Inversion of Dynamic Systems for Certain Classes of Signals

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Abstract. Methods of inversion of dynamic systems are widely used for solving problems of control of mechanical and electrical systems. Solving the inversion problems raises a number of difficulties related to the high sensitivity of the results with respect to the accuracy of setting the parameters of a mathematical model of object parameters, instability in controlling non-minimum phase objects, and violation of the conditions of physical realizability.

In this paper, an approximate method of solving the inversion problem for linear stationary dynamic systems is proposed which is largely free from those disadvantages. The method is based on the representation of the input and output signals by their approximations in the linear space of specially selected *D*-functions of time. The feature of the proposed method of inversion of dynamic systems is the representation of multidimensional polynomials approximating the input and output signals as a product of rectangular matrices and a vector of powers of time.

Mathematical models of linear dynamic systems in the form of differential equations in the state space and in the equivalent input-output form, as well as SISO and MIMO dynamical systems are considered in the paper.

Keywords: dynamical systems, polynomial signals, quasi-harmonic functions, matrix equations.

1 Introduction

The inversion problem of dynamic systems has an extensive bibliography and a long history. The papers [1, 2] can be considered as the fundamental work in this direction, where the criteria and methods for constructing inverse operators are justified. A significant contribution to the development of the theory and practice of inversion of dynamic systems was made in the works [3–5]. In them new criteria for the inversibility of linear dynamic systems are proposed and specific ways to solve the inversion problem are given. A number of practical results of solving inversion problems with regard to electrical and mechanical systems are given in [6, 7].

The inversion problem become of particular importance due to the solution of the problem of the synthesis of combined automatic control systems. Various aspects of the inversion problem for combined control systems in the most general statement are presented in [4]. Despite significant progress in solving the inversion problem of dynamic systems, in practice there are a number of difficulties associated with the high sensitivity of the results to the accuracy of the parameters of the mathematical model of a controlled object, the instability in controlling non-minimally phase objects, and violation of conditions of physical realizability of inverse operators. Generally, the listed problems do not allow us to find a practically realizable solution of the problem of finding the inverse operator in the control problem. Nevertheless, for solving a number of practical problems, it seems natural to consider approximate mathematical models of a controlled object and signals at its inputs and outputs, for which the inversion problem has the correct solution. In combined control systems, these assumptions are compensated for by the deviation control loop.

Thus, the purpose of this work is to develop an approximate, practically realizable numerical method for solving the inversion problem for linear dynamic systems.

2 Statement of the research problem

We will consider linear stationary dynamical systems whose mathematical models are presented either in the form of equations of state

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{1}$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the control vector; $y \in R^s$ the output vector; A, B, C are matrices of the corresponding dimensions, or in the equivalent form "input-output":

$$A_0 y^{(p)} + A_1 y^{(p-1)} + \ldots + A_p y = B_0 u^{(q)} + B_1 u^{(q-1)} + \ldots + B_q u \,, \tag{2}$$

where $A_0, A_1, \ldots, A_p$ are $(s \times s)$ matrices, $B_0, B_1, \ldots, B_q$ are $(s \times m)$ matrices. In the following, we will assume that the controlled system under consideration is asymptotically stable, and the dimensions of the control vector and the output vector coincide, i.e. $s = m$.

Consider the linear space of continuous Φ of continuous differentiable vector functions $\varphi(t)$, which satisfy the condition

$$\frac{d\varphi(t)}{dt} \in \Phi \,. \tag{3}$$

These D-functions include the class of functions of the form

$$\varphi(t)=\sum_{k=1}^{N} e^{\alpha_k t}\left(R_k(t)\sin\omega_k t+Q_k(t)\cos\omega_k t\right), \tag{4}$$

where α_k, ω_k are some constants, but $R_k(t)$ and $Q_k(t)$ are vector polynomials of the degree not higher than l. It is not difficult to verify that a function of the form (4) satisfies condition (3). For different values of the parameters α_k, ω_k, $R_k(t)$, $Q_k(t)$ different classes of D-functions can be obtained: polynomials, trigonometric polynomials, quasi-harmonic functions.

Prove the following statement: if the input action $u(t)$ is a D-function of the Φ class, then the forced response of the dynamic system (1) is also a function of the Φ class.

To prove the statement, we will seek a solution of equation (1) in the form of an infinite series

$$x(t)=\sum_{k=1}^{\infty} C_k u^{(k-1)}(t), \tag{5}$$

where C_k are some $m \times n$ matrices to be determined.

After substitution (5) in (1) we get

$$\sum_{k=1}^{\infty} C_k u^{(k)}=\sum_{k=1}^{\infty} AC_k u^{(k-1)}+Bu\ . \tag{6}$$

By equating matrix coefficients of derivatives $u^{(k)}$ of the same order in the left and right sides of (6), we obtain a system of recurrence relations for calculating matrices C_k:

$$AC_1+B=0\,,\ AC_{k+1}=C_k\,,\ k=\overline{1,\infty}$$

of which directly follows

$$C_1=-A^{-1}B,\ C_2=-A^{-2}B,\ldots,C_k=-A^{-k}B,\ldots \tag{7}$$

Thus, the output response of the dynamic system (1) at zero initial conditions will take the form

$$y(t)=-C\sum_{k=1}^{\infty} A^{-k}Bu^{(k-1)}(t). \tag{8}$$

Since all $u^{(k-1)}(t)$ in (8) belong to the class Φ, their linear transformation $y(t)$ is also an element of Φ, i.e. the statement is proved.

Within these assumptions about the structures of the dynamic system and the signals at the inputs and outputs, the statement of the inversion problem can be formulated as follows: find an input action $u(t)$ on an interval $[t_0,t_1]$ belonging to a certain class Φ of D-functions, under which the output $y(t)$ of the system (1), (2) will be a specified D-function of the same class Φ at zero initial conditions.

3 Inverting dynamic systems in a class of polynomials

Let the set Φ of input and output signals be a set of vector polynomials of degree not higher than l

$$\varphi(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_l t^l , \tag{9}$$

where $a_0, a_1, \ldots, a_l$ are n-dimensional vectors of coefficients.

Instead of polynomials of the form (9), we will consider equivalent polynomials

$$\varphi(t) = b_0 + b_1 t + b_2 \frac{t^2}{2!} + b_3 \frac{t^3}{3!} + \ldots + b_l \frac{t^l}{l!} , \tag{10}$$

where vectors a_k and b_k are connected by the ratio $b_k = a_k k!,\ k = \overline{0,l}$.

Further, the vector polynomial (10) will be represented in a vector-matrix form.

$$\varphi(t) = BT , \tag{11}$$

where B is an $n \times (l+1)$-dimensional matrix, whose columns correspond to the vector coefficients of the polynomial (10), and $T = \left(1, t, \frac{t^2}{2!}, \ldots, \frac{t^l}{l!}\right)^T$ is the $(l+1)$-dimensional column vector.

Representation of multidimensional polynomials in the vector-matrix form (11) will allow in the future to effectively apply matrix methods to solving specific inversion problems of dynamic systems.

Consider some elementary operations with polynomials and their analogues in vector-matrix form:

- addition: $\varphi_1(t) + \varphi_2(t) = (B_1 + B_2)T$;
- multiplication by number: $\alpha\varphi(t) = (\alpha B)T$;
- multiplication by the matrix C on the left: $C\varphi(t) = CBT$;
- differentiation:

$$\frac{d\varphi(t)}{dt} = B\Lambda T , \tag{12}$$

where the elements of the $(l+1) \times (l+1)$ dimension matrix Λ are the form

$$\lambda_{ij} = \delta_{i,j+1} ,\ i, j = \overline{1, l+1} , \tag{13}$$

where δ_{ij} is the Kronecker delta.

Subsequent derivatives are found in accordance with the formula

$$\frac{d^k \varphi(t)}{dt^k} = B\Lambda^k T , \tag{14}$$

where the elements of the matrix Λ^k are found by the formula which is similar to (13)

$$\lambda_{ij}^k = \delta_{i,j+k} , \ i, j = \overline{1, l+1} . \tag{15}$$

Consider first the SISO system defined in the form of "input-output" (2)

$$a_0 y^{(p)} + a_1 y^{(p-1)} + ... + a_p y = b_0 u^{(q)} + b_1 u^{(q-1)} + ... + b_q u , \tag{16}$$

where $a_0, a_1, ..., a_p$, $b_0, b_1, ..., b_q$ are constant coefficients, $p \geq q$.

The polynomial signals at the input and at the output will be represented in a vector-matrix form

$$y(t) = YT , \ u(t) = UT , \tag{17}$$

where Y and U are $(l+1)$-dimensional row vector composed of the coefficients of the polynomials $y(t)$ and $u(t)$.

After substitution (17) into (16) we get

$$a_0 Y\Lambda^p T + a_1 Y\Lambda^{p-1} T + ... + a_p YT = b_0 U\Lambda^q T + b_1 U\Lambda^{q-1} T + ... + b_q UT . \tag{18}$$

From (18) directly follows

$$Y\left(a_0 \Lambda^p + a_1 \Lambda^{p-1} + ... + a_p E\right) = U\left(b_0 \Lambda^q + b_1 \Lambda^{q-1} + ... + b_q E\right) . \tag{19}$$

Introducing the corresponding notations, we rewrite (19) in the form

$$Y\overline{A} = U\overline{B} , \tag{20}$$

where $\overline{A}$ and $\overline{B}$ are the square $(l+1) \times (l+1)$ lower triangular matrices, which are formed in accordance with the following rules:

$$\begin{aligned} a_{ij} &= \begin{cases} 0 \ \text{if} \ \ j + p < i \vee j < j, \\ a_{j-1+p} \ \text{if} \ \ j + p \geq i \vee i \geq j, \end{cases} \\ b_{ij} &= \begin{cases} 0 \ \text{if} \ \ j + q < i \vee i < j, \\ b_{j-i+q} \ \text{if} \ \ j + q \geq i \vee i \geq j, \end{cases} \end{aligned} \quad i, j = \overline{1, l+1} .$$

and have the form

Table 1. View of the lower triangular matrix $\overline{A}$

		1	2	3	4	…	$l+1$
	1	a_p	0	0	0	…	0
	2	a_{p-1}	a_p	0	0	…	0
	3	a_{p-2}	a_{p-1}	a_p	0	…	0
$\overline{A} =$	⋮	⋮	⋮	⋮	⋮	…	⋮
	$p+1$	a_0	a_1	a_2	a_3	…	0
	⋮	⋮	⋮	⋮	⋮	…	⋮
	$l+1$	0	0	0	0	…	a_p

Table 2. View of the lower triangular matrix $\overline{B}$

		1	2	3	4	…	$l+1$
	1	b_q	0	0	0	…	0
	2	b_{q-1}	b_q	0	0	…	0
	3	b_{q-2}	b_{q-1}	b_q	0	…	0
$\overline{B} =$	⋮	⋮	⋮	⋮	⋮	…	⋮
	$q+1$	b_0	b_1	b_2	b_3	…	0
	⋮	⋮	⋮	⋮	⋮	…	⋮
	$l+1$	0	0	0	0	…	b_q

Let us analyze the relation (20). It is easy to see that (20) is a linear mapping between vectors Y and U, written in a symmetric form. Since $\overline{A}$ and $\overline{B}$ are lower triangular matrices with diagonal elements $a_{ii} = a_p$ and $b_{ii} = b_q \left(i = \overline{1, l+1}\right)$, then $\overline{A}$ and $\overline{B}$, in the general case, are non-degenerate and system (20) has a unique solution.

If a solution of the direct control problem is sought, then $Y = U\overline{B}\,\overline{A}^{-1}$.

In the case that is of interest to us, the solution of the inversion problem, the desired vector of coefficients U is found from the relation $U = Y\overline{A}\overline{B}^{-1}$.

The matrix $\overline{B}^{-1}$ can be calculated as follows.

$$\overline{B}^{-1} = \begin{pmatrix} \frac{1}{b_q} & 0 & 0 & \cdots \\ -\frac{\Delta_1}{b_q^2} & \frac{1}{b_q} & 0 & \cdots \\ \frac{\Delta_2}{b_q^3} & -\frac{\Delta_1}{b_q^2} & \frac{1}{b_q} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\Delta_1, \Delta_2, ..., \Delta_l$ is the sequence of minors of matrix $\overline{B}$, in which each successive one is formed as a result of the bordering of the previous minor, starting from $\Delta_1 = b_{q-1}$ – the element of the matrix b_{21}.

An effective method for calculating minors Δ_k is a method based on the consistent application of the Schur and Frobenius formulas [8] for calculating the determinant and inversion of block matrices. From the computational point of view, the solution (20) with respect to the vector U is easier to find by sequential calculation of the components U, starting from u_l. In this case, due to the triangular structure of the matrix $\overline{B}$, a linear equation with one unknown is formed at each step of the iterative process. Thus, the solution of the inversion problem is found for $l+1$ steps.

Consider the solution of the inversion problem for MIMO systems in the polynomial signals environment. Let the mathematical model of the system under consideration be of the form (1). Then the solution of the direct control problem in the general form is represented in the form (8). Substitute in (8) vector-matrix expressions for polynomials $y(t)$ and $u(t)$, taking into account that the powers of the matrix Λ satisfy the condition $\Lambda^k = 0$, $\forall k > l$.

As a result, we obtain the analytical relationship between the matrices Y and U:

$$Y = -\sum_{k=1}^{l+1} CA^{-k} BU\Lambda^{k-1} . \tag{21}$$

The result (21) for solving the direct control problem in the polynomial signal environment makes it easy to solve the inverse problem by solving the matrix equation (21) with respect to the matrix U under a specified matrix Y.

The solution of the matrix equation (21) can be obtained by vectoring matrices Y and U and constructions based on the Kronecker product of matrices [9], which leads to a linear system of algebraic equations of $m \times (l+1)$ dimension:

$$\sum_{k=1}^{l+1} CA^{-k} B \otimes \left(\Lambda^{k-1}\right)^T vecU + vecY = 0, \tag{22}$$

where the column vectors $vecU$ and $vecY$ are composed of transposed rows of matrices Y and U matched in ascending order of the row number.

4 Inverting dynamic systems in the class of quasiharmonic functions

Let the signals at the input and output of a dynamic system have the form

$$\varphi(t) = \sum_{k=1}^{N} \left(R_k(t) \sin \omega_k t + Q_k(t) \cos \omega_k t\right), \tag{23}$$

where $R_k(t)$ and $Q_k(t)$ are vector polynomials of the degree not higher than l, ω_k are some positive numbers.

Using the matrix-vector representation of polynomials, the relation (23) can be written as

$$\varphi(t) = \sum_{k=1}^{N} (R_k \sin \omega_k t + Q_k \cos \omega_k t) T , \tag{24}$$

where R_k and Q_k are matrices whose rows correspond to the coefficients of the components of vector polynomials in (23).

To find the derivatives of the function (24), we first consider the single-frequency function of the form (24)

$$\varphi(t) = (R \sin \omega t + Q \cos \omega) T . \tag{25}$$

The sequence of derivative functions of the form (25) can be written as

$$\frac{d\varphi}{dt} = [(R\Lambda - Q\omega)\sin \omega t + (R\omega + Q\Lambda)\cos \omega t]T,$$

$$\frac{d^2\varphi}{dt^2} = \left[(R\Lambda^2 - 2Q\Lambda\omega - R\omega^2)\sin \omega t + (Q\Lambda^2 + 2R\Lambda\omega - Q\omega^2)\cos \omega t\right]T,$$

....

$$\frac{d^k\varphi}{dt^k} = (R^k \sin \omega t + Q^k \cos \omega t)T;$$

where R^k and Q^k are calculated by the recurrence formula

$$\left(R^k \mid Q^k\right) = \left(R^{k-1} \mid Q^{k-1}\right)\Omega , \tag{26}$$

where Ω is a $2(l+1) \times 2(l+1)$ block matrix

$$\Omega = \left(\begin{array}{c|c} \Lambda & \omega E \\ \hline -\omega E & \Lambda \end{array}\right), \tag{27}$$

each block of which has a dimension of $(l+1) \times (l+1)$.

Based on the relations (26) and (27), the sequence of matrices of polynomial coefficients of derivatives of a single-frequency function (25) can be written as

$$\left(R^k \mid Q^k\right) = (R \mid Q)\Omega^k , \tag{28}$$

where the matrix Ω^k is calculated by successive raising to the power of the matrix (27).

Consider the most general case when an exponential multiplier is present at a specified output of the system, i. e.

$$\varphi(t)=\sum_{k=1}^{N} e^{\lambda_k t}\left(R_k(t)\sin\omega_k t+Q_k(t)\cos\omega_k t\right). \tag{29}$$

Spreading the method for finding the derivatives of a single-frequency quasi-harmonic signal, discussed earlier, we will write the sequence of derivatives of single-frequency functions (29) in the form

$$\frac{d^k\varphi}{dt^k}=e^{\lambda t}\left[R^k\sin\omega t+Q^k\cos\omega t\right]T,$$

where

$$\begin{aligned}
R^k &= R^{k-1}\Lambda+R^{k-1}\lambda-Q^{k-1}\omega,\\
Q^k &= Q^{k-1}\Lambda+Q^{k-1}\lambda+R^{k-1}\omega, \quad k=\overline{1,l}\\
\mathrm{R}^0 &= R,\ Q^0=Q.
\end{aligned}$$

or in a matrix form

$$\left(R^k \mid Q^k\right)=\left(R^{k-1} \mid Q^{k-1}\right)\left(\begin{array}{c|c}\Lambda+\lambda E & \omega E\\ \hline -\omega E & \Lambda+\lambda E\end{array}\right).$$

Introducing the notation

$$\overline{\Omega}=\left(\begin{array}{c|c}\Lambda+\lambda E & \omega E\\ \hline -\omega E & \Lambda+\lambda E\end{array}\right),$$

we obtain the final ratio to calculate the coefficients R^k and Q^k of derivatives $\varphi^{(k)}(t)$:

$$\left(R^k \mid Q^k\right)=\left(R \mid Q\right)\overline{\Omega}^k, \tag{30}$$

where the power of the matrix $\overline{\Omega}^k$ are calculated by successive raising $\overline{\Omega}$ to a power.

Let the matrix representation of the input and output signals be in the form

$$\begin{aligned}
u(t)&=e^{\lambda t}\left(R_u\sin\omega t+Q_u\cos\omega t\right)T,\\
y(t)&=e^{\lambda t}\left(R_y\sin\omega t+Q_y\cos\omega t\right)T,
\end{aligned} \tag{31}$$

where R_u, Q_u, R_y, Q_y are the $q\times(l+1)$-dimensional matrix of coefficients of polynomial multiplier.

Then the derivatives of these signals in accordance with (30) can be represented as

$$\begin{aligned}
\frac{d^k u}{dt^k}&=e^{\lambda t}\left(R_u^k\sin\omega t+Q_u^k\cos\omega t\right)T,\\
\frac{d^k y}{dt^k}&=e^{\lambda t}\left(R_y^k\sin\omega t+Q_y^k\cos\omega t\right)T,
\end{aligned} \tag{32}$$

where the matrix coefficients R_u^k, Q_u^k, R_y^k, Q_y^k are found in accordance with the formula (30).

After substituting the (31) and (32) into (2) and reducing by $e^{\lambda t}$ and T we get

$$\sum_{k=0}^{p} A_k Y \overline{\Omega}^{p-k} = \sum_{k=0}^{q} B_k U \overline{\Omega}^{q-k} , \tag{33}$$

where $Y = \left(R_y \mid Q_y\right)$ and $U = \left(R_u \mid Q_u\right)$ are matrix representations of polynomial factors in front of functions $\sin \omega t$ and $\cos \omega t$ for output and input signals.

The obtained matrix mapping (33) is symmetrical and allows us to find solutions to both direct and inverse control problems by vectoring the desired control of matrix Y and constructions based on the Kronecker product of matrices [9] by analogy with (22).

The obtained result is easily distributed to the SISO system. In this case, the matrices A_k and B_k are scalars a_k and b_k, accordingly, and the matrix equation (33) takes the form

$$Y \sum_{k=0}^{p} a_k \overline{\Omega}^{p-k} = U \sum_{k=0}^{q} b_k \overline{\Omega}^{q-k} ,$$

whose solution with respect to vectors Y or U (for the problem of inversion) is connected to the procedure of inversion of matrices $\sum_{k=0}^{p} a_k \overline{\Omega}^{p-k}$ or $\sum_{k=0}^{q} b_k \overline{\Omega}^{q-k}$ of $2(l+1) \times 2(l+1)$ dimension.

5 Software package for solving inversion problems

The software is fully developed in relation to polynomial models of input and output signals and contains the following basic structural blocks:

1. The block of input the initial information i.e. matrices A, B, C and their dimensions for the case of specifying the controlled object in the form (1), and coefficients $a_0, a_1, ..., a_p$, $b_0, b_1, ..., b_q$ for SISO systems specified in the form of "input-output" (16).

2. A task formation block that includes a random or fixed-step selection of N values of components of the output vector $y^*(t)$ at a fixed time interval, as well as setting the degree l of approximating polynomials $y^*(t)$ and calculating their coefficients using the least squares method. The approximation of signals by cubic splines is provided.

3. The block of formation of matrices $\overline{A}$ and $\overline{B}$ as well as the matrix of the system of linear equations (22) and the calculation of the condition number $cond(\cdot)$ of systems of linear equations based on the Euclidean norm. If $cond(\cdot) \leq 100$, then the solu-

tion of systems (20) or (22) is followed. Otherwise, the degree of approximating polynomials l is incremented by one.

4. The correctness of the inversion problem solution is controlled by numerical integration of the initial systems of differential equations with zero initial conditions and comparison of the result of integration $y(t)$ to the corresponding values of the initial output function $y^*(t)$.

6 Conclusion

Simplified mathematical models of signals based on polynomials and harmonic functions with polynomial varying amplitude are proposed in the paper.

A matrix representation of polynomial signals was proposed and substantiated, which made it possible to represent controlled dynamic processes as static linear transformations in the space of rectangular matrices. Rather simple and effective algorithms for the numerical solution of the inversion problem, as well as a method for estimating the degree of robustness of the results, are obtained.

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