem 7.5 we find the detailed structure of such bundle morphisms. The morphism Θ induces a connection on the classical bundle P, by keeping track of what happens to the horizontal path $\overline{\gamma}$. Thus the traditional connection A gives rise, through this process, to a new connection, and it is this generalized gauge transformation that we introduce and study in section 8.

In our concluding result, Theorem 8.5, we show that the action of the categorical gauge transformation on categorical connections leads to the following transformation of the classical gauge field A:

$$A \mapsto \mathrm{Ad}(\theta)A - (d\theta)\theta^{-1} + \tau\Lambda^H. \tag{1.5}$$

In [13, equation (1.2)] we used a different approach, in terms of local trivializations of bundles, to obtain a version of this result, with all forms pulled down to the base manifold. This transformation law (1.5) is also superficially resembles to the gauge transformation law in higher gauge theories obtained by Wang [52, equation (1.2)]), within a different framework.

In Section 8.6 we give an informal overview of the application of the framework developed here for higher categorical gauge transformations, which is to be pursued formally in our upcoming work.

1.2. Background in Higher Gauge Theory. Parallel-transport over path spaces have been studied in both the mathematics and physics literature. We shall mention just a few other works, though there is now quite a substantial body of literature on different approaches to higher gauge theories. Among the early works that directly or indirectly influenced the study of higher gauge theories is Migdal's work [35], wherein a loop-space formulation of quantum chromodynamics was used. Gross [20] developed a mathematically precise theory of connections over path spaces and derived results for Yang-Mills theory using this framework. Singer [42] made use of geometry over path bundles in the context of quantum Yang-Mills theory. Alvarez, Ferreira, and Sánchez-Guillén [1], and later [2], studied the problem of finding conserved quantities in integrable field-theoretic systems. Here they considered parallel transport over higher-dimensional geometric objects, and used multiple higher forms, beyond the usual 1-form, for such parallel transport processes. Pfeiffer [38] and Girelli and Pfeiffer [19] used category-theoretic methods. Other works with a heavier category-theoretic focus include Baez and Schreiber [5], Baez, Schreiber and Huerta [4, 5], Baez and Wise [6], Martins and Picken [32–34], Parzygnat [36], and Sati, Schreiber, and Stasheff [39]. Higher gauge transformations have been studied within other frameworks by Breen and Messing [9], Schreiber and Waldorf [40, 41], Soncini and Zucchini [43], Waldorf [48–50], and Wang

[51–53]. In our approach we don't use Čech cohomology or gerbes, and we don't make any use of local trivializations. We do not assume any familiarity with higher categories [27, 31], which does provide structures and language relevant for higher gauge theory. Currently the most extensive online resource related to higher gauge theories is available at https://ncatlab.org/nlab/show/HomePage

Our approach to categorical principal bundles, developed in [11,12], is an *application of category theory to geometry.* This approach, including gauge transformations as developed in this paper, can be extended to higher path/surface spaces, but in this paper we have focused on just path space categories.

1.3. Results and organization. We review the basic notions of categorical groups as well as our framework for categorical bundle theory in sections 2 to 4. In section 4.1, we introduce a categorical bundle $\mathbf{P}^{\bullet\bullet}$ that views path categorical bundles in terms of endpoints: this structure will be very useful for the rest of the paper.

In section 5 we give a self-contained account of categorical bundles, following the framework we have introduced in earlier works, and *introduce the notion of pushing forward a categorical connection* in the context of a specific type of bundle. In Theorem 5.8 we prove that this process does produce a categorical connection.

In section 6 we introduce pushforwards of categorical connections in the general case. The main result in this section is *Theorem 6.3, which shows that the procedure of pushing forward does indeed produce a categorical connection on a new bundle.*

Next, in section 7 *we use the notion of pushforwards of categorical connections to study functorial gauge transformations of categorical connections.* The main result here, Theorem 7.5, gives a concrete description of such functorial gauge transformations.

Finally, in section 8 we specialize to the case of certain decorated categorical bundles arising from classical bundle with connections. Here we can interweave the categorical bundle theory, with its notion of pushforwards of connections, with classical bundles and connections, thereby obtaining a transformation of connections on classical bundles. This extended gauge transformation on a classical principal bundle $\pi : P \to M$ is specified by a function $\theta : P \to G$ and a 1-form Λ_H on P with values in a Lie algebra $L(H)$. The main result in this section, *Theorem 8.5, shows that, working with a classical connection A on a principal G-bundle $P \to M$, the categorical gauge transformation results in a new connection given by*

$$\mathrm{Ad}(\theta)A - (d\theta)\theta^{-1} + \tau\Lambda^H. \tag{1.6}$$

In this paper we will not go beyond connection 1-forms but the investigation may be extended to connections given by higher order forms within the general framework described in section 5.

1.4. A NOTE ON NOTATION. We will use a convenient but nonstandard convention of displaying maps or morphisms from right to left rather than left to right. Thus

$$q \xleftarrow{\;\gamma\;} p \tag{1.7}$$

is a map (or morphism or path) with domain (source or initial point) p and codomain (target or terminal point) q. The advantage of this display convention is that a composition $\delta \circ \gamma$ is displayed in the same order as δ and γ appear in $\delta \circ \gamma$:

$$r \xleftarrow{\;\delta\;} q \xleftarrow{\;\gamma\;} p$$

This convention has been used extensively by Parzygnat (for example, in [36]).

2. Categorical groups

A categorical group is a category $\mathbf{G}$ along with a functor

$$\mathbf{G} \times \mathbf{G} \to \mathbf{G}$$

that makes both $\mathrm{Obj}(\mathbf{G})$ and $\mathrm{Mor}(\mathbf{G})$ groups. A categorical Lie group is a categorical group $\mathbf{G}$ for which $\mathrm{Obj}(\mathbf{G})$ and $\mathrm{Mor}(\mathbf{G})$ are both Lie groups, the source and target maps

$$s, t : \mathrm{Mor}(\mathbf{G}) \to \mathrm{Obj}(\mathbf{G})$$

are smooth and so is the identity-assigning morphism

$$\mathrm{Obj}(\mathbf{G}) \to \mathrm{Mor}(\mathbf{G}) : a \mapsto 1_a.$$

Associated to a categorical group $\mathbf{G}$ is a crossed module (G, H, α, τ), where

$$\begin{aligned} G &= \mathrm{Obj}(\mathbf{G}) \\ H &= \ker s : \mathrm{Mor}(\mathbf{G}) \to \mathrm{Obj}(G). \end{aligned} \tag{2.1}$$

(The correspondence between categorical groups and crossed modules is credited to George Janelidze by Mac Lane [31, sec XII.8]; see also Janelidze [26]. The structure seems to appear also in Turing [47].) Thus any element of H is a morphism $e \to x$ for some $x \in G$, with e being the identity in G. The homomorphism

$$\tau : H \to G$$

is just the target map t restricted to H, and

$$\alpha : G \times H \to H : (g, h) \mapsto \alpha_g(h)$$

is given by

$$\alpha_g(h) = 1_g h 1_{g^{-1}}.$$

The categorical group $\mathbf{G}$ can be reconstructed from (G, H, α, τ) by taking $\mathbf{G}$ to have object group G and morphism group the semi-direct product $H \rtimes_\alpha G$. Henceforth,

we will write $(h, g) \in H \rtimes_\alpha G$ *as* hg*;*

(this notation carries a slight risk of confusion but is very convenient). In particular, we identify $g \in G$ with $(e, g) \in H \rtimes_\alpha G$ and $h \in H$ with (h, e). Then

$$\alpha_g(h) = ghg^{-1}. \tag{2.2}$$

We note the Peiffer identities [37]:

$$\begin{aligned} \tau\big(\alpha_g(h)\big) &= g\tau(h)g^{-1} \\ \alpha_{\tau(h)}(h') &= hh'h^{-1} \end{aligned} \tag{2.3}$$

for all $g \in G$ and $h \in H$.

With $(h, g) \in H \rtimes_\alpha G$ viewed as an element of $\mathrm{Mor}(\mathbf{G})$, the source and targets are

$$s(h, g) = g, \qquad \text{and} \qquad t(h, g) = \tau(h)g. \tag{2.4}$$

Composition of morphisms, viewed as an operation on $H \times G$, is given by

$$(h_2, g_2) \circ (h_1, g_1) = (h_2 h_1, g_1), \tag{2.5}$$

where $g_2 = \tau(h_1)g_1$ for the composition to be meaningful.

The categorical group $\mathbf{G}$ is a categorical Lie group if and only if G and H are Lie groups and α and τ are smooth.

2.1. The categorical group $\mathbf{G}^{\bullet\bullet}$. For any group G, let $\mathbf{G}^{\bullet\bullet}$ be the categorical group whose objects are the elements of G and for which there is a unique morphism $g_0 \to g_1$ for any $g_0, g_1 \in G$. Following the notational convention in (1.7), we display this unique morphism as

$$g_1 \leftarrow g_0,$$

where g_0 is the source of the morphism and g_1 is the target. In the crossed module notation $H \rtimes_\alpha G$, the group H is the same as G, with τ being just the identity map and α given by conjugation. The target map t is given by

$$g_1 = t(k, g_0) = kg_0,$$

for $k \in H = G$. Thus

$$g_1 \hookleftarrow g_0 \qquad \text{corresponds to } (g_1 g_0^{-1}, g_0) \in G \rtimes_\alpha G. \tag{2.6}$$

Thus the object group of $\mathbf{G}^{\bullet\bullet}$ is G (not to be confused with the category whose only object is G), and whose morphism group is $G \times G$. If $\mathbf{G}$ is a categorical group whose object group is G then we have the functor

$$S : \mathbf{G} \to \mathbf{G}^{\bullet\bullet}, \tag{2.7}$$

which is just the identity map $G \to G$ on objects, and takes any $\phi \in \mathrm{Mor}(\mathbf{G})$ to the morphism

$$t(\phi) \leftarrow\!\shortmid\, s(\phi) \tag{2.8}$$

in $\mathrm{Mor}(\mathbf{G}^{\bullet\bullet})$. It is readily checked that S is indeed a functor, and, moreover, it is a homorphism of groups at the object and at the morphism levels. If $\mathbf{G}$ is a categorical Lie group then so is $\mathbf{G}^{\bullet\bullet}$ in the obvious way, and S is smooth both at the object and at the morphism levels.

3. Categorical path spaces

In this section we describe the framework of path space categories that we will use. This framework, using specific parametrizations of paths, is described specifically in sections 3.2 and 3.4. Alternative frameworks, such as the thin-homotopy approach, are mentioned in section 3.5. For more on the background and ideas, in the context of string theory, connected with the description of paths in terms of parametrization see Stasheff [46] and references therein such as the papers [21, 22].

3.1. Smooth spaces. We will not need any details concerning smooth structures on path spaces but we note here some minimal background. For our purposes it is convenient to use the framework of *diffeological spaces*, introduced by Souriau [44] and discussed further by several authors [3, 23, 30]; however, we will use the term *smooth space*, which is used by Baez and Hoffnung [3] in a broader sense. There are several other approaches to smooth structures, such as the one by Fröhlicher [18]; Batubenge, Iglesias-Zemmour, Karshon and Watts [7] and Stacey [45] provide overviews and comparisons of different approaches to smoothness.

Very briefly, we take a smooth space to be a non-empty set X along with a *diffeology*, a family S_X of maps $U \to X$, called *plots*, with U running over all open subsets of all finite-dimensional spaces $\mathbb{R}^n$ with $n \geq 0$, such that: (i) all maps from the one-point space $\mathbb{R}^0$ to X are in S_X; (ii) if $\phi : U \to X$ is in S_X and if $g : V \to U$ is C^∞, where V is an open subset of some $\mathbb{R}^n$, then $\phi \circ g \in S_X$; (iii) if $\phi : U \to X$ is such that the restriction of ϕ to every member of an open covering of U is in S_X then $\phi \in S_X$.

If X is a smooth space and X_0 is a non-empty subset of X, then a diffeology on X_0 is obtained by taking as plots all the plots $U \to X$ whose images lie in X_0. In particular, a closed interval $[a, b]$ is a smooth space.

If X and Y are smooth spaces then a map $F : X \to Y$ is said to be smooth if $F \circ \phi \in S_Y$ for all $\phi \in S_X$. Thus the condition is that for any plot of X the composition with F is a plot of Y.

A surjective map $h : X \to Y$ pushes forward a diffeology S_X on X to a diffeology h_*S_X on Y, with h_*S_X consisting of all maps $h \circ \phi$ with ϕ running over S_X.

3.2. Path spaces. Let $a < b$ be real numbers, and $C_0^\infty([a,b];X)$ the set of all smooth maps $[a,b] \to X$, where X is a smooth space, that are constant near a and near b. We define a plot for $C_0^\infty([a,b];X)$ to be a smooth variation of paths on X in the following sense. Let U be any nonempty open subset of some $\mathbb{R}^m$, with $m \geq 0$. Consider a map of the form

$$U \to C_0^\infty([a,b];X) : u \mapsto \phi_u,$$

such that

$$U \times [a,b] \to X : (u,t) \mapsto \phi_u(t)$$

is smooth, and there is an $\epsilon > 0$ such that, for each $u \in U$, the path ϕ_u is constant on $[a, a+\epsilon)$ and on $(b-\epsilon, b]$. We take all such maps as the plots specifying a diffeology on $C_0^\infty([a,b];X)$. Varying a and b, gives a smooth space that is the disjoint union

$$\mathcal{P}_1(X) = \cup_{a,b\in\mathbb{R},a<b} C_0^\infty([a,b];X), \tag{3.1}$$

on which the smooth structure (diffeology) is the union of the ones on each $C_0^\infty([a,b];X)$. On $\mathcal{P}_1(X)$ there is an action of $\mathbb{R}$ by time-translation: if $u \in \mathbb{R}$ and $\gamma \in C_0^\infty([a,b];X)$ then we have the path

$$\gamma_{+u} \in C_0^\infty([a-u,b-u];X) : t \mapsto \gamma(t+u).$$

Then there is a natural surjection of $\mathcal{P}_1(X)$ onto the quotient space $\mathcal{P}_1(X)/\mathbb{R}$. The plots on $\mathcal{P}_1(X)$ composed with this projection give plots on $\mathcal{P}_1(X)/\mathbb{R}$, and make the latter into a smooth space.

Given a path $\delta : [a,b] \to X$ and a path $\gamma : [b,c] \to X$, with $\delta(b) = \gamma(b)$, we can form a composite path

$$\gamma * \delta : [a,c] \to X : u \mapsto \begin{cases} \delta(u) & \text{if } t \in [a,b]; \\ \gamma(u) & \text{if } u \in [b,c]. \end{cases} \tag{3.2}$$

The following result shows that composition of paths is a smooth operation.

3.3. Proposition. *Let $a, b, c \in \mathbb{R}$ with $a < b < c$, and let $\mathcal{K}$ be the subset of the product $C_0^\infty([b,c];X) \times C_0^\infty([a,b];X)$ consisting of all pairs (γ, δ) for which $\gamma(b) = \delta(b)$. Then the composite map*

$$K : \mathcal{K} \to C_0^\infty([a,c];X) : (\gamma,\delta) \mapsto \gamma * \delta \tag{3.3}$$

is smooth.

Proof. A plot of $\mathcal{K}$ is a plot of $C_0^\infty([b,c];X) \times C_0^\infty([a,b];X)$ that takes values in the subset $\mathcal{K}$. Thus this plot is of the form

$$(\phi, \psi)$$

where there is a non-empty open subset U of $\mathbb{R}^m$, for some $m \geq 0$, and $\phi : U \to C_0^\infty([b,c];X)$ and $\psi : U \to C_0^\infty([a,b];X)$ are smooth. Then $K \circ (\phi, \psi)$ maps $u \in U$ to $\kappa_u = \phi_u \circ \psi_u : [a,c] \to X$. Now κ_u is smooth on $[b,c]$, being the same as ϕ_u on that interval, and also smooth on $[a,b]$; furthermore, it is constant near b. It follows, by using property (iii) of plots, that κ_u is smooth on $[a,c]$. Moreover, there is an $\epsilon_1 > 0$ such that each ϕ_u is constant within an ϵ_1-neighborhood of the points b and c, and there is an $\epsilon_2 > 0$ such that each ψ_u is constant within an ϵ_2-neighborhood of a and b. Hence, κ_u is constant within an ϵ-neighborhood of a and c, where $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Thus $K \circ (\phi, \psi)$ is a plot of $C_0^\infty([a,c];X)$. This means that K is smooth. ∎

That the operation of composition $*$ is associative, without any need for using homotopy equivalence, makes it convenient for our purposes as it makes it easy to consider paths as morphisms. The framework of *Moore paths*, where continuous maps $[0,r] \to X$, with varying $r \in [0,\infty)$, are the paths and composition is done by translating the domain appropriately first, is in the same spirit as our approach.

3.4. Categorical path spaces. From a smooth space X we can construct a category $\mathbb{P}_1(X)$ as follows. The object set of $\mathbb{P}_1(X)$ is X, and the morphism set is $\mathcal{P}_1(X)/\mathbb{R}$. Let us look at the morphisms in more detail. The morphisms of $\mathbb{P}_1(X)$ arise from smooth paths $[a,b] \to X : u \mapsto \gamma(u)$, for any $a,b \in \mathbb{R}$ with $a \leq b$, constant near the endpoints a and b, with two such paths identified if one is obtained from the other by a constant translation of the parameter u. A morphism's source is the initial point, which we often denote γ_0, and the target is the terminal point, which we denote γ_1. The identity morphism at an object $p \in X$ is the equivalence class of the point-path $[a,a] \to X : a \mapsto p$. Composition of morphisms corresponds to composition of paths, with the first path terminating at the source of the second. The set of morphisms $\mathrm{Mor}(\mathbb{P}_1(X))$ has a natural smooth space structure, and the source and target maps

$$s, t : \mathrm{Mor}(\mathbb{P}_1(X)) \to X$$

are smooth.

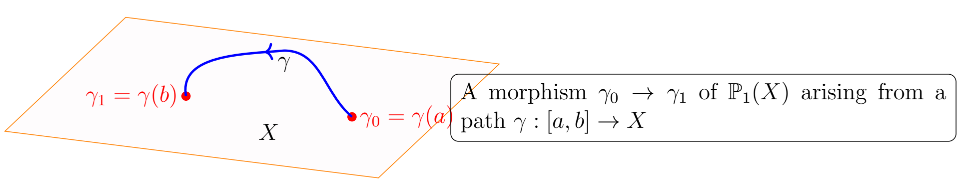


Figure 1: The category $\mathbb{P}_1(X)$

If $f : X \to Y$ is a smooth map between smooth spaces then $\gamma \mapsto f \circ \gamma$ induces a smooth map

$$\mathbb{P}_1(f) : \mathbb{P}_1(X) \to \mathbb{P}_1(Y).$$

In fact, $\mathbb{P}_1$ is a functor from the category of smooth spaces and smooth maps into itself. Lastly, let us note that $\mathbb{P}_1$ can be composed with itself multiple times to yield "higher" path spaces. There is an alternative, technically easier, way to work with higher path spaces, by viewing them as being obtained from smooth maps $[a_1, b_1] \times \ldots [a_k, b_k] \to X$ that are suitably constant near the boundary of the domain. We mention this for cultural context; we will not work with such higher path spaces in this paper.

By a *categorical space* we will mean a category for which both object set and morphism set are equipped with smooth space structures such that the following maps are smooth: (i) source and target maps; (ii) the identity assigning map $a \mapsto 1_a$; (iii) the composition of morphisms, defined on the set of all composable pairs of morphisms.

There is a special case that we use frequently, for which it is convenient to use a simpler notation. For a smooth manifold M we denote the path space category $\mathbb{P}_1(M)$ by $\mathbf{M}$.

3.5. Backtrack and Thin Homotopy Equivalences equivalence. There are several reasonable choices for the path space category. One we have used before [11] involves identifying paths that are the same except for some pieces that are backtracked. More precisely, for a path $\gamma : [a, b] \to X$ let $\gamma_{-1} : [b, b + b - a] \to X$ be given by $\gamma_{-1}(t) = \gamma(2b - t)$. It is reasonable to identify the composite $\gamma_{-1} * \gamma : [a, 2b - a] \to X$ with the constant path at $\gamma(a)$. Next we identify two paths that differ by a finite number of compositions of the type $\gamma_{-1} * \gamma$. We call this *backtrack equivalence*. With this equivalence, the paths with a fixed initial point form a group under composition. In Singer [42] a principal bundle is constructed informally for which this group of loops based at a fixed point serves as the structure group of a principal bundle. In some approaches to categorical gauge theory or geometry one uses "thin homotopy" classes of paths. Smooth paths $\gamma, \delta : [a, b] \to M$ are thin homotopy equivalent if there is a smooth map $F : [0, 1] \times [a, b] \to M : (t, s) \mapsto F_t(s)$, with $F_0 = \gamma$ and $F_1 = \delta$, such that the differential dF has rank ≤ 1 everywhere (intuitively, the image of F has zero area). Our framework of categorical bundle theory works more naturally and easily using reparametrization equivalence classes of paths rather than thin homotopy classes, just as a technical matter.

4. Categorical principal bundles

This section is devoted to the framework of categorical bundles we use. There are multiple approaches to and formulations of the notion of categorical bundles; this section focuses on the framework we have developed in earlier works such as [11], which we use in this paper.

4.1. The categorical bundle $\mathbf{P}^{\bullet\bullet}$. Let

$$\pi : P \to M$$

be a principal G-bundle, where G is a Lie group. Specifically, P and M are smooth spaces, π is a surjective submersion, and there is a smooth free right action of G on P:

$$P \times G \to P : (p, g) \mapsto pg = R_g p$$

which preserves the fibers of π. We will construct categorical spaces from this bundle. Intuitively, the category $\mathbf{P}^{\bullet\bullet}$ will have the points of P as objects, and morphisms are paths on M connecting the projections on M of the source and the target.

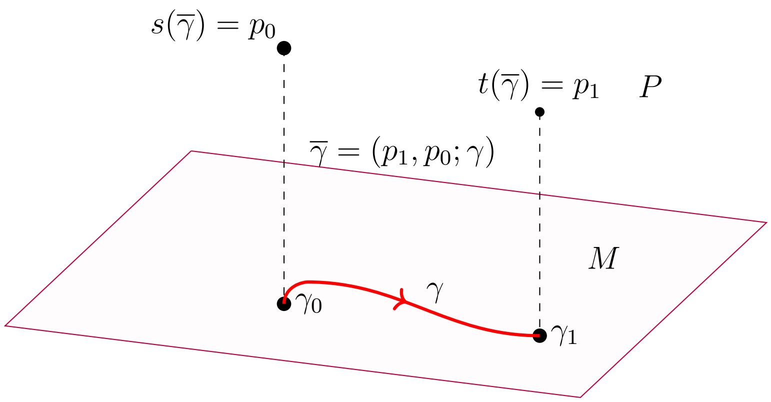
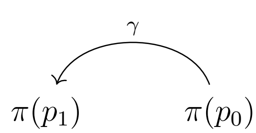


Figure 2: A morphism $\overline{\gamma} = (p_1, p_0; \gamma)$ of $\mathbf{P}^{\bullet\bullet}$.

More precisely, we define $\mathbf{P}^{\bullet\bullet}$ to be the category whose object set is P and whose morphisms are of the form

$$(p_1, p_0; \gamma) \in P \times P \times \mathrm{Mor}(\mathbf{M}),$$

with γ having source $\pi(p_0)$ and target $\pi(p_1)$:



Source and targets are given by

$$s(p_1, p_0; \gamma) = p_0 \qquad \text{and} \qquad t(p_1, p_0; \gamma) = p_1. \tag{4.1}$$

Composition is given by

$$(p_2, p_1; \delta) \circ (p_1, p_0; \gamma) = (p_2, p_0; \delta \circ \gamma). \tag{4.2}$$

The identity morphism at p is $(p, p; 1_{\pi(p)})$, where 1_u is the point-path at u.

A *categorical right action* of a categorical group $\mathbf{G}$ on a category $\mathbf{X}$ is a functor

$$\mathbf{R} : \mathbf{X} \times \mathbf{G} \to \mathbf{X},$$

which is a group right action at both object and morphism levels. In the context of categorical Lie groups and categorical spaces we also require that the actions, both at the object level and the morphism level, be smooth.

The categorical group $\mathbf{G}^{\bullet\bullet}$ has a categorical right action on $\mathbf{P}^{\bullet\bullet}$ given on objects by the action of G on P and on morphisms by

$$(p_1, p_0; \gamma)(g_1 \leftarrow g_0) = (p_1 g_1, p_0 g_0; \gamma). \tag{4.3}$$

We have the projection functor

$$\pi : \mathbf{P}^{\bullet\bullet} \to \mathbf{M},$$

given on objects by the bundle projection $\pi : P \to M$ and on morphisms by

$$\pi(p_1, p_0; \gamma) = \gamma.$$

4.2. The decorated categorical bundle $\mathbf{P}^{A,\text{dec}}$. Now consider a connection A on a principal G-bundle $\pi : P \to M$, and let $\mathbf{G}$ be a categorical Lie group with associated Lie crossed module (G, H, α, τ). From this we can construct a categorical principal $\mathbf{G}$-bundle that we call *decorated* bundle and denote

$$\pi : \mathbf{P}^{A,\text{dec}} \to \mathbf{M}. \tag{4.4}$$

The object space of $\mathbf{P}^{A,\text{dec}}$ is P. A morphism of $\mathbf{P}^{A,\text{dec}}$ is to be thought of as an A-horizontal path on P equipped with a decorating element drawn from H. More precisely, a morphism of $\mathbf{P}^{A,\text{dec}}$ is of the form

$$\overline{\gamma} = (\tilde{\gamma}, h),$$

where $\tilde{\gamma}$ is a morphism of $\mathbf{P}$ coming from an A-horizontal path on P, smooth and constant near its initial and terminal points, and $h \in H$. Source and target maps are defined by:

$$\begin{aligned} s(\overline{\gamma}; h) &= s(\tilde{\gamma}) \\ t(\overline{\gamma}; h) &= t(\tilde{\gamma})\tau(h), \end{aligned} \tag{4.5}$$

where $s(\tilde{\gamma}) = \tilde{\gamma}_0$ is the initial point of $\tilde{\gamma}$ and $t(\tilde{\gamma}) = \tilde{\gamma}_1$ is the terminal point. We call (4.4) the *decorated bundle* corresponding to the bundle $\pi : P \to M$ and connection A.

The categorical group $\mathbf{G}$ acts on $\mathbf{P}^{A,\text{dec}}$ on objects by the action of G on P and on morphisms by:

$$(\tilde{\gamma}; h)h'g' = (\tilde{\gamma}g'; \gamma; {g'}^{-1}hh'g'). \tag{4.6}$$

Here it is useful to recall that the notation hg is really a short form of (h, g). For $(\tilde{\delta}; h_2), (\tilde{\gamma}; h_1) \in \text{Mor}(\mathbf{P}^{A,\text{dec}})$, with

$$\tilde{\delta}_0 = \tilde{\gamma}_1 \tau(h_1), \tag{4.7}$$

the composition of morphisms is defined by

$$(\tilde{\delta}; h_2) \circ (\tilde{\gamma}; h_1) = \big(\tilde{\delta}\tau(h_1)^{-1} \circ \tilde{\gamma}; h_1 h_2\big). \tag{4.8}$$

The source of either side is $\tilde{\gamma}_0$ and the target of either side is $\tilde{\delta}_1 \tau(h_2)$. Moreover, the identity morphism at p is $(1_p; e)$, where 1_p is the constant point-path at p.

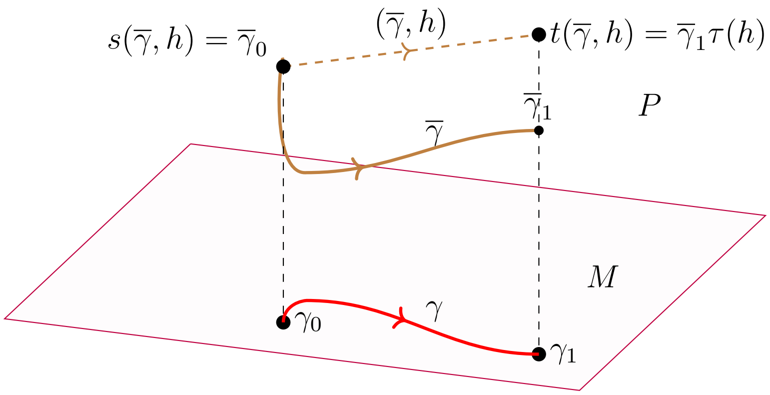


Figure 3: The decorated bundle, showing the source and target of a morphism $(\overline{\gamma}, h)$. Only $\left(\overline{\gamma}_1, \overline{\gamma}_0; \gamma\right)$ is needed about $\overline{\gamma}$.

We check the behavior of compositions under the action of the categorical group. Consider the composition (4.8) under the action of a composition $(h_2'g_2') \circ (h_1'g_1')$; for the latter to be meaningful we have $g_2' = \tau(h_1')g_1'$. With this, we have:

$$\begin{aligned}
&(\tilde{\delta}; h_2)(h_2'g_2') \circ (\tilde{\gamma}; h_1)(h_1'g_1') \\
&= \left(\tilde{\delta}g_2'; {g'}_2^{-1}h_2h_2'g_2'\right) \circ \left(\tilde{\gamma}g_1'; {g'}_1^{-1}h_1h_1'g_1'\right) \\
&= \left(\tilde{\delta}\tau(h_1)^{-1}g_1' \circ \tilde{\gamma}g_1';\ {g'}_1^{-1}h_1h_2h_2'h_1'g_1'\right),
\end{aligned} \tag{4.9}$$

which agrees with

$$\begin{aligned}
&\big((\tilde{\delta}; h_2) \circ (\tilde{\gamma}; h_1)\big)\big((h_2'g_2') \circ (h_1'g_1')\big) \\
&= \left(\tilde{\delta}\tau(h_1)^{-1} \circ \tilde{\gamma};\ h_1h_2\right)(h_2'h_1'g_1') \\
&= \left(\left(\tilde{\delta}\tau(h_1)^{-1} \circ \tilde{\gamma}\right)g_1';\ {g'}_1^{-1}h_1h_2h_2'h_1'g_1'\right) \\
&= \left(\tilde{\delta}\tau(h_1)^{-1}g_1' \circ \tilde{\gamma}g_1';\ {g'}_1^{-1}h_1h_2h_2'h_1'g_1'\right).
\end{aligned} \tag{4.10}$$

This means that compositions of morphisms in $\mathbf{G}$ and $\mathbf{P}^{A,\mathrm{dec}}$ commute with the action of $\mathrm{Mor}(\mathbf{G})$ on $\mathrm{Mor}(\mathbf{P}^{A,\mathrm{dec}})$.

4.3. Categorical principal bundles. We have discussed $\mathbf{P}^{\bullet\bullet}$ and $\mathbf{P}^{A,\mathrm{dec}}$. These are both examples of the general notion of a categorical principal bundle as defined in our earlier work [11]. Let $\mathbf{G}$ be a categorical Lie group, and let $\mathbf{M}$ be a categorical space; we have in mind the usual case where $\mathbf{M}$ arises from a manifold M. A *categorical principal bundle* with *structure categorical group* $\mathbf{G}$ is comprised of categorical spaces $\mathbf{P}$ and $\mathbf{M}$, a functor

$$\pi : \mathbf{P} \to \mathbf{M}$$

that is smooth and surjective both on the level of objects and on the level of morphisms, along with a functor

$$\mathbf{P} \times \mathbf{G} \to \mathbf{P}$$

that is a free smooth right action both on objects and on morphisms, such that $\pi(pg) = \pi(p)$ for all objects/morphisms p of $\mathbf{P}$ and all objects/morphisms g of $\mathbf{G}$. In practice we are only concerned with the case where $\mathbf{G}$ is a categorical Lie group, $\mathrm{Obj}(\mathbf{P})$ and $\mathrm{Obj}(\mathbf{M})$ are smooth manifolds, and the object bundle

$$\mathrm{Obj}(\mathbf{P}) \to \mathrm{Obj}(\mathbf{M})$$

is a principal G-bundle, where $G = \mathrm{Obj}(\mathbf{G})$.

4.4. The functor $\mathbb{S}$. Let $\mathbf{P} \to \mathbf{M}$ be a categorical principal $\mathbf{G}$-bundle, and $\mathbf{P}^{\bullet\bullet} \to \mathbf{M}$ the categorical $\mathbf{G}^{\bullet\bullet}$-bundle discussed earlier, obtained from the object principal G-bundle $\pi : P \to M$. Let

$$\mathbb{S} : \mathbf{P} \to \mathbf{P}^{\bullet\bullet}, \tag{4.11}$$

be given on objects by $p \mapsto p$ and on morphisms by

$$\mathbb{S}(\tilde{\gamma}) = (\tilde{\gamma}_1, \tilde{\gamma}_0; \pi(\tilde{\gamma})), \tag{4.12}$$

where the subscripts 0 and 1 signify source and target, respectively. It is readily verified that this is a functor (commutes with source and targets, respects compositions, and maps identities to identities). Moreover, for any $\phi \in \mathrm{Mor}(\mathbf{G})$, we also have

$$\mathbb{S}(\tilde{\gamma}\phi) = \big(\tilde{\gamma}_1\phi_1, \tilde{\gamma}_0\phi_0; \pi(\tilde{\gamma})\big) = \mathbb{S}(\tilde{\gamma})S(\phi), \tag{4.13}$$

where the first equality holds because of the functorial nature of the action of $\mathbf{G}$ on $\mathbf{P}$ and the second equality is verified from the definition of S given in (2.7) and (2.8).

4.5. The functor $\mathbb{S}$ for decorated bundles. Now we specialize to the case where $\mathbf{P} \to \mathbf{M}$ is the usual decorated bundle $\mathbf{P}^{A,\mathrm{dec}} \to \mathbf{M}$, with A being a connection on the underlying object bundle $P \to M$, as discussed in section 4.2. We recall that the $\mathbf{P}^{A,\mathrm{dec}}$ and $\mathbf{P}^{\bullet\bullet}$ both have P as object space, but morphisms of $\mathbf{P}^{\bullet\bullet}$ are of the form $(p_1, p_0; \gamma)$, where $\gamma \in \mathrm{Mor}(\mathbf{M})$ runs from source $\pi(p_0)$ to target $\pi(p_1)$, while morphisms of $\mathbf{P}^{A,\mathrm{dec}}$ are of the form $(\tilde{\gamma}; h)$, where now $\tilde{\gamma}$ is an A-horizontal morphism of $\mathbf{P}$ and h is a general element of H. Then we have the functor

$$\mathbb{S} : \mathbf{P}^{A,\mathrm{dec}} \to \mathbf{P}^{\bullet\bullet}, \tag{4.14}$$

given on objects by $p \mapsto p$ and on morphisms by

$$\mathbb{S}(\tilde{\gamma}; h) = (\tilde{\gamma}_1\tau(h), \tilde{\gamma}_0; \gamma), \tag{4.15}$$

where $\tilde{\gamma}_0 = s(\tilde{\gamma})$ and $\tilde{\gamma}_1 = t(\tilde{\gamma})$. Let us verify for this case the properties of $\mathbb{S}$ noted in the general context earlier.

4.6. Proposition. *The assignment* $\mathbb{S}$ *given above is a functor. Moreover, it is a morphism of categorical principal bundles in the following sense:*

$$\begin{aligned} \mathbb{S}(pg) &= \mathbb{S}(p)g \qquad \text{for all } (p,g) \in P \times G, \\ \mathbb{S}\big((\tilde{\gamma};h)h'g'\big) &= \mathbb{S}(\tilde{\gamma};h)S(h',g') \end{aligned} \tag{4.16}$$

for all $(\tilde{\gamma};h) \in \mathrm{Mor}\left(\mathbf{P}^{A,\mathrm{dec}}\right)$ *and* $(h',g') \in H \rtimes_\alpha G$*, and* S *is as given in (2.7) and (2.8).*

Proof. From (4.15) it is readily seen that $\mathbb{S}$ maps sources and targets properly. Moreover, for compositions we have:

$$\begin{aligned} \mathbb{S}\big((\tilde{\delta};h_2) \circ (\tilde{\gamma};h_1)\big) &= \mathbb{S}(\tilde{\delta}\tau(h_1)^{-1} \circ \tilde{\gamma}; h_1h_2) \\ &= \big(\tilde{\delta}_1\tau(h_2), \tilde{\gamma}_0;\ \delta \circ \gamma\big), \end{aligned} \tag{4.17}$$

which agrees with

$$\begin{aligned} \mathbb{S}(\tilde{\delta};h_2) \circ \mathbb{S}(\tilde{\gamma};h_1) &= \big(\tilde{\delta}_1\tau(h_2), \tilde{\delta}_0; \delta\big) \circ \big(\tilde{\gamma}_1\tau(h_1), \tilde{\gamma}_0; \gamma\big) \\ &= \big(\tilde{\delta}_1\tau(h_2), \tilde{\gamma}_0; \delta \circ \gamma\big), \end{aligned} \tag{4.18}$$

where we used the composition law (4.2).

The first equation in (4.16) is immediate from the definition of $\mathbb{S}$ in (4.14) acting on objects. Next, using the definition (4.15) of $\mathbb{S}$ on morphisms and the action given by (4.6), we have

$$\begin{aligned} \mathbb{S}\big((\tilde{\gamma};h)h'g'\big) &= \mathbb{S}(\tilde{\gamma}g'; g'^{-1}hh'g') \\ &= \big(\tilde{\gamma}_1\tau(hh')g', \tilde{\gamma}_0 g'; \gamma\big), \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} \mathbb{S}(\tilde{\gamma};h)S(h'g') &= \big(\tilde{\gamma}_1\tau(h), \tilde{\gamma}_0; \gamma\big)\big(\tau(h')g' \leftarrow g'\big) \\ &= \big(\tilde{\gamma}_1\tau(hh')g', \tilde{\gamma}_0 g'; \gamma\big). \end{aligned} \tag{4.20}$$

Thus $\mathbb{S}$ satisfies (4.16). ∎

5. Connections on Categorical Bundles

Let $\mathbf{G}$ be a categorical Lie group with associated Lie crossed module (G, H, α, τ). We will now look at a counterpart of some of the previously discussed constructions, but with the traditional principal bundle $\pi : P \to M$, with classical connection A, replaced by a categorical bundle $\pi : \mathbf{P} \to \mathbf{M}$ with a categorical analog of a classical connection. To avoid confusion of terminologies we emphasize that this analogy should be interpreted in a very specific sense as defined below. Here we do not make any effort to construct a differential geometric connection on the principal bundle over a path space category. In fact a differential geometric connection on a bundle over a path space category provides an enriched version of the one we are interested in this paper, which we have studied in some of our earlier papers [11, 12]. The construction of such "higher connections" necessarily involves construction of differential forms on a path space, viewed as a diffeological space. Foundational work in this direction was initiated by K. T. Chen in a series of articles [14–17]. Among more recent works on connection structures on bundles over path spaces we mention the papers by Kohno [29], Igusa on superconnections on graded vector spaces over a path space [25], and by Block and Smith on Riemann-Hilbert correspondence [8]. While in this paper we do not explore such higher connections, and focus mainly on the study of gauge transformations of categorical connections, the framework we will be developing here can be adapted for a higher structure (see Section 8.6 and the concluding remarks).

5.1. Categorical connections. A connection A on a principal G-bundle specifies a special path $\tilde{\gamma}_p^A$ on P, starting at any given point p on the fiber over γ_0, and is called the A-horizontal lift of γ starting at p. This generalizes then readily to categorical bundles. A *categorical connection* $\mathbf{A}$ on a categorical principal $\mathbf{G}$-bundle $\pi : \mathbf{P} \to \mathbf{M}$ assigns to each $\gamma \in \mathrm{Mor}(\mathbf{M})$ and each $p \in \mathrm{Obj}(\mathbf{P})$ with $\pi(p) = s(\gamma)$, a morphism, called the *horizontal lift*,

$$\tau_{\mathbf{A}}(\gamma; p) \in \mathrm{Mor}(\mathbf{P})$$

with source p and whose π-projection is γ, such that the following conditions hold:

(CC1) If $\gamma = 1_u$, the identity at $u = \pi(p) \in \mathrm{Obj}(\mathbf{M})$, then $\tau_{\mathbf{A}}(\gamma; p) = 1_p$;

(CC2) $\tau_{\mathbf{A}}(\gamma; pg) = \tau_{\mathbf{A}}(\gamma; p)1_g$ for all $g \in G$;

(CC3) If $\gamma, \delta \in \mathrm{Mor}(\mathbf{M})$ are such that the composite $\delta \circ \gamma$ is defined then the horizontal lift $\tau_{\mathbf{A}}(\delta \circ \gamma; p)$ is the composite of the horizontal lift $\tau_{\mathbf{A}}(\gamma; p)$ followed by the horizontal lift of δ:

$$\tau_{\mathbf{A}}(\delta \circ \gamma; p) = \tau_{\mathbf{A}}\big(\delta; t\big(\tau_{\mathbf{A}}(\gamma; p)\big)\big) \circ \tau_{\mathbf{A}}(\gamma; p). \tag{5.1}$$

We also require that $(\gamma; p) \to \tau_{\mathbf{A}}(\gamma; p)$ be smooth, where, of course, $p \in \mathrm{Obj}(\mathbf{P})$ and $\gamma \in \mathrm{Mor}(\mathbf{M})$ are such that $\pi(p) = s(\gamma)$,

5.2. The standard example. Let A be a connection on a classical principal G-bundle $\pi : P \to M$, where $G = \mathrm{Obj}(\mathbf{G})$ is the object group of a categorical Lie group $\mathbf{G}$ associated to the crossed module (G, H, α, τ). Then we can construct a categorical connection $\mathbf{A}^{\bullet\bullet}$ on the categorical $\mathbf{G}^{\bullet\bullet}$-bundle $\pi : \mathbf{P}^{\bullet\bullet} \to \mathbf{M}$ by setting

$$\tau_{\mathbf{A}^{\bullet\bullet}}(\gamma; p) = (q, p; \gamma), \tag{5.2}$$

where q is the point obtained by parallel transporting p along γ by A.

Intuitively, every categorical connection on $\mathbf{P}^{\bullet\bullet}$ arises in this way from a classical connection on P.

5.3. The horizontal bundle $\mathbf{P}^{\mathbf{A}}$. With setting as above, by a *horizontal morphism* we shall mean a morphism of the form $\tau_{\mathbf{A}}(\gamma; p)$. Property (CC3) implies that the composition of horizontal lifts is horizontal. Thus we have a category $\mathbf{P}^{\mathbf{A}}$, whose object set is $\mathrm{Obj}(\mathbf{P})$ and whose morphisms are all the horizontal morphisms. The categorical group involved for this bundle has object group G and the only morphisms are the identity morphisms 1_g for all $g \in G$.

5.4. The decorated bundle $\mathbf{P}^{\mathbf{A},\mathrm{dec}}$. In section 4.2 we saw how a classical connection A on a principal G-bundle $\pi : P \to M$, along with a categorical Lie group $\mathbf{G}$ whose object group is G, lead to a categorical $\mathbf{G}$-bundle $\pi : \mathbf{P}^{A,\mathrm{dec}} \to \mathbf{M}$. Here we shall see how this process generalizes to a categorical connection on a categorical principal bundle. Let $\mathbf{G}$ and $\mathbf{G}_1$ be categorical Lie groups, with

$$\mathrm{Obj}(\mathbf{G}_1) = \mathrm{Obj}(\mathbf{G}) = G.$$

We have in mind the case where $\mathbf{G}_1$ is $\mathbf{G}^{\bullet\bullet}$. As before we have taken (G, H, α, τ) to be the Lie crossed module associated to $\mathbf{G}$.

Let $\mathbf{A}$ be a categorical connection on a categorical principal $\mathbf{G}_1$-bundle $\pi : \mathbf{P} \to \mathbf{M}$. Thus to each $\gamma \in \mathrm{Mor}(\mathbf{M})$ and $p \in \mathrm{Obj}(\mathbf{P})$ there is the associated horizontal lift $\tau_{\mathbf{A}}(\gamma; p)$. The *decorated* categorical principal $\mathbf{G}$-bundle

$$\mathbf{P}^{\mathbf{A},\mathrm{dec}}, \tag{5.3}$$

has

$$\mathrm{Obj}(\mathbf{P}^{\mathbf{A},\mathrm{dec}}) = P, \qquad \mathrm{Mor}(\mathbf{P}^{\mathbf{A},\mathrm{dec}}) = \mathrm{Mor}(\mathbf{P}^{\mathbf{A}}) \times H. \tag{5.4}$$

Source and targets are given by

$$s(\overline{\gamma}; h) = \overline{\gamma}_0, \qquad t(\overline{\gamma}; h) = \overline{\gamma}_1 \tau(h), \tag{5.5}$$

where, as always, the subscripts 0 and 1 signify source and target, respectively. We have the functor

$$\mathbb{S} : \mathbf{P}^{\mathbf{A},\mathrm{dec}} \to \mathbf{P}^{\bullet\bullet}, \tag{5.6}$$

given on objects by $p \mapsto p$ and on morphisms by

$$\mathbb{S}(\overline{\gamma}; h) = \big(\overline{\gamma}_1 \tau(h), \overline{\gamma}_0; \pi(\overline{\gamma})\big). \tag{5.7}$$

This is the more abstract form of the functor $\mathbb{S}$ introduced in (4.16) in the context of the decorated categorical bundle arising from a classical principal G-bundle with connection for A.

5.5. Lifting a connection. Our goal here and in the next subsection is to transfer connections from one bundle to another. First, here, we see how a connection on $\mathbf{P}^{\bullet\bullet}$ can be lifted to a connection on $\mathbf{P}^{\mathbf{A},\mathrm{dec}}$, by decorating each horizontal morphism in $\mathbf{P}^{\bullet\bullet}$ with the identity $e \in H$.

5.6. Proposition. *With notation as above, let* $\mathbf{A}$ *be a categorical connection on the categorical* $\mathbf{G}^{\bullet\bullet}$*-bundle* $\mathbf{P}^{\bullet\bullet}$*. For any* $\gamma \in \mathrm{Mor}(\mathbf{M})$ *and* $p \in P$ *on the fiber over* $s(\gamma)$*, let*

$$\tau_{\mathbf{A}_d}(\gamma; p) = \big(\tau_{\mathbf{A}}(\gamma; p), e\big) \in \mathrm{Mor}(\mathbf{P}^{\mathbf{A},\mathrm{dec}}). \tag{5.8}$$

Then $\mathbf{A}_d$ *is a categorical connection on* $\mathbf{P}^{\mathbf{A},\mathrm{dec}}$*.*

Proof. First we note that $\big(\tau_{\mathbf{A}}(\gamma; p), e\big)$ is indeed in $\mathrm{Mor}(\mathbf{P}^{\mathbf{A},\mathrm{dec}})$. The condition (CC1) is readily verified. Condition (CC2) follows by applying the definition (4.6) of the action of $H \rtimes_\alpha G$ on $\mathrm{Mor}(\mathbf{P}^{\mathbf{A},\mathrm{dec}})$. Lastly, (CC3) follows by using the composition specification (4.8). Smoothness of $(\gamma; p) \mapsto \big(\tau_{\mathbf{A}}(\gamma; p), e\big)$ follows from smoothness of $\tau_{\mathbf{A}}$. ∎

5.7. Pushing forward a connection. Let $\pi : \mathbf{P} \to \mathbf{M}$ be a categorical $\mathbf{G}$-bundle and $\pi : \mathbf{P}^{\bullet\bullet} \to \mathbf{M}$ the corresponding categorical $\mathbf{G}^{\bullet\bullet}$-bundle. We have then the functor

$$\mathbb{S} : \mathbf{P} \to \mathbf{P}^{\bullet\bullet}, \tag{5.9}$$

introduced in (4.11). Now suppose $\mathbf{A}$ is a categorical connection on $\mathbf{P} \to \mathbf{M}$. We construct a categorical connection $\mathbf{A}^{\bullet\bullet}$ on $\mathbf{P}^{\bullet\bullet} \to \mathbf{M}$ as follows. For $\gamma \in \mathrm{Mor}(\mathbf{M})$ and $p \in \mathrm{Obj}(\mathbf{P})$ on the fiber above the source γ_0, we set

$$\tau_{\mathbf{A}^{\bullet\bullet}}(\gamma; p) = \mathbb{S}\big(\tau_{\mathbf{A}}(\gamma; p)\big). \tag{5.10}$$

Because of the properties of $\mathbb{S}$ this assignment defines a categorical connection $\mathbf{A}^{\bullet\bullet}$.

Specializing this pushforward process to a connection on $\mathbf{P}^{\mathbf{A},\mathrm{dec}}$ produces a connection on $\mathbf{P}^{\bullet\bullet}$, which we verify in the following result.

5.8. Theorem. *With notation as above, suppose* $\mathbf{A}_1$ *is a categorical connection on* $\mathbf{P}^{\mathbf{A},\mathrm{dec}}$*. If* $\tau_{\mathbf{A}_1}(\gamma; p) = \big(\tilde{\gamma}, h\big)$*, where* $\gamma \in \mathrm{Mor}(\mathbf{P})$ *and* $p \in \mathrm{Obj}(\mathbf{P})$*, with* $\pi(p) = s(\gamma)$*, we set*

$$\tau_{\mathbf{A}_1^{\bullet\bullet}}(\gamma; p) = \big(q\tau(h), p; \gamma\big) \in \mathrm{Mor}(\mathbf{P}^{\bullet\bullet}), \tag{5.11}$$

where $q = t(\tilde{\gamma})$*. Then* $\mathbf{A}_1^{\bullet\bullet}$ *is a categorical connection on* $\mathbf{P}^{\bullet\bullet}$*.*

Proof. For notational convenience let us write $\mathbf{A}_2$ for $\mathbf{A}_1^{\bullet\bullet}$.

For (CC1) we note that if $\gamma = 1_u$, where $u = \pi(p)$, then

$$\tau_{\mathbf{A}_1}(1_u; p) = (1_p, e), \tag{5.12}$$

and so

$$\tau_{\mathbf{A}_2}(1_u; p) = \big(p, p; 1_u\big). \tag{5.13}$$

Next, for (CC2), if $\tau_{\mathbf{A}_1}(\gamma; p) = (\tilde{\gamma}, h)$, where $\tilde{\gamma}$ runs from p to q, then

$$(\tilde{\gamma}, h)1_g = (\tilde{\gamma}1_g, g^{-1}hg) \tag{5.14}$$

and so, noting that $\tilde{\gamma}1_g$ runs from pg to pq, we have

$$\tau_{\mathbf{A}_2}(\gamma; pg) = (qg\, g^{-1}\tau(h)g, pg; \gamma) = (q\tau(h)g, pg; \gamma). \tag{5.15}$$

This agrees with:

$$\tau_{\mathbf{A}_2}(\gamma;p)1_g = \big(q\tau(h),p;\gamma\big)1_g = \big(q\tau(h)g,pg;\gamma\big). \tag{5.16}$$

Thus (CC2) holds.

Now, for (CC3), we consider a composite $\delta\circ\gamma$ and $p\in P$ with $\pi(p)=u=s(\gamma)$. Let

$$\tau_{\mathbf{A}_1}(\gamma;p)=(\tilde\gamma,h_1) \qquad \text{and} \qquad \tau_{\mathbf{A}_1}(\delta;q)=(\tilde\delta,h_2), \tag{5.17}$$

where $q=t(\tilde\gamma)$.

Then

$$\begin{aligned}\tau_{\mathbf{A}_1}(\delta\circ\gamma;p) &= \tau_{\mathbf{A}_1}\big(\delta;t\big(\tau_{\mathbf{A}_1}(\gamma;p)\big)\big)\circ\tau_{\mathbf{A}_1}(\gamma,p)\\ &= \tau_{\mathbf{A}_1}\big(\delta;q\tau(h_1)\big)\circ(\tilde\gamma,h_1)\\ &= \big(\tilde\delta\tau(h_1),\tau(h_1)^{-1}h_2\tau(h_1)\big)\circ(\tilde\gamma,h_1)\\ &= \big(\tilde\delta\tau(h_1),h_1^{-1}h_2h_1\big)\circ(\tilde\gamma,h_1)\\ &= \big(\tilde\delta\circ\tilde\gamma,h_2h_1\big)\end{aligned} \tag{5.18}$$

where we used the second of the Peiffer identities (2.3). Hence,

$$\tau_{\mathbf{A}_2}(\delta\circ\gamma;p) = \big(r\tau(h_2h_1),p;\delta\circ\gamma\big). \tag{5.19}$$

On the other hand,

$$\begin{aligned}&\tau_{\mathbf{A}_2}\big(\delta;t\big(\tau_{\mathbf{A}_2}(\gamma;p)\big)\big)\circ\tau_{\mathbf{A}_2}(\gamma,p)\\ &= \tau_{\mathbf{A}_2}\big(\delta;q\tau(h_1)\big)\circ\big(p\tau(h_1),p;\gamma\big)\\ &= \big(r\tau(h_1)\tau(h_1)^{-1}\tau(h_2)\tau(h_1),q\tau(h_1);\delta\big)\circ\big(p\tau(h_1),p;\gamma\big)\\ &= \big(r\tau(h_2h_1),q\tau(h_1);\delta\big)\circ\big(p_1\tau(h_1),p;\gamma\big)\\ &= \big(r\tau(h_2h_1),p;\delta\circ\gamma\big),\end{aligned} \tag{5.20}$$

which agrees with the right side in (5.18). Thus (CC3) holds.

Smoothness of $\tau_{\mathbf{A}_2}$ follows from smoothness of $\tau_{\mathbf{A}_1}$ and of τ. ■

6. Pushforwards of Categorical Connections

In this section we present a more abstract construction of the pushforward discussed in section 5.7.

Let us briefly recall how classical connections can be pushed forward from one bundle to another. For a detailed and more general account we refer to Kobayashi and Nomizu [28, section II.6]; there is also a pullback process for classical bundles that we will not discuss here. Recall that in the classical geometric framework a connection A on a principal G-bundle $P\to M$ is given by an $L(G)$-valued 1-form; more geometrically, we understand the connection A as providing parallel-transports or *horizontal paths*, which are paths in P whose tangent vectors are annihilated by A. In the categorical bundles framework we also have a pullback in the special situation discussed in section 5.5.

6.1. Pushforwards of classical connections. Let G and K be Lie groups, and $s: G \to K$ a smooth homomorphism. Now consider a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{S} & Q \\ & {\scriptstyle \pi_P}\searrow \quad \swarrow{\scriptstyle \pi_Q} & \\ & M & \end{array} \tag{6.1}$$

where $\pi_P : P \to M$ is a principal G-bundle and $\pi_Q : Q \to M$ is a principal K-bundle, and suppose that the smooth map S is equivariant in the following sense

$$S(pg) = S(p)s(g), \tag{6.2}$$

for all $p \in P$ and $g \in G$.

With this setting there is a way to push forward a connection A on P to a connection f_*A on Q as follows. Let $q \in Q$ and $v \in T_qQ$. We pick any point $p \in P$ on the fiber over $\pi_Q(q)$; then $q = S(p)k$, for some $k \in K$. Then the horizontal space $\ker(f_*A)_q$ for f_*A at q is $f_*(\ker A_p)k$. More geometrically, *f_*A-horizontal paths in Q are K-translates of the images under f of A-horizontal paths in P.*

6.2. Pushforwards of categorical connections. We now construct a categorical counterpart of this pushforward process. Let $\mathbf{P}$ and $\mathbf{Q}$ be, respectively, a $\mathcal{G}$- and an $\mathcal{K}$- categorical principal bundle over the same base category $\mathbf{M}$. Here $\mathcal{G}$ and a $\mathcal{K}$ are categorical Lie groups. Suppose $\mathbb{S} : \mathbf{P} \to \mathbf{Q}$ and $S : \mathcal{G} \to \mathcal{K}$ are functors satisfying

$$\begin{aligned} \mathbb{S}(pg) &= \mathbb{S}(p)S(g) \\ \mathbb{S}(\tilde{\gamma}\phi) &= \mathbb{S}(\tilde{\gamma})S(\phi) \end{aligned} \tag{6.3}$$

for all $p \in \mathrm{Obj}(\mathbf{P})$, $g \in \mathrm{Obj}(\mathcal{G})$, $\tilde{\gamma} \in \mathrm{Mor}(\mathbf{P})$ and $\phi \in \mathrm{Mor}(\mathcal{G})$. We assume also that

$$\pi_{\mathbf{Q}}\big(\mathbb{S}(p)\big) = \pi_{\mathbf{P}}(p) \tag{6.4}$$

for all $p \in \mathrm{Obj}(\mathbf{P})$. We say that the pair $\mathbb{S}$ and S, satisfying (6.3) and (6.4), is a morphism from the bundle $\mathbf{P}$ to the bundle $\mathbf{Q}$.

A categorical connection is known when all the horiziontal morphisms are known. Starting from a categorical connection $\tau_{\mathbf{P}}$ on $\mathbf{P}$ we *specify a categorical connection $\mathbf{Q}$ by requiring that the $\mathbb{S}$-images of $\tau_{\mathbf{P}}$-horizontal morphisms be $\tau_{\mathbf{Q}}$-horizontal.* The following result ensures that $\tau_{\mathbf{Q}}$ is a categorical connection.

To avoid notational clutter we will write $\overline{\gamma}1_b$ as $\overline{\gamma}b$, for $\overline{\gamma}$ a morphism in $\mathbf{P}$ or $\mathbf{Q}$ and b an object of $\mathcal{G}$ or $\mathcal{K}$. This is consistent with standard practice for the case where $\overline{\gamma}$ corresponds to a horizontal path.

6.3. Theorem. *We use the notation as above. Let $\tau_{\mathbf{P}}$ be a categorical connection on $\mathbf{P}$, and, for any $\gamma \in \mathrm{Mor}(\mathbf{M})$ and any $q \in \mathrm{Obj}(\mathbf{Q})$ on the $\mathbf{Q}$-fiber over the source $s(\gamma)$, let*

$$\tau_{\mathbf{Q}}(\gamma, q) := \mathbb{S}(\tau_{\mathbf{P}}(\gamma, p))k_{p,q}, \tag{6.5}$$

where $p \in \mathrm{Obj}(\mathbf{P})$ is any point on the $\mathbf{P}$-fiber over $s(\gamma)$ and $k_{pq} \in \mathrm{Obj}(\mathcal{K})$ is specified by requiring that $q = \mathbb{S}(p)k_{p,q}$. Then $\tau_{\mathbf{Q}}$ satisfies conditions (CC1)-(CC3) for categorical connections, PROOF. First, let us verify that (6.5) is independent of the choice of the point p. Suppose $a \in \mathrm{Obj}(\mathcal{G})$ is such that $\mathbb{S}(pa) = \mathbb{S}(p) = q$. Then

$$S(a) = e,$$

the identity in the group $\mathrm{Obj}(\mathcal{K})$. We observe that

$$q = \mathbb{S}(p)k_{p,q} = \mathbb{S}(pa)k_{p,q},$$

and so

$$k_{pa,q} = k_{p,q}. \tag{6.6}$$

Then

$$\begin{aligned} \mathbb{S}(\tau_{\mathbf{P}}(\gamma, pa))k_{pa,q} &= \mathbb{S}(\tau_{\mathbf{P}}(\gamma, p)1_a)k_{pa,q} \\ &= \mathbb{S}(\tau_{\mathbf{P}}(\gamma, p))S(1_a)k_{p,q} \\ &= \mathbb{S}(\tau_{\mathbf{P}}(\gamma, p))1_{S(a)}k_{p,q} \\ &= \mathbb{S}(\tau_{\mathbf{P}}(\gamma, p))k_{p,q}, \end{aligned} \tag{6.7}$$

since $S(a) = e$. Thus (6.5) is independent of the choice of p.

Next we verify the conditions (CC1-3) from section 5.1. Condition (CC1) is readily verified and we omit the argument.

For (CC2) consider $\gamma \in \mathrm{Mor}(\mathbf{M})$, $q \in \mathrm{Obj}(\mathbf{Q})$ a point lying on the fiber above the source $s(\gamma)$. We choose any $p \in \mathrm{Obj}(\mathbf{P})$ on the fiber over $s(\gamma)$, and let $k_{p,q} \in \mathrm{Obj}(\mathcal{K})$ be such that $q = \mathbb{S}(p)k_{p,q}$. Then, for $k \in \mathrm{Obj}(\mathcal{K})$, we have

$$qk = \mathbb{S}(p)k_{p,q}k,$$

and so

$$k_{p,qk} = k_{p,q}k. \tag{6.8}$$

Hence

$$\begin{aligned} \tau_{\mathbf{Q}}(\gamma, qk) &= \mathbb{S}(\tau_{\mathbf{P}}(\gamma, p))k_{p,q}k \\ &= \tau_{\mathbf{Q}}(\gamma, q)k. \end{aligned} \tag{6.9}$$

Thus, the translate of any $\tau_{\mathbf{Q}}$-horizontal morphisms by any $k \in \mathrm{Obj}(\mathcal{K})$ is $\tau_{\mathbf{Q}}$-horizontal.

Next, we verify that the condition (CC3) holds; this condition means that compositions of $\tau_{\mathbf{Q}}$-horizontal morphisms are $\tau_{\mathbf{Q}}$-horizontal. By definition of $\tau_{\mathbf{Q}}$, as given in (6.5), a $\tau_{\mathbf{Q}}$-horizontal morphism is of the form

$$\tau_{\mathbf{Q}}(\gamma, q) = \mathbb{S}\big(\tau_{\mathbf{P}}(\gamma, p)\big)k_{p,q}.$$

Let $\gamma_1, \gamma_2 \in \mathrm{Mor}(\mathbf{M})$ and $p_1, p_2 \in \mathrm{Obj}(\mathbf{P})$ be such that the composition

$$\mathbb{S}\big(\tau_{\mathbf{P}}(\gamma_2, p_2)\big)k_{p_2,q_2} \circ \mathbb{S}\big(\tau_{\mathbf{P}}(\gamma_1, p_1)\big)k_{p_1,q_1} \tag{6.10}$$

is defined. Let p_1' be the terminal point (target) of $\tau_{\mathbf{P}}(\gamma_1, p_1)$. Since the composition (6.10) is defined we have

$$\mathbb{S}(p_2)k_{p_2,q_2} = \mathbb{S}(p_1')k_{p_1,q_1}. \tag{6.11}$$

Applying $\pi_{\mathbf{Q}}$ shows that

$$\pi_{\mathbf{P}}(p_2) = \pi_{\mathbf{P}}(p_1'),$$

and so the composition $\gamma_2 \circ \gamma_1$ is defined. Moreover,

$$p_2 = p_1' a,$$

for some $a \in \mathrm{Obj}(\mathcal{G})$. Then, applying $\mathbb{S}$ and using (6.11), we see that

$$S(a)k_{p_2,q_2} = k_{p_1,q_1}. \tag{6.12}$$

Using this relation in (6.10) turns it into

$$\big[\mathbb{S}\big(\tau_{\mathbf{P}}(\gamma_2, p_2)\big) \circ \mathbb{S}\big(\tau_{\mathbf{P}}(\gamma_1, p_1)\big)S(a)\big]k_{p_2,q_2}. \tag{6.13}$$

The term within $[\cdots]$ is equal to

$$\mathbb{S}\big(\tau_{\mathbf{P}}(\gamma_2, p_2)\big) \circ \mathbb{S}\big(\tau_{\mathbf{P}}(\gamma_1, p)\big), \tag{6.14}$$

where

$$p = p_1 a.$$

Since $\mathbb{S}$ is a functor, we have

$$\begin{aligned} \mathbb{S}\big(\tau_{\mathbf{P}}(\gamma_2, p_2)\big) \circ \mathbb{S}\big(\tau_{\mathbf{P}}(\gamma_1, p)\big) &= \mathbb{S}\big(\tau_{\mathbf{P}}(\gamma_2, p_2) \circ \tau_{\mathbf{P}}(\gamma_1, p)\big) \\ &= \mathbb{S}\big(\tau_{\mathbf{P}}(\gamma_2 \circ \gamma_1, p)\big). \end{aligned} \tag{6.15}$$

Combining this with the expression in (6.13) shows that the original composition of $\tau_{\mathbf{Q}}$-horizontal morphisms given in (6.10) is equal to

$$\mathbb{S}\big(\tau_{\mathbf{P}}(\gamma_2 \circ \gamma_1, p)\big)k_{p_2,q_2}. \tag{6.16}$$

By definition of $\tau_{\mathbf{Q}}$, as given in (6.5), we have

$$\mathbb{S}\big(\tau_{\mathbf{P}}(\gamma_2 \circ \gamma_1, p)\big)k_{p_2,q_2} = \tau_{\mathbf{Q}}(\gamma_2 \circ \gamma_1, q_2'), \tag{6.17}$$

where $q_2' = \mathbb{S}(p)k_{p_2,q_2}$. Thus, the composition of $\tau_{\mathbf{Q}}$-horizontal morphisms given in (6.10) is the $\tau_{\mathbf{Q}}$-horizontal morphism $\tau_{\mathbf{Q}}(\gamma_2 \circ \gamma_1, q_2')$. ∎

7. Gauge Transformations

By a *gauge transformation* on the categorical $\mathbf{G}$-bundle $\pi : \mathbf{P} \to \mathbf{M}$ we mean a functor

$$\Theta : \mathbf{P} \to \mathbf{P}$$

that commutes with the action of $\mathbf{G}$, satisfies $\pi \circ \Theta = \pi$, and is smooth on both objects and morphisms, Θ^{-1} exists and is also smooth on objects and morphisms. In [13] we have studied gauge transformations mainly in the context of decorated categorical bundles and twisted product bundles.

7.1. Gauge transformations for objects and morphisms. For any $p \in \mathrm{Obj}(\mathbf{P})$ the point (object) $\Theta(p)$ lies on the same fiber as p and so there is a $\theta_p \in G$ such that

$$\Theta_p = p\theta_p.$$

Next, for the same reason, for any $\overline{\gamma} \in \mathrm{Mor}(\mathbf{P})$ we have

$$\Theta(\overline{\gamma}) = \overline{\gamma}\theta(\overline{\gamma}),$$

for some $\theta(\overline{\gamma}) \in \mathrm{Mor}(\mathbf{G})$. Now since $\mathrm{Mor}(\mathbf{G}) \simeq H \rtimes_\alpha G$ we can write any element of $\mathrm{Mor}(\mathbf{G})$ as a product of an element of H and an element of G, in either order, with both of these groups being viewed as subgroups of $\mathrm{Mor}(\mathbf{G})$. Moreover, gh, with $g \in G$ and $h \in H$, has source g and target $g\tau(h)g^{-1}$.

Thus we can write

$$\theta(\overline{\gamma}) = h_{\overline{\gamma}}g_{\overline{\gamma}}$$

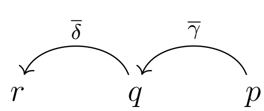
for some $g_{\overline{\gamma}} \in G$, which is the source of $\theta(\overline{\gamma}) \in \mathrm{Mor}(\mathbf{G})$, and $h_{\overline{\gamma}} \in H$. Thus,

$$\begin{aligned}\Theta(p) &= p\theta_p \\ \Theta(\overline{\gamma}) &= \overline{\gamma}h_{\overline{\gamma}}g_{\overline{\gamma}}\end{aligned} \tag{7.1}$$

for some $g_{\overline{\gamma}} \in G$, $h_{\overline{\gamma}} \in H$. The source of $\Theta(\overline{\gamma})$ is then $\overline{\gamma}_0 g_{\overline{\gamma}}$. For this to match $\Theta(\overline{\gamma}_0)$, which is $\overline{\gamma}_0\theta_{\overline{\gamma}_0}$, we have

$$g_{\overline{\gamma}} = \theta_{\overline{\gamma}_0}. \tag{7.2}$$

7.2. Gauge transformations for morphism compositions. Next consider a composition $\overline{\delta} \circ \overline{\gamma}$ of morphisms $\overline{\delta}$ and $\overline{\gamma}$ in $\mathbf{P}$.



For Θ to be a functor, Θ applied to $\overline{\delta} \circ \overline{\gamma}$ must agree with $\Theta(\overline{\delta}) \circ \Theta(\overline{\gamma})$. For the latter we compute

$$\begin{aligned}\Theta(\overline{\delta}) \circ \Theta(\overline{\gamma}) &= \overline{\delta}h_{\overline{\delta}}\theta_q \circ \overline{\gamma}h_{\overline{\gamma}}\theta_p \\ &= (\overline{\delta} \circ \overline{\gamma})\big((h_{\overline{\delta}}, \theta_q) \circ (h_{\overline{\gamma}}, \theta_p)\big) \\ &= (\overline{\delta} \circ \overline{\gamma})(h_{\overline{\delta}}h_{\overline{\gamma}}, \theta_p).\end{aligned} \tag{7.3}$$

For this to agree with $\Theta(\overline{\delta} \circ \overline{\gamma})$, whose value is given by

$$\Theta(\overline{\delta} \circ \overline{\gamma}) = (\overline{\delta} \circ \overline{\gamma})(h_{\overline{\delta}\circ\overline{\gamma}}, \theta_p),$$

the condition is

$$h_{\overline{\delta}\circ\overline{\gamma}} = h_{\overline{\delta}} h_{\overline{\gamma}}. \tag{7.4}$$

We can summarize this discussion in the following conclusion.

7.3. Theorem. *Let $\Theta : \mathbf{P} \to \mathbf{P}$ be a gauge transformation for the categorical $\mathbf{G}$-bundle $\pi : \mathbf{P} \to \mathbf{M}$, where the categorical Lie group $\mathbf{G}$ has associate Lie crossed module (G, H, α, τ). Then*

$$\begin{aligned} \Theta(p) &= p\theta_p \\ \Theta(\overline{\gamma}) &= \overline{\gamma} h_{\overline{\gamma}} g_{\overline{\gamma}}, \end{aligned} \tag{7.5}$$

for all $p \in \mathrm{Obj}(\mathbf{P})$ and $\overline{\gamma} \in \mathrm{Mor}(\mathbf{P})$, where $\theta_p, g_{\overline{\gamma}} \in G$ and $h_{\overline{\gamma}} \in H$, and, furthermore,

$$\begin{aligned} g_{\overline{\gamma}} &= \theta_p \\ h_{\overline{\delta}\circ\overline{\gamma}} &= h_{\overline{\delta}} h_{\overline{\gamma}}, \end{aligned} \tag{7.6}$$

whenever $\overline{\gamma}, \overline{\delta} \in \mathrm{Mor}(\mathbf{P})$ are composable morphisms with $p = s(\overline{\gamma})$, and

$$\begin{aligned} \theta_{pa} &= a^{-1}\theta_p a \\ h_{\overline{\gamma}a} &= a^{-1} h_{\overline{\gamma}} a \\ h_{\overline{\gamma}b} &= h_{\overline{\gamma}}\theta_p b \theta_p^{-1} \end{aligned} \tag{7.7}$$

for all $a \in G$ and $b \in H$, with p being $s(\overline{\gamma})$.

Conversely, if $\Theta : \mathbf{P} \to \mathbf{P}$ is such that conditions (7.5), (7.6) and (7.7) hold then Θ is a functor that commutes with the projection $\pi_{\mathbf{P}}$ and with the action of the categorical group $\mathbf{G}$.

Proof. We have already established all claims except for (7.7). Next, we have, for $p \in P$ and $a \in G$,

$$\Theta(pa) = \Theta(p)a = paa^{-1}\theta_p a, \tag{7.8}$$

which proves the first equation in (7.7). Using this, we see that for $\overline{\gamma} \in \mathrm{Mor}(\mathbf{P})$ with source p, and for any $a \in G$, we have

$$\Theta(\overline{\gamma}a) = \Theta(\overline{\gamma})a = \overline{\gamma} h_{\overline{\gamma}}\theta_p a = \overline{\gamma} a a^{-1} h_{\overline{\gamma}} a \theta_{pa}, \tag{7.9}$$

which implies the second relation in (7.7). Finally, for $b \in H$, noting that

$$s(\overline{\gamma}b) = s(\overline{\gamma})s(b) = pe = p,$$

we have

$$\Theta(\overline{\gamma}b) = \overline{\gamma} h_{\overline{\gamma}}\theta_p b = \overline{\gamma}(h_{\overline{\gamma}}\theta_p b {\theta_p}^{-1})\theta_p. \tag{7.10}$$

This shows that $h_{\overline{\gamma}b} = h_{\overline{\gamma}}\theta_p b {\theta_p}^{-1}$, which is the third equation in (7.7). ∎

7.4. Gauge transformations at the differential level. We work now with a classical connection A on a principal G-bundle $\pi : P \to M$, and the decorated categorical principal bundle $\mathbf{P}^{A,\mathrm{dec}}$. Let (G, H, α, τ) be a Lie crossed module with associate categorical group denoted $\mathbf{G}$. Let $\theta : P \to G$ be a smooth mapping that satisfies

$$\theta_{pa} = a^{-1}\theta_p a \qquad \text{for all } a \in G \text{ and } p \in P. \tag{7.11}$$

Let Λ^H be a smooth 1-form on P with values in the Lie algebra $L(H)$ satisfying

$$\Lambda^H_{pa}(va) = \alpha(a^{-1})\Lambda^H_p(v), \tag{7.12}$$

for all $p \in P$, $a \in G$, and $v \in T_pP$, and vanishes on vertical vectors (that is, vectors in the kernel of $d\pi$).

For any A-horizontal path $\tilde{\gamma} : [a, b] \to P$ we *define*

$$h_{\tilde{\gamma}} = h(b), \tag{7.13}$$

where $h : [a, b] \to H$ is the solution to the differential equation

$$\frac{dh(u)}{du}h(u)^{-1} = -\Lambda^H\big(\tilde{\gamma}'(u)\big) \tag{7.14}$$

In the following result we show how Λ^H and θ give rise to a gauge transformation Θ on the categorical bundle $\mathbf{P}^{A,\mathrm{dec}} \to \mathbf{M}$. We denote a typical A-horizontal path on P by $\tilde{\gamma}$ or $\tilde{\delta}$.

7.5. Theorem. *With θ and Λ^H, and other notation as above, let*

$$\Theta : \mathbf{P}^{A,\mathrm{dec}} \to \mathbf{P}^{A,\mathrm{dec}} \tag{7.15}$$

be given on objects and on morphisms by

$$\begin{aligned} \Theta(p) &= p\theta_p \\ \Theta(\overline{\gamma}) &= \overline{\gamma}h_{\tilde{\gamma}}\theta_p h, \end{aligned} \tag{7.16}$$

for all $p \in \mathrm{Obj}(\mathbf{P})$ and $\overline{\gamma} \in \mathrm{Mor}(\mathbf{P}^{A,\mathrm{dec}})$ of the form

$$\overline{\gamma} = (\tilde{\gamma}, h),$$

where $p = s(\tilde{\gamma})$. Then Θ is a gauge transformation.

The second condition in (7.5) is motivated by the third equation in (7.7) holds. It is easier to understand if we write the condition as

$$\Theta(\overline{\gamma}) = \overline{\gamma}h_{\overline{\gamma}}\theta_p, \tag{7.17}$$

with

$$h_{\overline{\gamma}} = h_{\tilde{\gamma}}\theta_p h\theta_p^{-1}. \tag{7.18}$$

In the conclusion what is missing from saying that Θ is a gauge transformation is the smoothness.

PROOF. We will verify the conditions given in Theorem 7.3 hold.

Consider smooth A-horizontal paths $\tilde{\gamma} : [a, b] \to P$ and $\tilde{\delta} : [b, c] \to P$, constant near the initial and terminal points, that are composable; this means,

$$\tilde{\delta}(b) = \tilde{\gamma}(b).$$

On the interval $[a, c]$ let us consider the map

$$f : [a, c] \to H : u \mapsto \begin{cases} h_1(u) & \text{if } u \in [a, b]; \\ h_2(u)h_1(b) & \text{if } u \in [b, c], \end{cases} \tag{7.19}$$

where $h_1(\cdot)$ is a solution to (7.14) for $u \in [a, b]$ with $h_1(a) = e$, and $h_2(\cdot)$ is a solution to (7.14) for $u \in [b, c]$ with $h_2(b) = e$. Then f solves the equation (7.14) for h, with $f(a) = e$. Thus

$$\frac{df(u)}{du} f(u)^{-1} = -\Lambda^H\big((\tilde{\delta} \circ \tilde{\gamma})'(u)\big) \tag{7.20}$$

for all $u \in [a, c]$ and $f(u) = e$. Hence

$$f(c) = h_{\tilde{\delta}\circ\tilde{\gamma}},$$

and so

$$h_{\tilde{\delta}\circ\tilde{\gamma}} = h_2(c)h_1(b) = h_{\tilde{\delta}} h_{\tilde{\gamma}}. \tag{7.21}$$

The condition (7.12) ensures that

$$h_{\tilde{\gamma}a} = \alpha(a^{-1})h_{\tilde{\gamma}} = a^{-1}h_{\tilde{\gamma}}a, \tag{7.22}$$

Recall from (7.17) and (7.18) that for a morphism $\overline{\gamma}$ of $\mathbf{P}^{A,\mathrm{dec}}$ given by $(\tilde{\gamma}, h)$, with source p, we have defined

$$h_{\overline{\gamma}} = h_{\tilde{\gamma}}\theta_p h\theta_p^{-1}. \tag{7.23}$$

For any $b \in H$ we have

$$\overline{\gamma}b = (\tilde{\gamma}, hb),$$

and so

$$h_{\overline{\gamma}b} = h_{\tilde{\gamma}}\theta_p hb\theta_p^{-1} = h_{\tilde{\gamma}}\theta_p h\theta_p^{-1}\theta_p b\theta_p^{-1} = h_{\overline{\gamma}}\theta_p b\theta_p^{-1}, \tag{7.24}$$

which shows that the third equation in (7.7) holds.

Finally, for $a \in G$, we have

$$\overline{\gamma}a = (\tilde{\gamma}a, a^{-1}ha), \tag{7.25}$$

and so

$$\begin{aligned} h_{\overline{\gamma}a} &= h_{\tilde{\gamma}a}\theta_{pa}(a^{-1}ha)\theta_{pa}^{-1} && \text{(from the definition (7.18))} \\ &= h_{\tilde{\gamma}a}a^{-1}\theta_p h\theta_p^{-1}a \\ &= a^{-1}h_{\tilde{\gamma}}aa^{-1}\theta_p h\theta_p^{-1}a && \text{(using (7.22))} \\ &= a^{-1}h_{\tilde{\gamma}}\theta_p h\theta_p^{-1}a \\ &= a^{-1}h_{\overline{\gamma}}a && \text{(using (7.23)).} \end{aligned} \tag{7.26}$$

Thus we have verified that the functions $p \mapsto \theta_p$ and $\overline{\gamma} \mapsto h_{\overline{\gamma}}$ satisfy all the properties, other than smoothness, listed in Theorem 7.3 associated with a categorical gauge transformation. Smoothness of Θ on objects follows from the smoothness of the mapping $\theta : P \to G$. Smoothness of Θ on morphisms follows from the fact that if $(u, v) \mapsto \tilde{\gamma}(u; v) \in P$ is smooth for $u \in [a, b]$ and v running over some $\mathbb{R}^k$, then the solution $u \mapsto h(u; v)$ to the differential equation (7.14), with $\tilde{\gamma}'(u; v)$ instead of $\tilde{\gamma}'(u)$ on the right side, depends smoothly also on the parameter v. ■

8. Categorical gauge transformations of classical connections

Let A be a connection on a principal G-bundle $\pi : P \to M$. We then have the categorical connection $A^{\bullet\bullet}$ on the categorical $\mathbf{G}^{\bullet\bullet}$-bundle $\pi : \mathbf{P}^{\bullet\bullet} \to \mathbf{M}$ and we have the decorated categorical $\mathbf{G}$-bundle $\pi : \mathbf{P}^{A,\mathrm{dec}} \to \mathbf{M}$. Let $\gamma \in \mathrm{Mor}(\mathbf{M})$. Then the A-horizontal lift $\tilde{\gamma}_p^A$ of γ, with initial point p, gives a morphism

$$\overline{\gamma}_p^A = (\tilde{\gamma}_p^A; e) \in \mathrm{Mor}\big(\mathbf{P}^{A,\mathrm{dec}}\big).$$

Applying a gauge transformation

$$\Theta : \mathbf{P}^{A,\mathrm{dec}} \to \mathbf{P}^{A,\mathrm{dec}} \tag{8.1}$$

to the morphism $\overline{\gamma}_p^A$ yields the morphism

$$\Theta(\overline{\gamma}_p^A) = \big(\tilde{\gamma}\theta_p; h_{\tilde{\gamma}}\big), \tag{8.2}$$

where, for notational simplicity, we are writing $\tilde{\gamma}_p^A$ simply as $\tilde{\gamma}$, and we are using other notation as before. This then pushes down to the morphism

$$\big(qg_q\tau(h_{\tilde{\gamma}}), p\theta_p; \gamma\big) \in \mathrm{Mor}(\mathbf{P}^{\bullet\bullet}), \tag{8.3}$$

where q is the terminal point of $\tilde{\gamma}_p^A$. Let us assume that the new categorical connection on $\mathbf{P}^{\bullet\bullet}$ also arises from a classical connection on P. Thus the resulting connection on P parallel transports the point p along γ to the point $q\theta_q\tau(h_{\tilde{\gamma}})\theta_p{}^{-1}$.

Thus the new horizontal path is

$$t \mapsto \overline{\gamma}(t) = \tilde{\gamma}_t\theta_{\tilde{\gamma}_t}\tau(h_{\tilde{\gamma}_t})\theta_p^{-1}, \tag{8.4}$$

where, of course, $\tilde{\gamma}$ is A-horizontal, with initial point p.

8.1. Parallel transport with respect to a shifted connection.

Suppose A is a connection form on $\pi : P \to M$, and C is an $L(G)$-valued 1-form on P that satisfies the Ad_G-equivariance

$$C_{pg}(v_pg) = \mathrm{Ad}(g)^{-1}C_p(v_p) \qquad \text{for all } p \in P,$$

and vanishes on vertical vectors:

$$C_p(V_p) = 0 \qquad \text{for all } V_p \in \ker d\pi_p \text{ and all } p \in P.$$

Then $A + C$ is also a connection form. We note how $(A + C)$-horizontal paths are given by suitable right-translations of A-horizontal paths.

8.2. Lemma. *Let A be a connection on a principal G-bundle $\pi : P \to M$, and C an Ad_G-equivariant 1-form on P with values in $L(G)$ that vanishes on vertical vectors. Suppose $[a,b] \to P : t \mapsto \tilde{\gamma}(t)$ is an A-horizontal path. Then $[a,b] \to P : t \mapsto \tilde{\gamma}(t)\xi(t)$ is horizontal with respect to the connection $A + C$, if $t \mapsto \xi(t)$ satisfies the differential equation*

$$\xi'(t)\xi(t)^{-1} = -C\big(\tilde{\gamma}'(t)\big) \qquad \text{for all } t \in [a,b]. \tag{8.5}$$

Proof. We apply $A + C$ to $t \mapsto \tilde{\gamma}(t)\xi(t)$ to compute:

$$\begin{aligned}(A+C)\big(\tilde{\gamma}'(t)\xi(t) + \tilde{\gamma}(t)\xi'(t)\big) &= 0 + \xi(t)^{-1}\xi'(t) + \xi(t)^{-1}C\big(\tilde{\gamma}'(t)\big)\xi(t) + 0 \\ &= \xi(t)^{-1}\left[\xi'(t)\xi(t)^{-1} + C\big(\tilde{\gamma}'(t)\big)\right]\xi(t).\end{aligned} \tag{8.6}$$

Thus the path $[a,b] \to P : t \mapsto \tilde{\gamma}(t)\xi(t)$ is $(A+C)$-horizontal if and only if equation (8.5) holds. ∎

8.3. Traditional gauge transformation of A. Let $\phi : P \to P$ be a traditional gauge transformation; this means that it is a smooth G-equivariant map for which $\pi \circ \phi = \pi$. Since p and $\phi(p)$ are on the same π-fiber, there is a unique $\theta(p) \in G$ such that

$$\phi(p) = p\theta(p) \qquad \text{for all } p \in P. \tag{8.7}$$

Local triviality can be used to show that θ is smooth, and G-equivariance of ϕ is equivalent to the condition

$$\theta(pg) = g^{-1}\theta(p)g \qquad \text{for all } p \in P \text{ and } g \in G. \tag{8.8}$$

(Proofs may be found in any standard text on bundle theory such as [28].) Then

$$(\phi^{-1})^*A = \mathrm{Ad}(\theta_p^{-1})A_p - (d\theta|_p)\theta_p^{-1}. \tag{8.9}$$

If $\tilde{\gamma}$ is A-horizontal then $\phi \circ \tilde{\gamma}$ is $(\phi^{-1})^*A$-horizontal:

$$(\phi^{-1})^*A\big((\phi\circ\tilde{\gamma})'(t)\big) = A\big(\phi_*^{-1}(\phi\circ\tilde{\gamma})'(t)\big) = A\big(\tilde{\gamma}'(t)\big) = 0. \tag{8.10}$$

8.4. Horizontal paths and generalized gauge transformations. In view of (8.10) and Lemma 8.2, if $\tilde{\gamma}$ is an A-horizontal path on P, and $\theta : P \to G$ is as above, associated to a gauge transformation $\phi : P \to P$, then the path

$$[a,b] \to P : t \mapsto \tilde{\gamma}(t)\theta\big(\tilde{\gamma}(t)\big)\xi(t) \tag{8.11}$$

is horizontal with respect to the connection

$$\mathrm{Ad}(\theta)A - (d\theta)\theta^{-1} + \tau\Lambda^H, \tag{8.12}$$

where

$$\xi'(t)\xi(t)^{-1} = -\tau\Lambda^H\big(\tilde{\gamma}'(t)\big) \tag{8.13}$$

for all $t \in [a,b]$. We can write ξ as

$$\xi(t) = \tau\big(h_{\tilde{\gamma}}(t)\big), \tag{8.14}$$

where $t \mapsto h_{\tilde{\gamma}}(t)$ solves

$$\frac{dh_{\tilde{\gamma}}(t)}{dt}h_{\tilde{\gamma}}(t)^{-1} = -\Lambda^H\big(\tilde{\gamma}'(t)\big). \tag{8.15}$$

Thus, we have established the following result.

8.5. Theorem. *With notation as above, and* $\tilde{\gamma} : [a,b] \to P$ *the* A*-horizontal path on* P *with initial point* p*, the path*

$$[a,b] \to P : t \mapsto \tilde{\gamma}(t)\theta_{\tilde{\gamma}(t)}\tau\big(h_{\tilde{\gamma}}(t)\big)\theta(p)^{-1} \tag{8.16}$$

has initial point p *and is horizontal with respect to the connection*

$$\mathrm{Ad}(\theta)A - (d\theta)\theta^{-1} + \tau\Lambda^H. \tag{8.17}$$

Looking back at (8.4) we conclude that the connection form induced by the gauge transformation Θ on $\mathbf{P}^{A,\mathrm{dec}}$ is given by (8.17).

8.6. Higher order gauge transformations. The purpose of this section is to give a heuristic description of how the results and constructions developed in this paper can be used for studying higher gauge transformations. It is not our intention here to give a formal or a very rigorous treatment of higher gauge transformations, a project we leave to a future work.

In section 3.4 we have briefly recalled the construction of the path space category $\mathbb{P}_1(X)$ for a given smooth space X, whose object space is X and morphisms are smooth paths $[a,b] \to X$ constant on neighborhoods of a and b, with two such paths identified if they are reparametrizations of each other under a constant translation:

$$\mathrm{Mor}(\mathbb{P}_1(X)) := \mathcal{P}_1(X)/\mathbb{R}. \tag{8.18}$$

The space $\mathcal{P}_1(X)/\mathbb{R}$ has a natural smooth structure [12, section 2]. We take $\mathbb{P}_0(X)$ to be the discrete category, with both object space and morphism space being X, each morphism being thought of as the constant path at a point. Inductively, then, we have a hierarchy of (smooth) categorical spaces such that the object space at any level is the morphism space of the preceding one,

$$\mathrm{Obj}(\mathbb{P}_i(X)) = \mathrm{Mor}(\mathbb{P}_{i-1}(X)), \tag{8.19}$$

and

$$\mathrm{Mor}(\mathbb{P}_i(X)) = \mathcal{P}_1\big(\mathrm{Obj}(\mathbb{P}_{i-1}(X))\big)/\mathbb{R}, \qquad i \geq 1. \tag{8.20}$$

We will only be interested for the cases $i \leq 2$. Now consider a classical connection A on a classical principal G-bundle $\pi\colon P \to M$. Let

$$\mathcal{P}_A P := \{\ A\text{-horizontal paths on } P\} \subset \mathcal{P}_1(P). \tag{8.21}$$

be the smooth space obtained from A-horizontal lifts of paths in $\mathcal{P}_1(M)$. We note that a horizontal path remains horizontal when the parametrization is changed by a translation. The quotient space $\mathcal{P}_A P/\mathbb{R} := \mathcal{P}_A$ projects smoothly onto $\mathcal{P}_1(M)/\mathbb{R}$ under the projection map $\tilde{\gamma} \mapsto \pi(\tilde{\gamma})$. Moreover, the Lie group G has a natural right action on $\mathcal{P}_A P/\mathbb{R}$ by constant vertical shifts along the fiber:

$$(\widetilde{\gamma}g)(t) := \widetilde{\gamma}(t)g. \tag{8.22}$$

In turn $\mathcal{P}_A P/\mathbb{R} = \mathcal{P}_A$ can be formally viewed as a smooth principal G-bundle over the smooth space $\mathcal{P}_1(M)/\mathbb{R}$. This principal G-bundle and connections on it have been extensively studied in several of our previous works [10–12]. In particular, given a classical connection $\overline{A}$ on the principal G-bundle $\pi\colon P \to M$, and an $L(H)$-valued 2-form B on P satisfying

$$\begin{aligned} &B_{pg}(ug, vg) = \alpha_{g^{-1}}\big(B_p(u, v)\big), \\ &B(u, v) = 0, \qquad \text{if } u \text{ or } v \text{ is vertical,} \end{aligned} \tag{8.23}$$

we constructed a connection on the principal G-bundle $\mathcal{P}_A P/\mathbb{R} \to \mathcal{P}_1(M)/\mathbb{R}$ given by (see Sections 2 and 3 of [12]):

$$\omega_{\widetilde{\gamma}} := \overline{A}_{\widetilde{\gamma}(0)} + \tau \int B(\cdot, \widetilde{\gamma}'), \qquad \forall \widetilde{\gamma} \in \mathcal{P}_A P/\mathbb{R}, \tag{8.24}$$

where (G, H, τ, α) is a Lie crossed module, and the second term on the right hand side is a first order *Chen integral* [14, 15]. It is possible to find a more general form of a connection involving higher degree differential forms and higher order Chen integrals. Other than Chen's original works, the reader may consult the works of Igusa [24, 25], and Block and Smith [8] for a very interesting construction of connections on a graded vector space over a path space using higher order Chen integrals.

Once we have such a connection ω on the principal G-bundle $\mathcal{P}_A P/\mathbb{R} \to \mathcal{P}_1(M)/\mathbb{R}$, the decoration process described in section 4.2 will produce a categorical principal $\mathbf{G}$-bundle ${\mathbf{P}_A}^{\omega,\mathrm{dec}}$ over the category $\mathbb{P}_2(M)$, where morphisms of ${\mathbf{P}_A}^{\omega,\mathrm{dec}}$ are of the form

$$(\tilde{\Gamma}, h)$$

for ω-horizontal lift $\widetilde{\Gamma}$ of a path $\Gamma\colon [a, b] \to \mathcal{P}_1(M)/\mathbb{R}$. See Section 8 of [11] for a detailed discussion on the horizontal lift and parallel transport by the connection ω. Likewise, one can adapt the construction of section 4.1 to obtain a categorical principal $\mathbf{G}^{\bullet\bullet}$-bundle ${\mathbf{P}_A}^{\bullet\bullet}$ over $\mathbb{P}_2(M)$. Henceforth all the results, most importantly the methods of pushforward and pullback and the gauge transformations, till Theorem 7.3 would be applicable to the principal bundles over $\mathbb{P}_2(M)$. The study of gauge transformations of such higher bundles at a differential level is expected to yield the higher gauge transformations of the higher differential forms (such as the $L(H)$-valued 2-form B in the definition of ω). It would be particularly interesting to re-examine the results of section 8 through successive applications of pullbacks and pushforwards in the context of higher principal bundles explained above.

8.7. Concluding remarks. In this work we have constructed pushforwards of categorical connections. Applying this to a decorated categorical bundle, whose ingredients are a classical principal G-bundle $\pi : P \to M$ with a connection and a second Lie structure group H, intertwined with G in a specific way, we obtain a transformation of A of the form $\phi^* A + \tau\Lambda^H$, where $\phi : P \to P$ is a classical gauge transformation and $\tau\Lambda^H$ arises from a particular type of $L(H)$-valued 1 form on P.

We have applied the pushforward method only to bundles over path spaces. However, as described in section 5 in the context of the categorical bundles $\mathbf{P}^{\bullet\bullet}$ and $\mathbf{P}^{\mathbf{A},\mathrm{dec}}$, we have constructed a more general framework for categorical bundles. That framework permits construction of higher bundles over path spaces, and iterations thereof to higher dimensional 'paths', and corresponding connections that involve higher-degree differential forms. We expect that the ideas and constructions developed in this paper to extend to higher gauge theories with higher path spaces, and the corresponding gauge transformations would involve not only the classical transformation ϕ and the 1-form Λ^H but also higher-order forms with values in other Lie algebras. We leave the development of such a theory involving higher dimensional structures for future work.

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