Pointwise Circumscription in Description Logics

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Abstract
Circumscription is one of the major approaches to bring non-monotonic (common-sense) reasoning features to first-order logic and related formalisms, and it has already received attention in Description Logics (DLs), with the focus on understanding the computational complexity of reasoning. Those studies revealed that circumscription causes a dramatic increase in computational complexity in a broad range of DLs. In this paper, we consider a new notion of circumscription in DLs, aiming to preserve the key ideas and advantages of classical circumscription while mitigating its impact on the computational complexity of reasoning. Our main idea is to replace the second-order quantification step with a series of (pointwise) local checks on all domain elements and their immediate neighborhood. This approach provides a sound approximation of classical circumscription and is closely related to the notion of pointwise circumscription proposed by Lifschitz for first-order logic. Our main achievement is to show that, under certain syntactic restrictions, standard reasoning problems like subsumption testing or concept satisfiability for $\mathcal{ALCIO}$ KBs with pointwise circumscription are (co)NExpTime-complete.

1. Introduction

As fragments of first-order logic, Description Logics (DLs) inherit many of its features, including *monotonicity*. In a monotonic logic, adding new knowledge to a knowledge base does not invalidate the previously derived knowledge. Humans in their everyday life often resort to non-monotonic reasoning, e.g., when dealing with incomplete information. Adding non-monotonic features to monotonic formalisms is a big challenge, and it often causes undecidability or a significant increase in the complexity of reasoning. Several non-monotonic extensions of DLs have been proposed, aiming to balance the computational cost and the expressiveness (see, e.g., [1, 2, 3, 4]). A prominent research line here is *circumscribed DLs* [5, 6, 7, 4]. Circumscription is a powerful tool that was first introduced by McCarthy as an extension of first-order logic. In its basic form, the intended (or *preferred*) models of a circumscribed theory are obtained by considering its classical models and additionally minimizing the extensions of some selected predicates [8, 9, 10]. In general, additionally to the predicates to be minimized, one may specify—by means of a *circumscription pattern*—the predicates whose extensions must remain fixed and the predicates that may vary freely. Circumscription in DLs has different expressiveness benefits, but unfortunately the complexity of reasoning in circumscribed DLs is often very high, and undecidability is easily encountered [11, 4]. The key reason for the high complexity is the *second-order quantification* that is needed in order to identify the preferred models of a circumscribed

DL 2022: 35th International Workshop on Description Logics, August 7–10, 2022, Haifa, Israel
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DL knowledge base (KB). The main goal of our work is to lower the computational complexity of reasoning by considering an alternative (weaker) notion of circumscription that is useful for knowledge representation and does not use such a powerful second-order quantification.

We introduce *pointwise circumscription* in DLs. The basic idea here is to replace the *single global* minimality check of classic circumscription by *multiple local* minimality checks at all domain elements and their immediate neighborhood. Since the local minimality checks of pointwise circumscription are not capable of detecting self-justification cycles that span two or more domain elements—something easily detectable using second-order quantification of classic circumscription—pointwise circumscription is strictly weaker than (and it thus provides a sound approximation of inferences w.r.t.) classic circumscription. That is, given a KB and a circumscription pattern, the preferred models under pointwise circumscription will subsume the preferred models under classic circumscription, but not necessarily the other way around.

The term "pointwise circumscription" was coined by Lifschitz in [12] for full first-order logic, where the second-order quantification over predicate extensions is replaced with a series of individual additions or removals of tuples (*points*) in predicates, performed *pointwise.* To get some intuition behind this, consider a theory φ consisting of the single formula $(\exists x, y.A(x, y)) \wedge (\forall x, y.A(x, y) \leftrightarrow (B(y, x) \vee C(y, x))$ and suppose we want all predicates A, B, C to be minimized. Consider three models of φ over the domain $\{c, d\}$ represented in the obvious way as sets of atoms $I_1 = \{A(c, d), B(d, c)\}$, $I_2 = \{A(c, d), B(d, c), A(d, c), B(c, d)\}$, $I_3 = \{A(c, d), B(d, c), A(d, c), B(c, d), C(c, d)\}$. To check whether a structure above is an intended model according to Lifschitz's pointwise (resp., classic) circumscription we need to see if we can remove *one* atom (resp., a *set* of atoms) such that the resulting structure remains a model of φ. Observe that given I_1 or I_2, we cannot obtain a new model of φ by deleting a single atom from them thus I_1, I_2 are both minimal models of φ in the sense of pointwise circumscription. However, I_2 is not a minimal model in the sense of classic circumscription, because by deleting the pair $A(d, c), B(c, d)$ from I_2 we obtain a model of φ. I_1 is clearly an intended model under classic circumscription. I_3 is not an intended model under either notions of circumscription because the deletion of $C(c, d)$ still leads to a model of φ. Observe that pointwise circumscription did not eliminate I_2 because it did not identify a self-justification cycle: $A(d, c)$ survives in I_2 because of the presence of $B(c, d)$, and $B(c, d)$ survives because of the presence of $A(d, c)$. The variant of circumscription we consider in this paper is closely related in spirit but is technically orthogonal to the one of Lifschitz. We consider a restricted fragment of first-order logic (concretely, the DL $\mathcal{ALCIO}$) and perform a single model "improvement" step that allows to modify the configuration of a domain element, i.e., its membership in concept names and roles.

The main contributions of this paper are the following.

- We define the framework of *pointwise circumscription* in DLs. To do this, we introduce a simple notion of *comparability relations* for pairs of interpretations, which allows defining different variants of pointwise circumscription. These variants correspond to different ways we are allowed to make local improvements to a given model of a DL KB. For further study, we settle on a "star-based" variant where at each domain object e we are allowed to simultaneously change the participation of the object e in concept names and roles.

- We provide a method for reasoning under pointwise circumscription in TBoxes in the very expressive DL $\mathcal{ALCIO}$. The reasoning tasks include concept satisfiability, concept subsumption, and entailment of assertions. In this preliminary work, we consider TBoxes with inclusions of a specific shape, similar to various 'normalized' TBoxes in the literature, and where concept expressions are limited to modal depth 1. We show that in this setting, reasoning is feasible in (co)NExpTime. The upper bounds are obtained by applying the mosaic technique and integer programming. The basic idea of the mosaic technique [13] is that showing the existence of a model is equivalent to showing the existence of a set of fragments of models. We call these fragments star types. We define a minimality condition over star types and following the work done in [14] and [15], we provide algorithms by means of integer programming.

- We show that satisfiability of concept names in pointwise circumscribed $\mathcal{ALCIO}$ TBoxes in our restricted form is already hard for NExpTime. This is done by providing a reduction from the tiling problem for exponential grids, and it shows that the upper bounds of this paper are worst-case optimal.

2. Preliminaries

We denote with N_C, N_R, and N_I countably infinite sets of *concept names* (unary predicates), *role names* (binary predicates), and *individual names* (constants). The *inverse role* of a role name $r \in N_R$ is r^-. We define the set of *roles* $N_R^+ = N_R \cup \{r^- | \; r \in N_R\}$. We let $r^{--} = r$, and given a set R of roles, we define $R^- = \{r^- | \; r \in R\}$. In $\mathcal{ALCIO}$, *concepts* are defined according to the syntax $C := \top | \; \bot | \; A | \; \{a\} | \; \neg C | \; C \sqcup C | \; C \sqcap C | \; \exists r.C | \; \forall r.C$, with $A \in N_C$, $r \in N_R^+$, and $a \in N_I$. An expression of the form $C \sqsubseteq D$, with C and D concepts, is a *concept inclusion*. An expression of the form $A(a)$, where $A \in N_C$ and $a \in N_I$, is a *concept assertion*. An expression of the form $r(a, b)$, with $r \in N_R$ and $a, b \in N_I$, is a *role assertion*. A TBox $\mathcal{T}$ in $\mathcal{ALCIO}$ is a collection of concept inclusions, and an ABox $\mathcal{A}$ is a collection of concept and role assertions. A knowledge base $\mathcal{K}$ in $\mathcal{ALCIO}$ is a pair $(\mathcal{T}, \mathcal{A})$, with $\mathcal{T}$ a TBox and $\mathcal{A}$ an ABox. Given a TBox, we denote with $N_C(\mathcal{T})$, $N_R(\mathcal{T})$, and $N_I(\mathcal{T})$ the sets of concept names, role names, and individual names occurring in $\mathcal{T}$. We denote with $N_C^+ = N_C \cup \{\{a\} | \; a \in N_I\} \cup \{\top, \bot\}$ the set of *basic concepts*. Given a TBox $\mathcal{T}$, $N_C^+(\mathcal{T})$ denotes the set of basic concepts occurring in $\mathcal{T}$ and $N_R^+(\mathcal{T})$ the set of roles occurring in $\mathcal{T}$. The semantics is defined by means of *interpretations* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$. The two components $\Delta^\mathcal{I}$ and $\cdot^\mathcal{I}$ are called *domain* and *interpretation function*. The latter associates to each $a \in N_I$ a unique element in the domain, to each $A \in N_C$ a set $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$ and to each $r \in N_R$ a set $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$. The semantics for $\mathcal{ALCIO}$ concepts is defined as usual [16]. The notions of *model* of an inclusion, a TBox, a KB are also standard. Given an interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, for any $d \in \Delta^\mathcal{I}$, we define the *unary type* of d as $ut^\mathcal{I}(d) = \{A \in N_C^+ \mid d \in A^\mathcal{I}\}$, and for any $e \in \Delta^\mathcal{I}$, we define the *binary type* of (d, e) as $bt^\mathcal{I}(d, e) = \{r \in N_R^+ \mid (d, e) \in r^\mathcal{I}\}$.

3. From Circumscription to Pointwise Circumscription

The framework of *circumscribed DLs* was introduced in [17, 4]. To combine circumscription with DLs, the authors introduce the notion of *circumscription patterns*, declaring how predicates are handled during the minimization process. A circumscription pattern is a triple $\mathcal{P} = (M, V, F)$ declaring three mutually disjoint sets of *minimized* predicates M, *varying* predicates V, and *fixed* predicates F. A circumscription pattern $\mathcal{P}$ induces a *preference relation* $\leq_{\mathcal{P}}$ between interpretations. Given two interpretations $\mathcal{I}$ and $\mathcal{J}$, we write $\mathcal{I} \leq_{\mathcal{P}} \mathcal{J}$ if the following hold: (i) $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$ and $o^{\mathcal{I}} = o^{\mathcal{J}}$, for all $o \in N_I$, (ii) for all $p \in M$, $p^{\mathcal{I}} \subseteq p^{\mathcal{J}}$, (iii) for all $p \in F$, $p^{\mathcal{I}} = p^{\mathcal{J}}$. We write $\mathcal{I} <_{\mathcal{P}} \mathcal{J}$, if $\mathcal{I} \leq_{\mathcal{P}} \mathcal{J}$ and $p^{\mathcal{I}} \subset p^{\mathcal{J}}$ for some $p \in M$. A *circumscribed knowledge base* $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$ is a knowledge base $\mathcal{K}$ equipped with a circumscription pattern $\mathcal{P}$. A model of $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$ is a model of $\mathcal{K}$ that is minimal w.r.t. $<_{\mathcal{P}}$. In [4], the authors provided a characterization of the complexity of reasoning in different fragments of $\mathcal{ALCIOQ}$. In particular, allowing roles to be minimized, reasoning turns out to be undecidable even in circumscribed $\mathcal{ALC}$.

We now introduce *pointwise circumscription* for DLs. To capture some of the possible variants of (local) minimization, we use a family of *comparability relations* between interpretations. Note that any interpretation $\mathcal{I}$ can be seen as a directed labeled graph, where each node is labeled with a collection of concept names and individuals, while each edge is labeled with a collection of role names. We start with the *basic comparability* relation.

Definition 1. *For a pair of interpretations $\mathcal{I}, \mathcal{J}$ we write $\mathcal{I} \sim^{\mathrm{B}} \mathcal{J}$ if $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$ and $a^{\mathcal{I}} = a^{\mathcal{J}}$ for all individuals a.*

The definition above encodes condition (i) of the preference relation $<_{\mathcal{P}}$. Two interpretations $\mathcal{I}$ and $\mathcal{J}$ are B-comparable if they have the same domain and interpret all individuals in the same way. We define the binary relation $\sim^{\mathrm{CL}}$ (resp. $\sim^{\mathrm{CLs}}$) that relates two interpretations that differ by at most one concept label (resp. a set of concept labels) decorating at most one node.

Definition 2. *Assume two interpretations $\mathcal{I}, \mathcal{J}$ with $\mathcal{I} \sim^{\mathrm{B}} \mathcal{J}$. We write $\mathcal{I} \sim^{\mathrm{CL}} \mathcal{J}$, if there exists an object e and a concept name A such that (i) $r^{\mathcal{I}} = r^{\mathcal{J}}$ for all role names r, (ii) $B^{\mathcal{I}} = B^{\mathcal{J}}$ for all concept names $B \neq A$, and (iii) $A^{\mathcal{I}} \setminus \{e\} = A^{\mathcal{J}} \setminus \{e\}$. We write $\mathcal{I} \sim^{\mathrm{CLs}} \mathcal{J}$ if there exists an object e such that (i) $r^{\mathcal{I}} = r^{\mathcal{J}}$ for all role names r, and (ii) $A^{\mathcal{I}} \setminus \{e\} = A^{\mathcal{J}} \setminus \{e\}$ for all concept names A.*

We introduce the binary relation $\sim^{\mathrm{RL}}$ (resp. $\sim^{\mathrm{RLs}}$) that relates two interpretation in terms of a modification of a single role label (resp. a collection of role labels) at one edge.

Definition 3. *Assume two interpretations $\mathcal{I}, \mathcal{J}$ with $\mathcal{I} \sim^{\mathrm{B}} \mathcal{J}$. We write $\mathcal{I} \sim^{\mathrm{RL}} \mathcal{J}$ if there exist objects e, e' and a role name r such that (i) $A^{\mathcal{I}} = A^{\mathcal{J}}$ for all concept names A, (ii) $p^{\mathcal{I}} = p^{\mathcal{J}}$ for all role names $p \neq r$, and (iii) $r^{\mathcal{I}} \setminus \{(e, e')\} = r^{\mathcal{J}} \setminus \{(e, e')\}$. We write $\mathcal{I} \sim^{\mathrm{RLs}} \mathcal{J}$ if there exist objects e, e' such that (i) $A^{\mathcal{I}} = A^{\mathcal{J}}$ for all concept names A, (ii) $r^{\mathcal{I}} \setminus \{(e, e'), (e', e)\} = r^{\mathcal{J}} \setminus \{(e, e'), (e', e)\}$ for all role names r.*

We introduce a *star* comparability relation, where $\mathcal{I}$ and $\mathcal{J}$ are comparable if there is at most one node e on which $\mathcal{I}$ and $\mathcal{J}$ disagree, i.e. for one node e, they might disagree on the concept labeling e or the roles labeling some edges involving e.

Definition 4. *Assume two interpretations $\mathcal{I}, \mathcal{J}$ with $\mathcal{I} \sim^{\mathrm{B}} \mathcal{J}$. We write $\mathcal{I} \sim^{\mathrm{ST}} \mathcal{J}$ if there exists an object e such that: (i) $A^{\mathcal{I}} \setminus \{e\} = A^{\mathcal{J}} \setminus \{e\}$ for all concept names A, and (ii) $r^{\mathcal{I}} \cap (\Delta \times \Delta) = r^{\mathcal{J}} \cap (\Delta \times \Delta)$ for all role names r, where $\Delta = \Delta^{\mathcal{I}} \setminus \{e\}$.*

Proposition 1. *Given two interpretations $\mathcal{I}$ and $\mathcal{I}'$: (i) if $\mathcal{I} \sim^{\mathrm{CL}} \mathcal{I}'$ then $\mathcal{I} \sim^{\mathrm{CLs}} \mathcal{I}'$; (ii) if $\mathcal{I} \sim^{\mathrm{RL}} \mathcal{I}'$ then $\mathcal{I} \sim^{\mathrm{RLs}} \mathcal{I}'$; (iii) if $\mathcal{I} \sim^{\mathrm{CLs}} \mathcal{I}'$ or $\mathcal{I} \sim^{\mathrm{RLs}} \mathcal{I}'$ then $\mathcal{I} \sim^{\mathrm{ST}} \mathcal{I}'$.*

We can now define various versions of pointwise circumscription, parametrized by a concrete comparability relation $\sim^{\circ}$. Circumscription patterns are defined as for classical circumscription. In the following sections, while comparing two interpretations $\mathcal{I}$ and $\mathcal{J}$ via a comparability relation $\sim^{\circ}$ to explicitly state the node e (resp. the pair of nodes (e, e')) at which they may differ in terms of labels, we write $\sim^{\circ}_{e}$ (resp. $\sim^{\circ}_{(e,e')}$).

Definition 5. *Assume a circumscription pattern $\mathcal{P} = (M, V, F)$, a pair of interpretations $\mathcal{I}, \mathcal{J}$, and let $\circ \in \{\mathrm{B}, \mathrm{CL}, \mathrm{CLs}, \mathrm{RL}, \mathrm{RLs}, \mathrm{ST}\}$. We write $\mathcal{I} \preceq^{\circ}_{\mathcal{P}} \mathcal{J}$ if the following conditions hold:*

(i) $Q^{\mathcal{I}} \subseteq Q^{\mathcal{J}}$ *for all* $Q \in M$*,*

(ii) $Q^{\mathcal{I}} = Q^{\mathcal{J}}$ *for all* $Q \in F$*, and*

(iii) $\mathcal{I} \sim^{\circ} \mathcal{J}$*.*

We write: (a) $\mathcal{I} \prec^{\circ}_{\mathcal{P}} \mathcal{J}$, if $\mathcal{I} \preceq^{\circ}_{\mathcal{P}} \mathcal{J}$ and $Q^{\mathcal{I}} \subset Q^{\mathcal{J}}$ for some $Q \in M$, and (b) $\mathcal{I} \preceq^{\circ}_{\mathcal{P},e} \mathcal{J}$ if $\mathcal{I} \sim^{\circ}_{e} \mathcal{J}$. We denote with $Circ^{\circ}_{\mathcal{P}}(\mathcal{K})$ a pointwise circumscribed KB $\mathcal{K}$, i.e., a knowledge base equipped with a circumscription pattern $\mathcal{P}$, where $\circ \in \{\mathrm{B}, \mathrm{CL}, \mathrm{CLs}, \mathrm{RL}, \mathrm{RLs}, \mathrm{ST}\}$. An interpretation $\mathcal{I}$ is a model of $Circ^{\circ}_{\mathcal{P}}(\mathcal{K})$, if $\mathcal{I} \models \mathcal{K}$ and there is no intepretation $\mathcal{J}$ s.t. $\mathcal{J} \models \mathcal{K}$ and $\mathcal{J} \prec^{\circ}_{\mathcal{P}} \mathcal{I}$.

We focus on the reasoning tasks of *concept satisfiability*, *concept subsumption* and *instance checking* w.r.t. *pointwise circumscribed KBs*. As shown for circumscribed DLs in [4], the aforementioned reasoning tasks can be polynomially reduced one into the other.

Definition 6. *Assume a KB $\mathcal{K}$ and a circumscription pattern $\mathcal{P}$. Given two concepts C_0 and D_0 in $\mathcal{ALCIO}$ and $a \in N_I$:*

- *C_0 is satisfiable w.r.t. $Circ^{\circ}_{\mathcal{P}}(\mathcal{K})$ if there exists a model $\mathcal{I}$ of $Circ^{\circ}_{\mathcal{P}}(\mathcal{K})$ such that $C_0^{\mathcal{I}} \neq \emptyset$;*
- *C_0 is subsumed by D_0 w.r.t. $Circ^{\circ}_{\mathcal{P}}(\mathcal{K})$, and we write $Circ^{\circ}_{\mathcal{P}}(\mathcal{K}) \models C_0 \sqsubseteq D_0$, if $C_0^{\mathcal{I}} \subseteq D_0^{\mathcal{I}}$ in any model $\mathcal{I}$ of $Circ^{\circ}_{\mathcal{P}}(\mathcal{K})$;*
- *a is an instance of a concept C_0 w.r.t $Circ^{\circ}_{\mathcal{P}}(\mathcal{K})$, and we write $Circ^{\circ}_{\mathcal{P}}(\mathcal{K}) \models C_0(a)$, if $a^{\mathcal{I}} \in C_0^{\mathcal{I}}$ in all models $\mathcal{I}$ of $Circ^{\circ}_{\mathcal{P}}(\mathcal{K})$.*

Example 1. *We reformulate the example in [18] on the* situs inversus*, a condition affecting those humans whose heart is located on the right-hand side of the body. Consider TBox $\mathcal{T}_{\mathrm{SI}}$ defined as follows*

$$\mathrm{Situs\_Inversus} \sqsubseteq \mathrm{Human} \qquad \mathrm{Situs\_Inversus} \sqsubseteq \exists \mathrm{has\_heart}.\mathrm{Right}$$

$$\mathrm{Human} \sqcap \neg \mathrm{Situs\_Inversus} \sqsubseteq \exists \mathrm{has\_heart}.\mathrm{Left} \qquad \mathrm{Left} \sqcap \mathrm{Right} \sqsubseteq \bot$$

with the circumscription pattern $\mathcal{P}$ with $M = \{\mathrm{Situs\_Inversus}, \mathrm{has\_heart}\}$ and $F = \{\mathrm{Human}, \mathrm{Right}, \mathrm{Left}\}$. Since the role has_heart *is minimized, no heart can be positioned on both sides of the body. As in classic circumscription, with the* ST *semantics, we derive that*

$$\mathrm{Circ}^{\mathrm{ST}}_{\mathcal{P}}(\mathcal{T}_{\mathrm{SI}}) \models \mathrm{Human} \sqsubseteq \exists \mathrm{has\_heart}.\mathrm{Left} \quad \textit{and}$$

$$\mathrm{Circ}^{\mathrm{ST}}_{\mathcal{P}}(\mathcal{T}_{\mathrm{SI}}) \models \mathrm{Human} \sqcap \exists \mathrm{has\_heart.Right} \sqsubseteq \mathrm{Situs\_Inversus}$$

i.e. normally humans have the heart positioned on the left-hand side of the body and humans affected by the situs inversus are all and only the humans whose heart is positioned on the right-hand side of the body.

Of these variants of pointwise circumscription, the one induced by $\sim^{\mathrm{ST}}$ is the strongest.

Proposition 2. *Assume a knowledge base $\mathcal{K}$ and a circumscription pattern $\mathcal{P}$. If $\mathcal{I}$ is a model of $\mathrm{Circ}^{\mathrm{ST}}_{\mathcal{P}}(\mathcal{K})$ then $\mathcal{I}$ is a model of $\mathrm{Circ}^{\mathrm{RLs}}_{\mathcal{P}}(\mathcal{K})$ and $\mathrm{Circ}^{\mathrm{CLs}}_{\mathcal{P}}(\mathcal{K})$.*

In what follows, we focus on pointwise circumscription induced by $\sim^{\mathrm{ST}}$, and given a circumscription pattern $\mathcal{P}$, by *minimality* we mean minimality with respect to $\prec^{\mathrm{ST}}_{\mathcal{P}}$.

4. A Method for Reasoning under Pointwise Circumscription

We now provide a method for reasoning under pointwise circumscription in TBoxes in a fragment of the very expressive DL $\mathcal{ALCIO}$ with no restrictions on the circumscription patterns, and show that in this setting, the computational complexity of reasoning is in (co)NExpTime.

We consider a pointwise circumscribed fragment of $\mathcal{ALCIO}$ that we call 1-$\mathcal{ALCIO}$, where we only allow for some concepts of depth at most one. We denote with $d(C)$ the *(modal) depth* of an $\mathcal{ALCIO}$ concept C (defined as the maximum number of nested quantifiers). In 1-$\mathcal{ALCIO}$ only the following axiom shapes are allowed:

$$D_1 \sqsubseteq D_2 \qquad D_1 \sqsubseteq \exists r.D_2 \qquad D_1 \sqsubseteq \forall r.D_2$$

with $d(D_1) = d(D_2) = 0$. For simplicity, we consider 1-$\mathcal{ALCIO}$ TBoxes in *normal form*, containing only axioms of the shapes:

$$D_1 \sqsubseteq D_2 \qquad A \sqsubseteq \exists r.B \qquad A \sqsubseteq \forall r.B$$

with $A, B \in N_C^+$. That is, existential and universal axioms involve only basic concepts. We can apply a *normalization* procedure to TBoxes in 1-$\mathcal{ALCIO}$, introducing fresh concept names for the boolean combinations in quantified axioms, e.g., $D_1 \sqsubseteq Qr.D_2$, with $Q \in \{\exists, \forall\}$, can be replaced by $A_1 \sqsubseteq Qr.A_2$, $A_1 \equiv D_1$, $A_2 \equiv D_2$, where A_1, A_2 are fresh concept names.

Proposition 3. *Assume a TBox $\mathcal{T}$ in 1-$\mathcal{ALCIO}$ and a circumscription pattern $\mathcal{P}$, let $\mathcal{T}'$ be the normalization of $\mathcal{T}$ and let $\mathcal{P}' = (M, V \cup N, F)$ be the circumscription pattern extended with the set N of fresh concept names introduced by the normalization. A concept name C_0 is satisfiable w.r.t. $Circ^{\mathrm{ST}}_{\mathcal{P}}(\mathcal{T})$ if and only if C_0 is satisfiable w.r.t. $Circ^{\mathrm{ST}}_{\mathcal{P}'}(\mathcal{T}')$.*

We note that, in general, this form of normalization does not preserve minimality even if the TBox is 'almost' in normal from. This is shown in the following example.

Example 2. *Consider the TBox $\mathcal{T} = \{A \sqsubseteq \exists r.B \sqcup C\}$ with the circumscription pattern $\mathcal{P} = (M, V, F)$ with $M = \{B\}$ and $F = \{A, C, r\}$. Consider $\mathcal{T}' = \{A \sqsubseteq E \sqcup C, E \equiv \exists r.B\}$ obtaianed renaming the complex concepts in $\mathcal{T}$. Let $\mathcal{P}' = (M, V \cup \{E\}, F)$, the interpretation $\mathcal{I}$ with $\Delta^{\mathcal{I}} = \{e_1, e_2\}$, $ut^{\mathcal{I}}(e_1) = \{B, E, C\}$, $ut^{\mathcal{I}}(e_2) = \{B\}$ and $bt^{\mathcal{I}}(e_1, e_2) = \{r\}$ is a model of $\mathrm{Circ}^{\mathrm{ST}}_{\mathcal{P}'}(\mathcal{T}')$. The interpretation $\mathcal{I}' = \mathcal{I}|_{sig(\mathcal{T})}$ obtained restricting $\mathcal{I}$ to the signature of $\mathcal{T}$ is not a model of $\mathrm{Circ}^{\mathrm{ST}}_{\mathcal{P}}(\mathcal{T})$. The same happens if E is minimized or fixed.*

In the rest of this section, we assume a fixed 1-$\mathcal{ALCIO}$ TBox $\mathcal{T}$ in normal form. Note that we do not consider ABoxes since in 1-$\mathcal{ALCIO}$ they can be incorporated into the TBox, adding (i) for each $A(a) \in \mathcal{A}$, the axiom $\{a\} \sqsubseteq A$, and (ii) for each $r(a, b) \in \mathcal{A}$, the axiom $\{a\} \sqsubseteq \exists r.\{b\}$. Our algorithm uses a *mosaic technique* [13] to show that the existence of a model can be established by finding a suitable set of small model fragments, which we call *star types*. Importantly, minimality can be guaranteed *locally* at the star types.

Given a model $\mathcal{I}$ of $\mathcal{T}$ and an element $d \in \Delta^{\mathcal{I}}$, we extract a structure containing the description of the unary type of the element and its direct neighbors in the structure, i.e. all those elements connected via some role r to d.

Definition 7. *Let x be a symbol, called the* center placeholder. *Let N_Y be a countably infinite set of symbols s.t. $x \notin N_Y$, where each $y \in N_Y$ is called a* spike placeholder. *A* star type *τ is an interpretation such that:*

(i) $\Delta^\tau = S(\tau) \cup \{x\}$, where $S(\tau) \subseteq N_Y$ (called the set of spike placeholders in τ),

(ii) if $y \in \Delta^\tau \cap N_Y$, then $(x, y) \in r^\tau$ for some role r,

(iii) for all roles r, $(c, d) \in r^\tau$ iff either $c = x$ or $d = x$.

We require the local satisfiability of TBox axioms via the notion of *suitable star type*; note that satisfaction of the existential axioms is required only at the center.

Definition 8. *A type τ is suitable for $\mathcal{T}$ if the following conditions are satisfied:*

(a) for all $D_1 \sqsubseteq D_2 \in \mathcal{T}$ and for any $d \in \Delta^\tau$, if $d \in D_1^\tau$ then $d \in D_2^\tau$,

(b) for all $A \sqsubseteq \exists r.B \in \mathcal{T}$, if $x \in A^\tau$, there exists $y \in S(\tau)$ such that $(x, y) \in r^\tau$ and $y \in B^\tau$,

(c) for all $A \sqsubseteq \forall r.B \in \mathcal{T}$, if $d \in A^\tau$, then for any $d' \in \Delta^\tau$ s.t. $(d, d') \in r^\tau$, $d' \in B^\tau$.

To ensure that star types are locally minimal, we need some additional book-keeping. Consider the set $Ex = \{\exists r.C \mid r \in N_R^+ \text{ and } C \in N_C^+\}$. We now extend the notion of star type with two additional components ℓ_c and ℓ_i, labeling the spikes with elements in Ex.

Definition 9. *An* adorned star type *is a structure (τ, ℓ_c, ℓ_i) such that τ is a suitable star type for $\mathcal{T}$ and $\ell_c, \ell_i \colon S(\tau) \to 2^{Ex}$ are defined as follows:*

(i) $\exists r.C \in \ell_c(y)$ if and only if

(a) there exists $A \sqsubseteq \exists r.C \in \mathcal{T}$ such that $x \in A^\tau$, $(x, y) \in r^\tau$, and $y \in B^\tau$, and

(b) there exists no $y' \in \Delta^\tau$ s.t. $y' \neq y$ and (1) $(x, y') \in r^\tau$, and (2) $y' \in B^\tau$;

(ii) $\exists r.B \in \ell_i(y)$ only if there exists $A \sqsubseteq \exists r.B \in \mathcal{T}$ such that $y \in A^\tau$, $(y, x) \in r^\tau$, and $x \in B^\tau$.

Definition 10. *Assume a circumscription pattern $\mathcal{P}$. A suitable adorned star type (τ, ℓ_c, ℓ_i) for $\mathcal{T}$ is* minimal *if there is no interpretation $\mathcal{J}$ such that (1) $\mathcal{J} \prec^{\mathrm{ST}}_{\mathcal{P},x} \tau$, (2) $\mathcal{J}$ satisfies conditions (a)-(c) in Definition 8, and (3) for each $y \in S(\tau)$ such that $\exists r.C \in \ell_i(y)$ then $(y, x) \in r^{\mathcal{J}}$ and $x \in C^{\mathcal{J}}$.*

We define *abstract types* as compact descriptions of *star types*. While the latter may have an arbitrary number of spikes, the former store only a bounded, relevant neighborhood. First, we define the set of all possible unary types for $\mathcal{T}$.

Definition 11. *$T \subseteq N_C^+$ is a type for $\mathcal{T}$ if $\top \in T$ and $\bot \notin \mathcal{T}$ and for all $a, b \in N_I(\mathcal{T})$, if $\{a\}, \{b\} \in T$, then $a = b$. We denote with $\mathrm{Types}(\mathcal{T})$ the set of types for $\mathcal{T}$.*

Now we define abstract types. Our approach is similar to the one in [15, 14], but we add more information to the description of the neighbors of an element, matching the meaning of the labeling functions ℓ_c and ℓ_i in the star types. To this end, we define the set $\mathfrak{L} = \{(\exists r.B, l) |$ with $l \in \{u, f, s\},\ r \in N_R^+$ and $B \in N_C^+\}$ where the letters u, f, s stand for *unique*, *first* and *second*, the meaning whereof we clarify below the following definition. Given a concept C such that $d(C) = 0$ and $T \in \mathrm{Types}(\mathcal{T})$, for some TBox $\mathcal{T}$, we write $T \models_0 C$ if intuitively seeing C as a formula and T as an interpretation, C is satisfied in T.

Definition 12. *Let $\mathcal{T}$ be a normalized TBox in $\mathcal{ALCIO}$. An* abstract type *is a pair (T, ρ) such that $T \in \mathrm{Types}(\mathcal{T})$ and ρ is a set of tuples (R, T', L_c, L_i), we call* abstract spikes, *with $\emptyset \subset R \subseteq N_R^+(\mathcal{T})$, $T' \subseteq N_C^+(\mathcal{T})$ and $L_c, L_i \subseteq \mathfrak{L}$, satisfying the following conditions:*

1) $|\rho| \leq 4\,\|\mathcal{T}\|$;

2) if $D_1 \sqsubseteq D_2 \in \mathcal{T}$ and $T \models_0 D_1$, then $T \models_0 D_2$;

3) for all $A \sqsubseteq \exists r.B \in \mathcal{T}$ s.t. $A \in T$, there is $(R, T', L_c, L_i) \in \rho$ s.t. $r \in R$ and $B \in T'$;

4) for all $(R, T', L_c, L_i) \in \rho$ the following hold:

(i) if $A \sqsubseteq \forall r.B \in \mathcal{T}$, $A \in T$ and $r \in R$, then $B \in T'$,

(ii) if $A \sqsubseteq \forall r.B \in \mathcal{T}$, $A \in T'$ and $r^- \in R$, then $B \in T$;

5) $(\exists r.B, l) \in L_c$ only if there exists $A \sqsubseteq \exists r.B \in \mathcal{T}$ such that $A \in T$, $r \in R$ and $B \in T'$;

6) $(\exists r.B, l) \in L_i$ only if there exists $A \sqsubseteq \exists r.B \in \mathcal{T}$ such that $A \in T'$, $r^- \in R$ and $B \in T$.

Given the set $Ex(T, \rho) = \{(r, B) |$ there exists $A \sqsubseteq \exists r.B \in \mathcal{T}$ and $A \in T\}$, we define a function $wit_{(T,\rho)} \colon Ex(T, \rho) \to 2^\rho$ such that $wit_{(T,\rho)}((r, B)) = \{(R, T', L_c, L_i) \in \rho |\ r \in R \wedge B \in T'\}$ and we require that the two sets L_c and L_i satisfy the following conditions:

(i) for all $(r, B) \in Ex(T, \rho)$, $(\exists r.B, u) \in L_c$ only if $wit_{(T,\rho)}((r, B)) = \{(R, T', L_c, L_i)\}$;

(ii) for all $(r, B) \in Ex(T, \rho)$ such that $|wit_{(T,\rho)}((r, B))| > 1$, there exists exactly one spike y_1 such that $(\exists r.B, f) \in L_c$ and exactly one spike y_2 such that $(\exists r.B, s) \in L_c$.

The bound on the number of spikes ensures that the number of abstract types is exponential in the size of the TBox. Conditions *(i)* and *(ii)* ensure the correct meaning of the labeling sets L_c and L_i and ensure the consistency between the intended information of the letter l and the actual content of ρ: whenever a spike has a label $(\exists r.B, s)$ (resp. f), for some basic role r and some basic concept B, we explicitly require that there exists another spike with the label $(\exists r.B, f)$ (resp. s). In a nutshell, any *second* 'witness' must have a *first* 'witness', and vice versa.

There is a clear parallelism between abstract types and star types. We rely on star types as concrete realizations of abstract types, associating to each abstract type (T, ρ) a specific star type instantiating all and only the information enclosed in it.

Definition 13. *Given an abstract type (T, ρ), we call* canonic instance *of (T, ρ) the adorned star type $(\tau_{(T,\rho)}, \ell_c, \ell_i)$ such that $ut(x) = T$ and*

(i) *for each abstract spike* $s = (R, T', L_c, L_i) \in \rho$, *there exists a unique spike* y_s *such that (1)* $\ell_c(y_s) = \{\exists r.B | \ (\exists r.B, u) \in L_c\}$ *and* $\ell_i(y_s) = \{\exists r.B | \ (\exists r.B, u) \in L_i\}$, *(2)* $bt(x, y_s) = R$ *and (3)* $ut(y_s) = T'$,

(ii) *there are no further spikes, that is* $|S(\tau_{(T,\rho)})| = |\rho|$.

Definition 14. *Assume a TBox* $\mathcal{T}$ *and a concept* C_0 *in* $\mathcal{ALCIO}$ *such that* $d(C_0) \leq 1$. *For any abstract type* (T, ρ), *we write* $(T, \rho) \models C_0$ *if and only if* $x \in C_0^{\mathcal{T}(T,\rho)}$.

Definition 15. *Given a circumscription pattern* $\mathcal{P}$ *and an* $\mathcal{ALCIO}$ *TBox* $\mathcal{T}$, *an abstract star type is minimal if and only if its canonical instance is minimal.*

We denote $\mathrm{ST}(\mathcal{T}, \mathcal{P})$ the set of all minimal abstract types for a TBox $\mathcal{T}$ w.r.t a circumscription pattern $\mathcal{P}$. Intuitively, we want to describe minimal models by putting together minimal abstract start types. Before characterizing when a set of minimal star types represents a minimal model, we need a compatibility condition that ensures that minimality of a type is preserved when attaching it to another one via a spike.

Definition 16. *Assume an* $\mathcal{ALCIO}$ *TBox* $\mathcal{T}$ *and a circumscription pattern* $\mathcal{P} = (M, F, V)$. *Let* $(T, \rho) \in \mathrm{ST}(\mathcal{T}, \mathcal{P})$ *and* $s = (R, T', L_c, \emptyset) \in \rho$. *Given* $(T', \rho') \in \mathrm{ST}(\mathcal{T}, \mathcal{P})$, ρ' *is compatible with* s *if and only if for all* $r \in R^-$ *and* $B \in T$ *and for all* $(R', T'', L'_c, L'_i) \in \rho'$, $(\exists r.B, u) \notin L'_c$.

Let $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ such that for all $n \in \mathbb{N}$, $n + \infty = \infty$, $n \cdot \infty = \infty$, $\infty + \infty = \infty$ and $\infty \cdot \infty = \infty$. Following [14, 15], the system of inequalities in Theorem 1 is called *enriched system of inequalities.* Inequalities of the shapes *(3)-(4)* form a so-called *positive enriched system.*

Theorem 1. *Assume a TBox* $\mathcal{T}$ *in normal form and a concept a* C_0 *with* $d(C_0) \leq 1$. *Given circumscription pattern* $\mathcal{P} = (M, V, F)$, C_0 *is satisfiable w.r.t.* $\mathrm{Circ}^{\mathrm{ST}}_{\mathcal{P}}(\mathcal{T})$ *if and only if there exists a function* $N \colon ST(\mathcal{T}, \mathcal{P}) \to \mathbb{N}^*$ *such that the following conditions are satisfied:*

(1) For all $a \in sig(\mathcal{T})$, *the following inequality holds:*

$$\sum_{(T,\rho) \in ST(\mathcal{T},\mathcal{P}) \wedge \{a\} \in T} N((T, \rho)) = 1$$

(2) The following inequality holds: $\displaystyle\sum_{(T,\rho) \in ST(\mathcal{T},\mathcal{P}) \wedge (T,\rho) \models C_0} N((T, \rho)) \geq 1$

(3) For any pair of types $T, T' \subseteq N_C^+(\mathcal{T})$ *and for any set of roles* $R \subseteq N_R^+$ *and any* $L_c, L_i \subseteq \mathfrak{L}$:

$$\sum_{\substack{(T,\rho) \in \mathrm{ST}(\mathcal{T},\mathcal{P}) \\ (R,T',L_c,L_i) \in \rho \wedge L_i \neq \emptyset}} N(T, \rho) \leq \sum_{\substack{(T',\rho') \in \mathrm{ST}(\mathcal{T},\mathcal{P}) \wedge \\ (R^-,T,L_i,L_c) \in \rho}} N(T', \rho')$$

(4) For any pair of types $T, T' \subseteq N_C^+(\mathcal{T})$ *and for any set of roles* $R \subseteq N_R^+$ *and any* $L_c \subseteq \mathfrak{L}$:

$$\sum_{\substack{(T,\rho) \in \mathrm{ST}(\mathcal{T},\mathcal{P}) \\ s=(R,T',L_c,\emptyset) \in \rho}} N(T, \rho) > 0 \ \textit{implies} \sum_{\substack{(T',\rho') \in \mathrm{ST}(\mathcal{T},\mathcal{P}) \\ \wedge \rho' \textit{ compatible with } s}} N(T', \rho') > 0$$

(a) Initial situation: the minimization of h and v alone does not enforce a grid.

(b) Requiring the satisfiability of C enforces the existence of a unique good node $(1, 1)$.

Figure 1: Enforcement of a 2×2 grid with pointwise circumscription. We bypass functionality, ensuring the existence of a unique horizontal (h-)predecessor and vertical (v-)predecessor via minimization.

The function N associates to each abstract type the number of instances needed for building a model. Conditions *(1)-(2)* ensure the correct encoding of nominals and the satisfiability of C_0. Conditions *(3)-(4)* guarantee that for each realized abstract type there exist a 'successor' for each spike. Intuitively in *(3)*, for each spike with $L_i \neq \emptyset$ in a realized abstract type, we require the realization of an abstract type *overlapping* at that spike. The inequality *(3)* ensures the existence of enough overlapping abstract types. Condition *(4)* specifically deals with preserving minimality while constructing the model.

Theorem 2 ([14]). *Deciding the existence of a solution for an enriched system of inequalities H is feasible in non-deterministic polynomial time in the size of H. If H contains only positive inequalities, the problem is solvable in polynomial time in the size of H.*

The corollary below directly follows from Theorem 1 and Theorem 2.

Corollary 1. *In $\mathcal{ALCIO}$, satisfiability of concepts of depth at most 1 w.r.t. pointwise circumscribed KBs in normal form is in NExpTime.*

The following corollary is a consequence of Proposition 3 and Corollary 1.

Corollary 2. *In pointwise circumscribed 1-$\mathcal{ALCIO}$, satisfiability of concepts of depth at most one is in NExpTime.*

Recalling that concept satisfiability, subsumption and instance checking can be polynomially reduced one into the other, the following corollary holds.

Corollary 3. *In 1-$\mathcal{ALCIO}$, for concepts of depth at most one, subsumption and instance checking w.r.t. pointwise circumscribed TBoxes are in coNExpTime.*

Observe that the TBox $\mathcal{T}_{\mathrm{SI}}$ of Example 1 is in 1-$\mathcal{ALCIO}$. In contrast with classically circumscribed DLs, $\mathcal{T}_{\mathrm{SI}}$ falls in decidable fragment of pointwise circumscribed DLs.

5. Lower Bound

We provide a reduction from the exponential grid tiling problem to concept satisfiability w.r.t. pointwise circumscribed TBoxes in $\mathcal{ALCIO}$. Differently from the reduction proposed for $\mathcal{ALCIOF}$ [16], we bypass functionality using minimality.

Theorem 3. *In 1-$\mathcal{ALCIO}$, concept name satisfiability w.r.t. pointwise circumscribed TBoxes is* NExp*-hard.*

We briefly discuss the key ideas here. Given an instance of the exponential grid tiling problem P, we build a TBox $\mathcal{T}_P$ and a circumscription pattern $\mathcal{P}$ such that, in any model $\mathcal{I}$ of $\mathrm{Circ}_{\mathcal{P}}(\mathcal{T}_P)$, each domain element d is uniquely associated to a position (x_d, y_d) in the $2^n \times 2^n$ grid. The origin $(0, 0)$ corresponds to a nominal o. With special care for boundary nodes of the grid, each element has exactly one h-successor (horizontal) and one v-successor (vertical). To enforce uniqueness, we require that h and v are minimized. Via *counting axioms* (see [16]), we encode a correct updating of the coordinates x and y.

We call *good node* any node intuitively corresponding to a *correct* node in the grid, i.e. with a unique incoming h and a unique incoming v. We denote good nodes with the minimized concept GN. The notion of good node is adapted to deal with nodes of coordinates $(0, y)$ and $(x, 0)$. For $n = 1$, we obtain the structure in Figure 1a, where $(x, y)^{GN}$ denotes that the node of coordinates (x, y) is a good node. We check the satisfiability of a *minimized* concept C. Introducing appropriate axioms, given a minimal model $\mathcal{I}$, we require that $d \in C^{\mathcal{I}}$ if and only if $(x_d, y_d) = (2^n - 1, 2^n - 1)$ and $d \in GN^{\mathcal{I}}$. Since GN is minimized and $x_d, y_d \neq 0$, we ensure that there must exist an h-predecessor and a v-predecessor for d that are good nodes too. Figure 1b represents a model of $\mathrm{Circ}_{\mathcal{P}}(\mathcal{T}_P)$ where C is satisfiable. The minimization of GN and the roles h, v with the condition on the nominal o imply that there cannot be any other extra good node as h-predecessor for d, enforcing the grid.

6. Discussion

In this paper, we have presented our preliminary work on pointwise circumscription in DLs, as an alternative to classic circumscription: we aimed at lowering the computational complexity, while still maintaining sufficient expressiveness.

Many directions remain open for future work. The results presented here are for TBoxes of restricted shape, and hence the first direction is to study a larger class of TBoxes. We believe that our technique and the upper bounds also apply to TBoxes in a relaxed normal form, where arbitrary inclusions between concepts with modal depth 1 are allowed. An interesting next step is to consider satisfiability and subsumption of general concepts (with unbounded modal depth) under TBoxes in both of these normal forms. Naturally, we plan to investigate the complexity of reasoning with general TBoxes (with unbounded modal depth), which seems to require significant new insights. Another interesting direction is to extend our formalism with *priorities*, which are important for knowledge representation and are supported in classic circumscription.

Pointwise circumscription in lightweight DLs and DLs beyond $\mathcal{ALCIO}$ also remains to be explored. In [11], classic circumscription is used to provide the semantics to defeasible inclusions in DLs. We plan to explore whether pointwise circumscription can be applied in this context as well. Another direction is to study the *data complexity* of entailment of assertions from pointwise circumscribed KBs.

Acknowledgments

This work was partially supported by the Vienna Business Agency and the Austrian Science Fund (FWF) projects P30360 and P30873.

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