Drawing some planar graphs with integer edge-lengths

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Abstract

In this paper, we study drawings of planar graphs such that all edge lengths are integers. It was known that such drawings exist for all planar graphs with maximum degree 3. We give a different proof of this result, which is based on a simple transformation of hexagonal drawings as created by Kant. Moreover, if the graph is 3-connected then the vertices have integer coordinates that are in $O(n)$. We then study some other classes of planar graphs, and show that planar bipartite, series-parallel graphs, and some other graphs also have planar drawings with integer edge lengths.

1 Introduction

A planar graph is a graph that can be drawn without crossing. Fáry, Stein and Wagner [4, 13, 15] proved independently that every planar graph has a drawing such that all edges are drawn as straight-line segments. Sometimes additional constraints are imposed on the drawings. The most famous one is to have integer coordinates while keeping the area small; it was shown in 1990 that this is always possible in $O(n^2)$ area [5, 12].

In this paper, we study a different restriction that was first posed by Kemnitz and Harborth [8]: Does every planar graph admit a straight-line drawing with integer edge lengths? This question remains open in general, but was answered in the positive for planar graphs with maximum degree 3 by Geelen, Guo and McKinnon [6].

In this paper, we first give a different proof of the result for planar graphs with maximum degree 3. In particular, our proof is constructive and yields a linear-time algorithm to find the drawing. (In contrast to this, Geelen, Guo and McKinnon require a theorem about rational distances that does not lend itself to an algorithm easily.) For 3-connected 3-regular graphs, our algorithm is very easy: Use the drawings with few slopes that are known to exist, and modify them so that all edge lengths are integers. In the resulting drawing all vertices are also at (integer) grid points, and the grid has width and height $O(n)$. For graphs that are not 3-connected, we split them into subgraphs, draw these separately, and paste them together suitably. The proof here is still algorithmic, but no bound on the grid-size of the resulting drawing is apparent.

We also study some other graphs classes, such as planar bipartite graphs and series-parallel graphs. As it turns out, the proof of Geelen et al. [6] actually works for these graphs as well, and so they also can be drawn with integer edge lengths. This was also shown independently (with a different proof) by Sun [14].

2 Drawing 3-connected 3-regular planar graphs

We first study graphs with maximum degree 3 that are also 3-connected, i.e., cannot be separated by removing at most 2 vertices. Such graphs are in fact *3-regular*, i.e., every vertex has degree 3. In 1993, Kant [7] showed how to create hexagonal grid drawings of 3-connected 3-regular graphs. His results (cf. Theorem 9 and Figure 5 of [7]) imply:

Theorem 1 *[7] Let G be a 3-connected 3-regular graph, and let v_o be an arbitrary vertex on the outer-face of G. Then $G - v_o$ has a straight-line drawing Γ such that*

- *all edges are drawn horizontally, vertically or with slope -1,*
- *the drawing is contained in a triangle with corners at $(0, 0)$, $(\frac{n-2}{2} - 1, 0)$ and $(0, \frac{n-2}{2} - 1)$,*
- *the three neighbours of v_o are placed at the three corners of the triangle.*

Such a drawing can be found in linear time.

See also Figure 1. A similar result was also proved later by Dujmovic et al. [3] using the so-called canonical ordering.

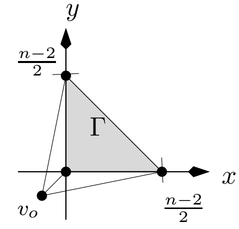

Figure 1: Kant's drawing (from [7].)

To convert Γ into a drawing with integer edge lengths, skew it suitably, and then add v_o. There are many ways

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to do such a skew. The most intuitive one would be to convert the drawing back to the hexagonal grid (with angle 0, $\pi/3$ and $2\pi/3$.) This would make the edge lengths integers, but vertices would not be at grid points (and in fact, would have irrational coordinates.)

We hence use a different skew that maps grid points to grid points and lines to slopes that are part of a Pythagorean triplet. More precisely, define the linear mapping $\psi : (x, y) \to (7x - 3y, 24y)$ and note that it maps grid points to grid points. Consider any line segment s in Γ; say s connects grid points (x_1, y_1) and (x_2, y_2):

- If s is horizontal, then $y_1 = y_2$, and hence $\psi(s)$ is also horizontal. Since grid points are mapped to grid points, $\psi(s)$ hence has integer length.
- If s has slope -1 then $x_2 - x_1 = y_1 - y_2$. Hence $\psi(s)$ has slope

$$\begin{aligned} & \frac{24y_1 - 24y_2}{(7x_1 - 3y_1) - (7x_2 - 3y_2)} \\ = \; & \frac{24(x_2 - x_1)}{7(x_1 - x_2) - 3(x_2 - x_1)} \\ = \; & \frac{24}{-10} = -\frac{12}{5} \end{aligned}$$

  Say $\psi(s)$ projects to length X in x-direction and length Y in y-direction, then $Y = \frac{12}{5}X$ and both X and Y are integers (since ψ maps grid points to grid points.) Hence $X = 5A$ for some integer A, and the length of $\psi(s)$ is $\sqrt{X^2 + Y^2} = \sqrt{(5A)^2 + (12A)^2} = \sqrt{169A^2} = 13A$, which is an integer.
- If s is vertical, then $x_1 = x_2$. Similar calculations show that $\psi(s)$ has slope $-24/7$, its x-projection has length $7A$ for some integer A, and the length of $\psi(s)$ is $\sqrt{(7A)^2 + (24A)^2} = 25A$, an integer.

Therefore $\psi(\Gamma)$ is a drawing of $G - v_o$ where all edges have integer length. Hence it only remains to add v_o suitably. In $\psi(\Gamma)$ the three neighbours of v_o are placed at $(0, 0)$, $(7\frac{n-2}{2}, 0)$ and $(-3\frac{n-2}{2}, 24\frac{n-2}{2})$. Place v_o at $(-3\frac{n-2}{2}, -24\frac{n-2}{2})$, which is a grid point. The three edges to its neighbours than have slope 12/5, 24/7 and $+\infty$, respectively. Since the neighbours are also at grid points, this implies (as above) that the edge lengths are integers.

Theorem 2 *Every 3-regular 3-connected planar graph has a planar straight-line drawing such that*

- *all edges have integer length,*
- *all edges are horizontal, vertical or have slope* $\pm\frac{5}{12}$ *or* $\pm\frac{7}{24}$,
- *the width is* $5(n-2)$ *and the height is* $24(n-2)$.

Such a drawing can be found in linear time.

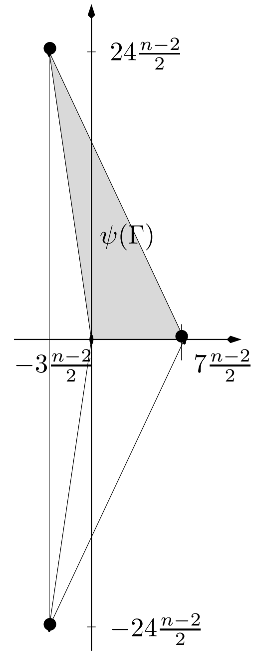

Figure 2: Applying ψ and adding v_o.

Without going into details, we note here that a different skew to apply would have been $\psi' : (x, y) \to (7x - 9y, 12y)$. This maps segments of slope -1 to segments of slope $-3/4$, and vertical segments to segments of slope $-4/3$; since $3^2 + 4^2 = 5^2$ one easily shows that edge lengths are then integers. The area of the final drawing then can be shown to be $8(n-2) \times 12(n-2)$, which is less than in Theorem 2 and also has a better aspect ratio. But this drawing would have a larger angle at the "top tip", which will be undesirable later.

In fact, any two Pythagorean triplets $a < b < c$ and $a' < b' < c'$ where $\{a, b\}$ and $\{a', b'\}$ have a term in common can provide a suitable skew. For example, if $a' = b$, then use the skew $(x, y) \to (b'x - ay, a'y)$ to obtain a drawing of area $(b' + a)a' \cdot ((n-2)/2)^2$. There are many such pairs of Pythagorean triples, and any of them give $O(n^2)$ area, but is ψ' the best one for the constant in the area-bound?

3 Max-degree-3 graphs

The previous section studied graphs that have maximum degree 3 and are 3-connected. In this section, we now extend this to all graphs of maximum degree 3, i.e., we explain how to deal with a *bridge* (an edge whose removal disconnects the graph) and with a *cutting edge-pair* (a pair of edges whose removal disconnected the graph.) Since the graph has maximum degree 3, it must have either a bridge or a cutting edge-pair or must be 3-connected.

Our approach is the "standard" approach in graph drawing, also used in [7]: Cut the graph apart at such an edge/pair, draw each part separately, and paste the drawings together suitably. The idea of this is very simple, but the details are a bit more complicated since we need to add invariants to the drawing to ensure that they can be merged.

We will only explain how to obtain a drawing with rational coordinates; this implies the result after scaling.

3.1 Bridges

We will show how to draw any planar graph G with maximal degree 3 with rational edge lengths such that additionally one pre-specified vertex w of G is on the convex hull of the resulting drawing. We proceed by induction on the number of bridges. In the base case, G has no bridge, so it is 2-connected. We will show an even stronger statement for 2-connected graphs in the next subsection.

For the induction step, assume now that G has a bridge $e = (v_1, v_2)$, and let G_1 and G_2 be the two subgraphs that result from removing e, with v_i in G_i. Assume that w belongs to graph G_1.

Draw G_1 recursively with rational edge lengths such that w is on the convex hull. In this drawing, find an open disk D that is inside the face where G_2 used to be, and such that any point inside D can be connected to v_1 without intersecting other edges of G_1. Furthermore, if v_1 is on the outer-face of G_1, choose D so small that w is still on the convex hull of the union of G_1 and D.

Draw G_2 with rational edge lengths such that v_2 is on the convex hull. Shrink the drawing of G_2, if necessary, so that it can fit inside the open disk D. Then connect v_2 and v_1, rotating G_2 so that the edge (v_1, v_2) does not intersect it, and shifting G_2 as needed to achieve rational length of the edge (v_1, v_2). See Figure 3.

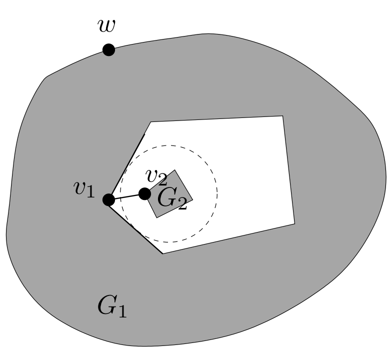

Figure 3: Merging the drawing of G_2 into the drawing of G_1.

3.2 Cutting edge-pairs

So now assume that G has no bridge, but it may have a cutting edge-pair.

In this case, we will show how to draw G with rational edge lengths such that additionally one pre-specified edge (v, w) on the outer-face of G is drawn *as the base of a strictly enclosing half-square*[1] . By this we mean that there exists a triangle T such that (v, w) is drawn on one edge of T, the angles of T at v and w are $\pi/4$, and the rest of the drawing is in the interior of T.

If G has no cutting edge-pair, then it is 3-connected and we can apply the construction of Section 2. Recall that Kant's algorithm allows to choose v_o. If we choose it to be the clockwise first of v and w, then these two vertices will be the leftmost vertices (connected vertically) in Figure 2. By choice of the slopes, the rays of slope -1 and $+1$ from these vertices will form a half-square that contains the whole drawing. So (v, w) is the base of a strictly enclosing half-square as desired.

Now assume that G has cutting edge-pair e_1, e_2 whose removal splits G into graphs G_1 and G_2. Furthermore, assume that $e_i = (v_i, w_i)$ and $v_i \in G_1$ while $w_i \in G_2$. Assume first that neither e_1 nor e_2 is the edge (v, w) to be drawn as base of a strictly enclosing half-square. After possible renaming, we can then assume that (v, w) belongs to G_1.

Let G_1' be the graph obtained from G_1 by adding (v_1, v_2), if it did not exist already, and draw G_1' recursively with rational edge lengths and with (v, w) as base of a strictly enclosing half-square. Locate a triangle T' with base at (v_1, v_2) and small enough that it fits inside the face of G_1 where G_2 used to be. Furthermore, if (v_1, v_2) was on the outer-face of G_1, choose T' so small it fits inside the half-square that strictly encloses the drawing and has (v, w) as base.

Let G_2' be the graph obtained from G_2 by adding (w_1, w_2), if it did not exist already, and draw G_2' recursively with rational edge lengths and with (w_1, w_2) as base of a strictly enclosing half-square.

Shrink the drawing of G_2' small enough so that it fits inside the interior of T with (w_1, w_2) is parallel to (v_1, v_2), and then connect v_1 to w_1 and v_2 to w_2. See Figure 4.

We can justify that this can be done with rational distances for all edges as follows. Let α be the smaller of the angles of T' at v_1 and v_2. Let $0 < \beta < \alpha$ be an angle such that $\sin(\beta)$ is rational. It is easy to find such a β: Since $(2i)^2 + (i^2 - 1)^2 = (i^2 + 1)^2$, simply choose $\beta = \arctan(2i/(i^2 - 1))$ for large enough integer i; then $\sin(\beta) = 2i/(i^2 + 1)$, which is rational.

Now form an isosceles trapezoid with base (v_1, v_2) and angle β at v_1 and v_2. Make the non-parallel sides to be of rational length. Then the *top edge* (the shorter of the parallel edges) has also rational length since $\sin(\beta)$ is rational and (v_1, v_2) has rational length. Also, choose the non-parallel sides long enough such the top edge is

[1]The term "isosceles right triangle" would be more accurate than "half-square", but also more cumbersome.

so short that a half-square with the top edge as base would still be inside T'. This is possible since $\beta < \alpha$.

Scale the drawing of G_2 such that the (rational-length) edge (w_1, w_2) fits onto the (rational-length) top-edge; hence all edges in G_2 are scaled by a rational as desired. The drawing of G_2 then fits entirely inside the half-square on top of the top-edge of the trapezoid, and hence is inside T' and creates no crossings. Therefore the resulting drawing is planar.

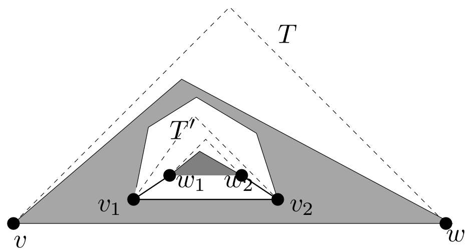

Figure 4: Merging the drawing of G_2 into the drawing of G_1.

A special case occurs if edge (v, w) is one of the edges of the cutting pair, say $v = v_1$ and $w = w_1$. Define the graphs G_1 and G_2 as before, and draw them recursively, drawing (v_1, v_2) and (w_1, w_2) as bases of their respective strictly enclosing half-squares.

Consider the drawing of G_1. Since it is strictly enclosed by a half-square, we can in fact enclose it in an isosceles triangle where the isosceles angles have size $\alpha_1 < \pi/4$. Similarly define $\alpha_2 < \pi/4$ as the isosceles angle of an isosceles enclosing triangle of G_2.

Choose β such that $\max\{\alpha_1, \alpha_2\} < \beta < \pi/4$ and such that $\sin(\beta)$ is rational. Such a β exists since there are infinitely many Pythagorean triplets for which the two shorter lengths differ by one [1].

Scale the drawings of G_1 and G_2 such that the edges (v_1, v_2) and (w_1, w_2) have length 1. Now place the drawings of G_1 and G_2 in a trapezoid where the angles at the larger parallel side are β, the non-parallel sides have length 1, and the parallel sides are rational. See Figure 5. One easily verifies all conditions.

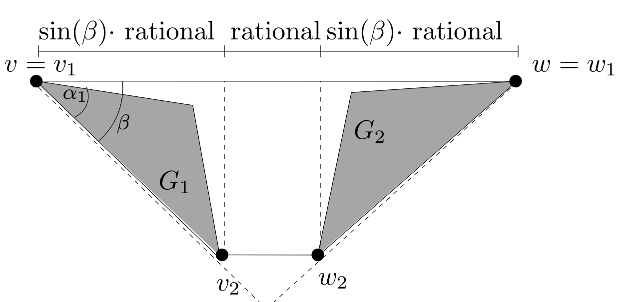

Figure 5: Merging the drawings of G_1 and G_2 if (v, w) is part of the cutting edge-pair.

This ends the proof that there exists an rational edge-length drawing in all cases. Since breaking apart a graph can be done in constant amortized time, and finding the appropriate coordinates for placing the subgraphs can be done in constant time, we hence have:

Theorem 3 *Any planar graph with maximum degree 3 has a planar drawing with integer edge lengths. Moreover, such a drawing can be found in $O(n)$ time.*

We note here that no bound on the coordinates required to achieve integer edge lengths are apparent if the graph has a bridge or a cutting edge-pair. The '$O(n)$' run-time hence only holds under the assumption that arbitrarily small rationals can be handled in constant time. Finding a bound for the coordinates remains an open problem.

4 Graphs with a 3-elimination order

The algorithm that we gave above for finding integer edge-length drawings very much relies on the graph having maximum degree 3. In contrast to this, the proof given by Geelen, Guo and McKinnon [6] only needs a much weaker property of graphs, which we paraphrase as follows:

Definition 1 *Let G be a graph. We say that G has a* 3-elimination order *$v_1, \ldots, v_n$ if*

- *G has only the 2 vertices v_1, v_2, or*
- *v_n has degree at most 2, and $v_1, \ldots, v_{n-1}$ is a 3-elimination order for $G - v_n$, or*
- *v_n has degree 3, and $v_1, \ldots, v_{n-1}$ is a 3-elimination order for $G' = G - v_n \cup (u, w)$, where u and w are two of the neighbours of v_n.*

We note here that the neighbours u and w of a vertex v_n of degree 3 can be chosen arbitrarily. Also, this edge is added only if $G - v_n$ does not contain it already.

This 3-elimination order is a stronger concept than the 3-acyclic edge orientation (see for example [2]), where it is only required that v_n has degree at most 3, but no edge between its neighbours is added. On the other hand, it is a weaker concept than the vertex order that defines a 3-tree. Since we will need this concept later, we define it briefly here. A graph is a *k-tree* if it has a vertex order $v_1, \ldots, v_n$ such that v_i, for $i > k$, has exactly k predecessors and they form a clique. A graph is a *partial k-tree* if it is a subgraph of a k-tree.

Geelen, Guo and McKinnon did not phrase their proof in terms of a 3-elimination order, but following their steps, one can see that this is in fact all they needed, and so they proved:

Theorem 4 *[6] Every graph G that has a 3-elimination order has a straight-line drawing Γ with rational edge* *lengths. Moreover, we can create Γ such that the vertices are placed arbitrarily close to a given drawing Γ' of G. In particular, if G is planar then Γ can be made planar.*

Geelen et al. then used the fact that every graph with maximum degree 3 has a 3-elimination order. They also point out that every partial 3-tree has such an order. But actually, this holds for even more graphs, as we will argue now.

Call a graph G (k,ℓ)-*sparse* if any induced subgraph H of G has $|E(H)| \leq \max\{0, k|V(H)| - \ell\}$. Independently of the research in this paper, Sun [14] showed that any $(2,3)$-sparse graph has an integer edge-length drawing, using results from rigidity theory. But in fact, it can even be shown for $(2,1)$-sparse graphs.

Lemma 5 *Any $(2,1)$-sparse graph G has a 3-elimination order.*

Proof: We prove this by induction; the claim is trivial if G has only 1 vertex. So assume $n \geq 2$. Since G has at most $2n-1$ edges, it has a vertex v of degree at most 3, and we choose this vertex as v_n. $G - v$ is $(2,1)$-sparse and so if $\deg(v) \leq 2$, we can find a 3-elimination order for $G - v$ by induction, add v_n to it and are done.

If $\deg(v) = 3$ then we add an edge between two neighbours u, w of v, so $(2,1)$-sparseness of $G' = G - v \cup \{(u,w)\}$ is not immediately obviously. But we claim that it holds as follows.

Let H' be any induced subgraph of G'. If H' does not contain both u and w, then H' is also an induced subgraph of G and hence $|E(H')| \leq 2|V(H')| - 1$. If H' does contain both u and w, then consider the graph H that is induced by the vertex $V(H') \cup \{v\}$ in graph G. Since G is $(2,1)$-sparse, $|E(H)| \leq 2|V(H)| - 1$. Therefore, $|E(H')| = |E(H)| + 1 - 3 \leq 2|V(H)| - 1 + 1 - 3 = 2|V(H)| - 3 = 2|V(H')| - 1$ as desired.

So G' is also $(2,1)$-sparse and we can find a 3-elimination order of it by induction. Adding $v = v_n$ to it gives the desired order. □

There are numerous graphs that are known to be $(2,1)$-sparse, and we list here just a few:

Theorem 6 *Any $(2,1)$-sparse graph G has a straight-line drawing with rational edge lengths that is planar if G is planar. This includes the following graph classes:*

1. *Connected graphs with maximum degree 4 that are not 4-regular.*
2. *Graphs with arboricity 2.*
3. *Planar bipartite graphs.*
4. *Series-parallel graphs, which are the same as graphs of treewidth 2, which are the same as partial 2-trees.*
5. *Outer-planar graphs.*

Proof: The main claim follows immediately by combining Theorem 4 with Lemma 5 and scaling to make edge lengths into integers. It remains to argue that the given graph classes are actually $(2,1)$-sparse.

1. Any connected graph G with maximum degree 4 is $(2,1)$-sparse unless it is 4-regular. For G itself has at most $2n-1$ edges, and any strict subgraph H of G has at least one vertex with an edge into $G - H$ and hence is also not 4-regular.
2. A graph of arboricity 2 is a graph whose edges can be split into two forests. It is well-known that this is the same as the set of $(2,2)$-sparse graphs [9], which are hence $(2,1)$-sparse.
3. Any planar bipartite graph has arboricity 2 [10].
4. A k-tree can easily be shown to have arboricity k: for each vertex v_i, assign the (at most) k edges to predecessors to different forests. Therefore a partial 2-tree has arboricity 2.
5. Every outer-planar graph is a series-parallel graph.

□

5 Conclusion and open problems

In this paper we studied planar straight-line drawings that have integer edge lengths for all edges. It was already known that this exists for all graphs with maximum degree 3; we provided a different proof of this, which is simpler and constructive, especially for 3-connected 3-regular graphs. Then we proved the same result for some other classes of planar graphs.

The most pressing open problem concerns whether such drawing exists for all planar graphs. Two subclasses of planar graphs where we strongly believe this to be true are the 4-regular graphs and graphs that are acyclic 3-orientable [2]. How can this be proved for them?

For graphs that have an integer edge-length drawing, can we assume that vertices are additionally at integer coordinates? (This holds for the drawings of Theorem 4, and hence for all graphs classes studied thus far.) If so, are the coordinates bounded as a function of n, and how small can they be made? (We proved linear bounds for 3-connected 3-regular graphs only.)

Other modification of this problem are possible. For example, we could define graphs on a given set of points by adding only those edges that are integers, or perhaps even only in a given set of integers (see also the work by Schnabel [11].) What kind of graphs define these, and do they include all planar graphs? In other words, can we draw any planar graph such that the distance between two points is an integer if and only if the edge between them exists?

References

[1] C.C. Chen and T.A. Peng. Classroom note: Almost-isosceles right-angled triangles. *Australasian Journal of Combinatorics*, 11:263–267, 1995.

[2] H. de Fraysseix and P. Ossona de Mendez. Regular orientations, arboricity and augmentation. In *Graph Drawing (GD'94)*, volume 894 of *Lecture Notes in Computer Science*, pages 111–118, 1994.

[3] V. Dujmovic, D. Eppstein, M. Suderman, and D. Wood. Drawings of planar graphs with few slopes and segments. *Computational Geometry: Theory and Applications*, 38:194–212, 2007.

[4] I. Fáry. On straight line representation of planar graphs. *Acta Scientiarum Mathematicarum (Szeged)*, 11:229–233, 1948.

[5] H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10:41–51, 1990.

[6] J. Geelen, A. Guo, and D. McKinnon. Straight line embeddings of cubic planar graphs with integer edge lengths. *Journal of Graph Theory*, 58(3):270–274, 2008.

[7] G. Kant. Hexagonal grid drawings. In *Graph-theoretic concepts in computer science (WG'93)*, volume 657 of *Lecture Notes in Comput. Sci.*, pages 263–276, 1993.

[8] A. Kemnitz and H. Harborth. Plane integral drawings of planar graphs. *Discrete Mathematics*, 236:191–195, 2001.

[9] C. St. J. Nash-Williams. Decomposition of finite graphs into forests. *J. London Math. Soc.*, 39:12, 1964.

[10] G. Ringel. Two trees in maximal planar bipartite graphs. *J. Graph Theory*, 17:755–758, 1993.

[11] K. Schnabel. Representation of graphs by integers. In *Topics in Combinatorics and Graph Theory*, pages 635–640. Physica-Verlag, 1990.

[12] W. Schnyder. Embedding planar graphs on the grid. In *ACM-SIAM Symposium on Discrete Algorithms (SODA '90)*, pages 138–148, 1990.

[13] S. Stein. Convex maps. In *Proceedings of the Amercian Mathematical Society*, volume 2, pages 464–466, 1951.

[14] T. Sun. Rigidity-theoretic construction of integral Fary embeddings. In *Canadian Conference on Computational Geometry (CCCG '11)*, 2011. In these proceedings.

[15] K. Wagner. Bemerkungen zum Vierfarbenproblem. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 46:26–32, 1936.