The Satisfiability Problem for Boolean Set Theory with a Rational Choice Correspondence⋆

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Abstract

We investigate the satisfiability problem for an elementary fragment of set theory denoted BSTC, which includes a choice operator along with Boolean set operators, singleton, membership, equality, inclusion, and propositional connectives. The intended interpretation of the choice operator is as a rationalizable choice within the framework of *Rational choice theory* (a model of social and economic behavior), namely a contractive map defined on nonempty subsets of a universe U that can be derived from a binary relation on U by selecting the maximal elements of each set. We establish a *small model property* for BSTC under the interpretation of the choice operator as a rationalizable choice. This property enables us to develop an algorithmic solution to the satisfiability problem for BSTC, which allows us to prove that the latter falls under the complexity classes NP-hard and NEXPTIME. By investigating the implications and characteristics of rational decision-making within this fragment of set theory, our research contributes to a better understanding of the interplay between rational choice theory and set theory.

1. Introduction

We continue our investigation of the satisfiability problem connected to fragments of set theory involving a choice operator within the framework of *Rational choice theory*, a model of social and economic behavior. A choice on a set U of alternatives is a correspondence $B \mapsto c(B)$ associating to "feasible menus" $B \subseteq U$ nonempty "choice sets" $c(B) \subseteq B$. A choice can be either *total* – i.e., defined for all nonempty subsets of the ground set U of alternatives – or *partial* – i.e., defined only for suitable subsets of U.

According to the *Theory of Revealed Preferences* initially explored by the economist Paul Samuelson [1], preferences of consumers can be inferred from their purchasing habits. In a nutshell, choice on menus is observed, and preferences – as summarized by a binary relation on U – are revealed. Technically, given a choice c on U, the "relation of revealed preference" $\precsim$ is

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defined by $x \precsim y$ if there is a menu $B \subseteq U$ such that $x \in c(B)$.[1]

Classically, a choice is considered *rationalizable* if the observed behavior can be univocally recovered by maximizing this relation of revealed preference: that is, $c(B) = \max(B, \precsim)$ for any feasible menu B.[2] This encoding of the concept of rationality yields a notable simplification of the observed behavior: in fact, rationalizability is equivalent to the possibility to represent a map from $\mathrm{pow}(U)$ into $\mathrm{pow}(U)$, which requires $\mathcal{O}(|U| \cdot 2^{|U|})$ space, by a subset of $U \times U$, which requires $\mathcal{O}(|U|^2)$ space (here $\mathrm{pow}(U)$ denotes the powerset of U).

Since Samuelson's groundbreaking paper, a significant amount of attention has been devoted to exploring various concepts of rationality within the framework of choice theory. The literature includes several influential contributions such as the classical works by authors like Houthakker [3], Arrow [4], Richter [5], Hansson [6], and Sen [7]. To delve deeper into the connections between choice, preference, and utility theories, one can refer to the book [8] and the quite recent paper [9]. Traditionally, the rationality of observed choice behavior is associated with the fulfillment of suitable *axioms of choice consistency*. These axioms establish rules for selecting items within menus and are expressed using second-order logic formulas universally quantified over menus. Noteworthy axioms introduced in the specialized literature include:

- *standard contraction consistency* (α), introduced by Chernoff [10];
- *standard expansion consistency* (γ) and *binary expansion consistency* (β), both proposed by Nobel Laureate Amartya Sen [7];
- *the weak axiom of revealed preference* (WARP), originally put forth by Samuelson [1].

It is well-known that, under appropriate assumptions on the domain, a choice can be considered rationalizable if and only if it satisfies the two standard consistency axioms (α) and (γ) (refer to Sen [7] for more details). Furthermore, the (complete) rationalizing preference exhibits transitivity if and only if axioms (α) and (β) are satisfied, which is equivalent to the WARP property. In such cases, the choice is referred to as *transitively rationalizable*. Section 2 provides the necessary background to choice theory.

In this paper, we focus on the satisfiability problem for unquantified formulae in an elementary fragment of set theory denoted as BSTC. This fragment includes essential elements such as the choice function symbol c, Boolean set operators like union $\cup$, intersection $\cap$, set difference $\setminus$, the singleton operator $\{\cdot\}$, predicates for membership $\in$, equality $=$, and inclusion $\subseteq$, as well as propositional connectives such as conjunction $\wedge$, disjunction $\vee$, negation $\neg$, implication $\implies$, etc.

In a previous work [11], we examined cases where the interpretation of the choice operator c was subject to combinations of consistency axioms, namely (α) and (β), whose conjunction is equivalent to the WARP property. In this paper, we focus on a different scenario where the choice operator c is interpreted as a rational choice. Specifically, we establish our decidability result by demonstrating that BSTC under rationalizability exhibits a *small model property*. The current approach contrasts with that used in [11], which relied on reduction techniques and various lifting results.

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[1] For our purposes, it will suffice to consider the asymmetric part $\prec$ of $\precsim$, defined by $x \prec y$ if $x \precsim y$ and $\neg(y \precsim x)$.

[2] In this case, "levels of rationality" are associated to the properties satisfied by the relation of reveled preference, e.g., transitivity: see [2].

By considering the interpretation of c as a rational choice, we aim to explore the implications and characteristics of rational decision-making within the framework of BSTC.

By depriving the BSTC-language of the choice function symbol c, we obtain the fragment 2LSS (here denoted BSTC^-) whose decidability was known since the birth of *Computable Set Theory* in the late 70's. The reader can find extensive information on Computable Set Theory in the monographs [12, 13, 14, 15].

The paper is organized as follows. Section 2 is devoted to the basis of choice theory, while the syntax and semantics of the BSTC-language are presented in Section 3. Then, in Section 4, we prove that the satisfiability problem for BSTC under rationalizability is decidable and belongs to the complexity classes NP-hard and NEXPTIME. Finally, in Section 5, we draw our conclusions and hint at future developments.

2. Preliminaries on choice theory

Hereafter, we fix a nonempty set U (the "universe"). Let $\mathrm{pow}(U)$ be the family of all subsets of U, and $\mathrm{pow}^+(U)$ the subfamily $\mathrm{pow}(U) \setminus \{\emptyset\}$. The next definition collects some basic notions in choice theory.

Definition 1. Let $\Omega \subseteq \mathrm{pow}^+(U)$ be nonempty. A *choice correspondence* on U is a contractive map $c\colon \Omega \to \mathrm{pow}^+(U)$ that is never empty-valued, namely such that $\emptyset \neq c(B) \subseteq B$, for every $B \in \Omega$.

In this paper, we denote a choice correspondence on U by $c\colon \Omega \rightrightarrows U$, and simply refer to it as a *choice*. The set Ω is the *choice domain* of c, sets in Ω are *(feasible) menus*, and elements of a menu are *items*. Further, we say that $c\colon \Omega \rightrightarrows U$ is *total* if $\Omega = \mathrm{pow}^+(U)$, and *partial* otherwise.

Given a choice $c\colon \Omega \rightrightarrows U$, the *choice set* $c(B)$ of a menu B collects the elements of B that are deemed selectable by an economic agent. Thus, in case $c(B)$ contains more than one element, the selection of a single element of B is deferred to a later time, usually with a different procedure (according to additional information or "subjective randomization", e.g., flipping a coin).

The next definition recalls the classical notion of rationalizable choices.

Definition 2. A choice $c\colon \Omega \rightrightarrows U$ is *rationalizable* if there exists a binary relation $\precsim$ over U such that, for all menus $B \in \Omega$, $c(B)$ is the set

$$\max(B, \precsim) \coloneqq \big\{a \in B : (\forall b \in B)(a \precsim b \Longrightarrow b \precsim a)\big\}$$

of the strictly $\precsim$-maximal members of B.

Rationalizable choices can also be characterized in terms of *asymmetric relations*, namely relations that contain no pair of elements that are mutually related to each other.

Lemma 1. *A choice is rationalizable if and only if it is induced by an asymmetric relation. In fact, a rationalizable total choice is induced by exactly one asymmetric relation.*

Proof. First assume that c is a choice rationalized by the binary relation $\precsim$ over U. Let $\prec$ be the asymmetric part of $\precsim$, thus:

$$x \prec y \iff x \precsim y \wedge y \not\precsim x.$$

Notice that $\max(A, \precsim) = \max(A, \prec)$ for all subset $A \subseteq U$, thus $\prec$ rationalizes c. On the counter-side plainly if c is induced by an asymmetric relation then c is rationalizable.

By contradiction, let us assume that a total choice $c\colon \mathrm{pow}^{+}(U) \rightrightarrows U$ is rationalized by two distinct asymmetric relations $\prec$ and $\prec'$ over U. Thus, there exist $x, y \in U$ such that

$$x \prec y \quad \iff \quad x \not\prec' y.$$

Let us assume, without loss of generality, that $x \prec y$ and $x \not\prec' y$. Then, we have

$$x \notin \max(\{x, y\}, \prec) \quad \text{and} \quad x \in \max(\{x, y\}, \prec'),$$

which contradicts that $\prec$ and $\prec'$ rationalize the same choice. □

For the rest of the paper, we will rely on the characterization of rationalizable choices in terms of asymmetric relations.

The rationalizability of choice is traditionally connected to the satisfaction of suitable *axioms of choice consistency*. These axioms codify rules of coherent behavior of an economic agent. Among the several axioms that are considered in the literature, the following are relevant to our analysis (a universal quantification on all the involved menus is implicit):

axiom (α) [*standard contraction*]: $A \subseteq B \implies A \cap c(B) \subseteq c(A)$
axiom (γ) [*standard expansion*]: $c(A) \cap c(B) \subseteq c(A \cup B)$
axiom (β) [*symmetric expansion*]: $\big(A \subseteq B \ \wedge\ c(A) \cap c(B) \neq \emptyset\big) \implies c(A) \subseteq c(B)$

Axiom (α) was studied by Chernoff [10], whereas axioms (γ) and (β) are due to Sen [7].

Upon reformulating these properties in terms of items, their semantics becomes clear. Chernoff's axiom (α) states that any item selected from a menu B is still selected from any submenu $A \subseteq B$ containing it. Sen's axiom (γ) says that any item selected from two menus A and B is also selected from the menu $A \cup B$ (if feasible). The expansion axiom (β) can be equivalently written as follows: if $A \subseteq B$, $x, y \in c(A)$ and $y \in c(B)$, then $x \in c(B)$. In this form, (β) says that if two items are selected from a menu A, then they are simultaneously either selected or rejected in any larger menu B.

Razionalizability is hereditary, as stated in the following lemma.

Lemma 2. *Let $\Omega \subseteq \mathrm{pow}^{+}(U)$. If a choice $c\colon \Omega \rightrightarrows U$ is rationalizable, then so is any of its restrictions $c|_{\Omega'}$, for $\emptyset \neq \Omega' \subseteq \Omega$.*

Proof. Trivially any relation $\prec$ that rationalizes c is such that $\max(A, \prec) = c(A) = c|_{\Omega'}(A)$, for all $A \in \Omega'$, thus $\prec$ rationalizes $c|_{\Omega'}$. □

Another useful preliminary result is that a rationalizable partial choice correspondence can be lifted to a rationalizable total choice if and only if each asymmetric relation $\prec$ that rationalizes it is devoid of infinite ascending $\prec$-sequences , namely there is no infinite sequence $a_0, a_1, a_2, \ldots$ of items such that

$$a_0 \prec a_1 \prec a_2 \prec \cdots .$$

This is proved in the following lemma.

Lemma 3 (Lifting rationalizability)**.** *A (rationalizable) partial choice c can be extended to a rationalizable total choice if and only if c is rationalizable by an asymmetric relation devoid of infinite ascending sequences.*

Proof. Let $c\colon \Omega \rightrightarrows U$ be a partial choice correspondence, with $\Omega \subseteq \mathrm{pow}^{+}(U)$. Let us assume that c can be extended to a rationalizable total choice $c'\colon \mathrm{pow}^{+}(U) \rightrightarrows U$, and let $\prec'$ be the asymmetric relation over U that rationalizes c', namely such that $c'(A) = \max(A, \prec')$ for every $A \in \mathrm{pow}^{+}(U)$. We claim that $\prec'$ is devoid of infinite ascending sequences. Indeed, if this were not the case, letting

$$a_0 \prec' a_1 \prec' a_2 \prec' \cdots$$

be an infinite ascending chain in U, the related menu $A \coloneqq \{a_0, a_1, a_2, \ldots\}$ would have no $\prec'$-maximal item and therefore $c'(A) = \max(A, \prec') = \emptyset$, which is impossible, as c' is a total choice. By Lemma 2, the choice correspondence c is rationalized by $\prec'$ as well.

Conversely, let us assume that our partial choice correspondence c can be rationalized by an asymmetric relation $\prec$ over U that is devoid of infinite ascending sequance. We claim that for every $A \in \mathrm{pow}^{+}(U)$ (not necessarily in Ω), the menu A must have some $\prec$-maximal item, namely $\max(A, \prec) \neq \emptyset$. Indeed, if not, then for every $a \in A$, there would be another item $a' \in A$ such that $a \prec a'$. Consequently, any finite $\prec$-sequence in A of length n could be extended to a $\prec$-sequence of length $n + 1$, for every $n \geq 1$. Since A is nonempty, this would imply the existence of infinite ascending $\prec$-sequences in $A \subseteq U$, leading to a contradiction. □

3. The Satisfiability Problem in the Presence of a Choice Pperator

We specify the syntax and semantics of the Boolean set-theoretic language extended with a choice correspondence, denoted by BSTC, of which we will study the satisfiability problem, under rationalizability.

3.1. Syntax of BSTC

Following [11], the language BSTC involves

- two disjoint denumerable collections $\mathcal{V}_0$ and $\mathcal{V}_1$ of *individual variables* (denoted by small final letters, such as x) and *set variables* (denoted by capital final letters, such as X), respectively;
- the constant $\varnothing$ (empty set);
- operation symbols: $\cdot \cup \cdot$, $\cdot \cap \cdot$, $\cdot \setminus \cdot$, $\{\cdot\}$, $\mathsf{c}(\cdot)$ (choice map);

- predicate symbols: $\cdot = \cdot$, $\cdot \subseteq \cdot$, $\cdot \in \cdot$.

Set terms of BSTC are recursively defined as follows:

- set variables and the constant $\varnothing$ are set terms;
- $\{x\}$ is a set term, for every individual variable x;
- if T, T_1, T_2 are set terms, then $T_1 \cup T_2$, $T_1 \cap T_2$, $T_1 \setminus T_2$, $\mathsf{c}(T)$ are set terms.

The *atomic formulae* (or *atoms*) of BSTC have one of the following forms $x = y$, $x \in T$, $T_1 = T_2$, $T_1 \subseteq T_2$, where T_1, T_2 are set terms.

Finally, BSTC-*formulae* are propositional combinations of BSTC-atoms by means of the usual logical connectives $\wedge$, $\vee$, $\neg$, $\implies$, $\iff$.

We regard $\{x_1, \ldots, x_k\}$ as a shorthand for the set term $\{x_1\} \cup \ldots \cup \{x_k\}$.

Choice terms are BSTC-terms of type $\mathsf{c}(T)$, whereas *choice-free terms* are BSTC-terms which do not involve the choice map c (at any level of nesting). We refer to BSTC-formulae containing only choice-free terms as BSTC^--formulae. With slight variations in syntax, BSTC^--formulae are essentially equivalent to 2LSS-formulae, which is known to have a decidable satisfiability problem (refer, for example, to [13, Exercise 10.5]).

We define the *size* (or *length*) $|\varphi|$ of a BSTC-formula φ as the number of the symbol occurrences (individual and set variables, set operators, propositional connectives) used to represent φ.

3.2. Semantics of BSTC under rationalizability

A *set assignment* is a pair $\boldsymbol{\mathcal{M}} = (U, M)$, where U is any nonempty collection of objects, called the *domain* or *universe* of $\boldsymbol{\mathcal{M}}$, and M is an interpretation over the variables and the choice map of BSTC such that

- $x^M \in U$, for each individual variable $x \in \mathcal{V}_0$;
- $\varnothing^M \coloneqq \emptyset$ and $X^M \subseteq U$, for each set variable $X \in \mathcal{V}_1$;
- c^M is a *rationalizable* total choice over U.

Then, we extend M over the terms by putting recursively:

- $(T_1 \otimes T_2)^M \coloneqq T_1^M \otimes T_2^M$, where $\otimes \in \{\cup, \cap, \setminus\}$;
- $\{x\}^M \coloneqq \{x^M\}$;
- $(\mathsf{c}(T))^M \coloneqq \mathsf{c}^M(T^M)$.

The *size* of a set assignment is the cardinality of its domain.

Satisfiability under rationalizability (Rtl-satisfiability, for short) of any BSTC-formula φ by $\boldsymbol{\mathcal{M}}$ (written $\boldsymbol{\mathcal{M}} \models_{\mathsf{Rtl}} \varphi$) is defined as follows:

$$\begin{array}{llll} \boldsymbol{\mathcal{M}} & \models_{\mathsf{Rtl}} & T_1 \star T_2 & \text{iff} \quad T_1^M \star T_2^M, \\ \boldsymbol{\mathcal{M}} & \models_{\mathsf{Rtl}} & x \in T & \text{iff} \quad x^M \in T^M, \\ \boldsymbol{\mathcal{M}} & \models_{\mathsf{Rtl}} & x_1 = x_2 & \text{iff} \quad x_1^M = x_2^M, \end{array}$$

for all BSTC-atoms $T_1 \star T_2$, $x \in T$, and $x = y$, where $\star \in \{=, \subseteq\}$. Finally, the logical connectives are interpreted according to their classical meaning.

For a BSTC-formula φ, if $\boldsymbol{\mathcal{M}} \models_{\mathsf{Rtl}} \varphi$ (i.e., $\boldsymbol{\mathcal{M}}$ *Rtl-satisfies* φ), then $\boldsymbol{\mathcal{M}}$ is an *Rtl-model for* φ. A BSTC-formula is *Rtl-satisfiable* if it has an Rtl-model. Two BSTC-formulae φ and ψ are *Rtl-equivalent* if they share exactly the same Rtl-models; they are *Rtl-equisatisfiable* if one is Rtl-satisfiable if and only if so is the other (possibly by different models).

The *Rtl-satisfiability problem* (or *Rtl-decision problem*) for BSTC asks for an effective procedure (or *decision procedure*) to establish whether any given BSTC-formula is Rtl-satisfiable or not.[3]

The satisfiability problem for BSTC has been addressed also under other semantics (see [11]): specifically, the (α)-*semantics*, the (β)-*semantics*, the WARP-*semantics*, and the *unrestricted semantics* (whose satisfiability relations are denoted by $\models_{\alpha}$, $\models_{\beta}$, $\models_{\mathsf{WARP}}$, and $\models$, respectively). These differ from the Rtl-semantics in that the interpreted choice map c^{M} is required to satisfy axiom (α) in the first case, axiom (β) in the second case, axioms (α) and (β) conjunctively (namely WARP) in the third case, and no particular consistency axiom in the latter case.

4. The Satisfiability Problem for BSTC under Rationalizability

In this section, we demonstrate that the satisfiability problem for the theory BSTC under rationalizability is decidable. We will prove this result by establishing a *small model property* for BSTC, which allows one to test the satisfiability of any BSTC-formula φ by verifying if it is satisfied by some set assignment whose 'size' is bounded by an exponential function of the size of φ. Since such bounded sets assignment can be generated effectively, their number is bounded, and it can be effectively checked whether any of them actually satisfies a given BSTC-formula φ, the decidability of the Rtl-satisfiability problem for BSTC follows.

We recall that the decidability results presented in [11] for the four previously mentioned semantics are based instead on a reduction technique. Such technique involves enriching a given BSTC-formula φ that needs to be tested for satisfiability by adding appropriate clauses, resulting in an extended BSTC-formula φ_1. Then, by systematically replacing the choice terms in φ_1 with newly introduced set variables, one obtains a (choice-free) BSTC^--formula $\overline{\varphi}_1$ that is equisatisfiable with φ. As a consequence, the decidability of the satisfiability problems for BSTC under the various four semantics follows from the known decidability of the satisfiability problem for BSTC^- (see [13, Exercise 10.5]).

Without compromising the expressivity of BSTC, we can limit ourselves to BSTC-formulae that consist solely of equality atoms $T_1 = T_2$, where T_1 and T_2 are set terms constructed using only the set difference operator '$\setminus$' and the singleton operator $\{\cdot\}$. Indeed, atoms of the form $x = y$, $x \in T$, and $T_1 \subseteq T_2$ can be just replaced by the equivalent atoms $\{x\} = \{y\}$, $\{x\} \subseteq T$, and $T_1 \cup T_2 = T_2$, respectively. Additionally, terms of the form $T_1 \cap T_2$ can be replaced by the equivalent term $T_1 \setminus (T_1 \setminus T_2)$. Finally, every term of the form $T_1 \cup T_2$ can be eliminated from a given BSTC-formula φ by replacing it by a newly introduced set variable $X_{T_1 \cup T_2}$ and by adding to φ the atoms $X_{T_1 \cup T_2} \setminus T_1 = T_2 \setminus T_1$ and $T_1 \setminus X_{T_1 \cup T_2} = \varnothing$ as conjuncts, since, as

[3] Since in this paper we are dealing we just one semantics for choice terms, we may occasionally omit the use of the shorthand Rtl when referring to expressions such as model, satisfiability, and so on.

observed in [16], it holds that

$$\models X_{T_1 \cup T_2} = T_1 \cup T_2 \iff (X_{T_1 \cup T_2} \setminus T_1 = T_2 \setminus T_1 \wedge T_1 \setminus X_{T_1 \cup T_2} = \varnothing).$$

Likewise, the set constant $\varnothing$ can be eliminated from a BSTC-formula φ by replacing each of its occurrences by a fresh set variable $X_\varnothing$ characterized by the atom $X_\varnothing = X_\varnothing \setminus X_\varnothing$.

It is important to note that the elimination of terms of type $T_1 \cup T_2$ and $\varnothing$ produces an equisatisfiable formula, rather than an equivalent formula as in the other cases considered, due to the introduction of new variables. However, this does not pose any problem for our satisfiability objectives. Note also that the size of the resulting formula is linear in the size of the initial formula.

Thus, let φ be an Rtl-satisfiable propositional combination of equality atoms of the form $T_1 = T_2$, where T_1 and T_2 are BSTC-terms, and let $\boldsymbol{\mathcal{M}} = (U, M)$ be a set model for φ under rationalizability, namely such that $\boldsymbol{\mathcal{M}} \models_{\mathsf{Rtl}} \varphi$.

Our aim is to demonstrate the possibility of 'extracting' from $\boldsymbol{\mathcal{M}}$ another set model for φ, denoted as $\overline{\boldsymbol{\mathcal{M}}}$, with a universe $\overline{U} \subseteq U$ of size $\mathcal{O}(2^{|\varphi|})$. This establishes a small model property for BSTC under rationalization, leading to the decidability of the Rtl-satisfiability problem for BSTC, as previously argued.

To this purpose, we will apply the following straightforward fact.

Lemma 4. *Given a* BSTC*-formula φ and two set assignments $\boldsymbol{\mathcal{M}}$ and $\overline{\boldsymbol{\mathcal{M}}}$ over the variables of φ that Rtl-satisfy the same equality atoms in φ, $\boldsymbol{\mathcal{M}}$ Rtl-satisfies φ if and only if $\overline{\boldsymbol{\mathcal{M}}}$ Rtl-satisfies φ.*

Thus, let $\mathsf{V}_0 \subseteq \mathcal{V}_0$ and $\mathsf{V}_1 \subseteq \mathcal{V}_1$ be the collections of the individual and set variables occurring in φ, respectively, and let $\mathcal{T}_\varphi$ be the collection of the set terms occurring in φ. Plainly, $|\mathcal{T}_\varphi| = \mathcal{O}(|\varphi|)$. We intend to show that the formula φ admits an Rtl-model over a universe of size $\mathcal{O}(2^{|\varphi|})$.

Let $\mathcal{T}_\varphi^M \coloneqq \{T^M : T \in \mathcal{T}_\varphi\}$ and let $\mathcal{R}_\varphi^M$ denote the *Euler-Venn partition* of $\mathcal{T}_\varphi^M$, namely the partition

$$\mathcal{R}_\varphi^M \coloneqq \left\{ \bigcap Q \setminus \bigcup (\mathcal{T}_\varphi^M \setminus Q) : \emptyset \neq Q \subseteq \mathcal{T}_\varphi^M \right\} \setminus \{\emptyset\}$$

of $\bigcup \mathcal{T}_\varphi^M$, where $\bigcap Q$ is the intersection of all the members of Q. Equivalently, the partition $\mathcal{R}_\varphi^M$ can be defined as the collection of the $\subseteq$-maximal nonempty subsets ρ of $\bigcup \mathcal{T}_\varphi^M$ such that, for every term $T \in \mathcal{T}_\varphi$, either $\rho \subseteq T^M$ or $\rho \cap T^M = \emptyset$.

Without loss of generality, we may assume that $U = \bigcup \mathcal{T}_\varphi^M$, since otherwise we could replace $\boldsymbol{\mathcal{M}}$ by the set assignment $\boldsymbol{\mathcal{M}}' = (U', M')$ in our analysis, where $U' = \bigcup \mathcal{T}_\varphi^M$, $x^{M'} = x^M$ for all $x \in \mathsf{V}_0$, $X^{M'} = X^M$ for all $X \in \mathsf{V}_1$, and $\mathsf{c}^{M'} = \mathsf{c}^M|_{U'}$. Indeed, by Lemma 2, $\mathsf{c}^{M'}$ is plainly a rationalizable total choice over U', and so $\boldsymbol{\mathcal{M}}' \models_{\mathsf{Rtl}} \varphi$.

A promising approach to define the universe $\overline{U} \subseteq U$, over which the sought-for set assignment $\overline{\boldsymbol{\mathcal{M}}}$ can be constructed, involves selecting an item from each block in $\mathcal{R}_\varphi^M$. It is natural then to define the interpretation $\overline{M}$ over the variables $\mathsf{V}_0 \cup \mathsf{V}_1$ occurring in φ by setting $x^{\overline{M}} \coloneqq x^M$ for each $x \in \mathsf{V}_0$ and $X^{\overline{M}} \coloneqq X^M \cap \overline{U}$ for each $X \in \mathsf{V}_1$. Notably, as $\{x\}^M \in \mathcal{R}_\varphi^M$ for every $x \in \mathsf{V}_0$, we then have $x^M \in \overline{U}$, enabling us to define $x^{\overline{M}}$ as x^M.

For the time being, we will not be specific on the interpretation by $\overline{\boldsymbol{\mathcal{M}}}$ of the choice map c. Under the hypothesis that φ is choice-free, meaning that it does not contain any choice term, we can readily prove that the set assignment $\overline{\mathcal{M}} = (\overline{U}, \overline{M})$ just defined is indeed a model for φ. In view of Lemma 4, it is enough to prove that, for every atom $T_1 = T_2$ in φ, we have

$$T_1^M = T_2^M \iff T_1^{\overline{M}} = T_2^{\overline{M}}.$$

In its turn, to prove the latter biconditional it suffices to show that $T^{\overline{M}} = T^M \cap \overline{U}$ holds for all set terms T occurring in φ, as proved in the following lemma.

Lemma 5. *If $T^{\overline{M}} = T^M \cap \overline{U}$ holds for all set terms T occurring in a* BSTC*-formula φ (not necessarily choice-free), then the biconditional*

$$T_1^M = T_2^M \iff T_1^{\overline{M}} = T_2^{\overline{M}} \tag{1}$$

holds for all *set terms T_1 and T_2 occurring in φ.*

Proof. Let T_1 and T_2 be any two terms in φ. The forward implication in (1) is straightforward, since if $T_1^M = T_2^M$ holds then

$$T_1^{\overline{M}} = T_1^M \cap \overline{U} = T_2^M \cap \overline{U} = T_2^{\overline{M}}.$$

As for the backward implication, let us assume, for the sake of contradiction, that $T_1^M \cap \overline{U} = T_2^M \cap \overline{U}$, but $T_1^M \neq T_2^M$. Since both T_1^M and T_2^M are unions of blocks in $\mathcal{R}_\varphi^M$, there must exist a block $\rho \in \mathcal{R}_\varphi^M$ such that

$$\rho \cap T_1^M \neq \emptyset \iff \rho \cap T_2^M = \emptyset.$$

For definiteness, let us assume that $\rho \cap T_1^M \neq \emptyset$ and $\rho \cap T_2^M = \emptyset$. Hence, we have $\rho \subseteq T_1^M$, and therefore:

$$\emptyset \neq \rho \cap T_1^M \cap \overline{U} = \rho \cap T_2^M \cap \overline{U} = \emptyset,$$

which is a contradiction. Thus, even the backward implication in (1) is true, and so the biconditional (1) holds, provided that $T^{\overline{M}} = T^M \cap \overline{U}$ <u>is true</u> for all set terms T occurring in φ. ☐

Thus, we are left with establishing the truth of the condition $T^{\overline{M}} = T^M \cap \overline{U}$, for all set terms T occurring in φ.

The case in which φ is choice-free can be handled quite straightforwardly by the following lemma.

Lemma 6. *If φ is* choice-free*, then the condition $T^{\overline{M}} = T^M \cap \overline{U}$ holds for all set terms T occurring in φ.*

Proof. We proceed by structural induction on T.

For the base case, when T is a set variable $X \in \mathsf{V}_1$, we can directly apply the definition of $\overline{M}$ to obtain:

$$T^{\overline{M}} = X^{\overline{M}} = X^M \cap \overline{U} = T^M \cap \overline{U}.$$

For the inductive step, we consider the following cases:

1. if T has the form $\{x\}$, where $x \in \mathsf{V}_0$, by recalling that $x^{\overline{M}} = x^M$ by definition, then we have:

$$T^{\overline{M}} = \{x\}^{\overline{M}} = \{x^{\overline{M}}\} = \{x^M\} = \{x\}^M = T^M;$$

2. if T has the form $T_1 \setminus T_2$, we have:

$$\begin{aligned} T^{\overline{M}} = (T_1 \setminus T_2)^{\overline{M}} = T_1^{\overline{M}} \setminus T_2^{\overline{M}} &= (T_1^M \cap \overline{U}) \setminus (T_2^M \cap \overline{U}) \\ &= (T_1^M \setminus T_2^M) \cap \overline{U} = (T_1 \setminus T_2)^M \cap \overline{U} = T^M \cap \overline{U}, \end{aligned}$$

where we used the inductive hypotheses $T_i^{\overline{M}} = T_i^M \cap \overline{U}$, for $i = 1, 2$, and the identity

$$(T_1^M \cap \overline{U}) \setminus (T_2^M \cap \overline{U}) = (T_1^M \setminus T_2^M) \cap \overline{U}.$$

The latter is a direct consequence of the fact that for any set s:

$$\begin{aligned} s \in (T_1^M \cap \overline{U}) \setminus (T_2^M \cap \overline{U}) &\iff s \in T_1^M \cap \overline{U} \wedge s \notin T_2^M \cap \overline{U} \\ &\iff s \in T_1^M \wedge s \in \overline{U} \wedge s \notin T_2^M \\ &\iff s \in T_1^M \setminus T_2^M \wedge s \in \overline{U} \\ &\iff s \in (T_1^M \setminus T_2^M) \cap \overline{U}. \end{aligned}$$

□

Remark 1. *We observe that Lemmas 4, 5, and 6 enable us to rediscover the decidability of the satisfiability problem for* BSTC^-*. It is noteworthy that, in the case of* BSTC^-*-formulae, one can significantly reduce the number of items needed to construct the universe* $\overline{U} \subseteq U$*. As demonstrated in [17, 18], the construction of* $\overline{U}$ *in this case only requires selecting one item from each block in a carefully chosen set of at most* $m - 1$ *blocks within* $\mathcal{R}_\varphi^M$*, where* m *represents the number of distinct terms occurring in* φ*. This observation is at the base of the NP-completeness of the decision problem for* BSTC^-*.*

If we drop from the statement of Lemma 6 the hypothesis that the formula φ is choice-free, another case must be taken into account in its inductive proof: the case in which the term T is of the form $\mathsf{c}(T_1)$. This would require us to prove, under the inductive hypothesis $T_1^{\overline{M}} = T_1^M \cap \overline{U}$, that $(\mathsf{c}(T_1))^{\overline{M}} = (\mathsf{c}(T_1))^M \cap \overline{U}$, namely:

$$\mathsf{c}^{\overline{M}}(T_1^M \cap \overline{U}) = \mathsf{c}^M(T_1^M) \cap \overline{U}. \tag{2}$$

Hence, special care must be taken in the definition of $\mathsf{c}^{\overline{M}}$ in order that the identity (2) holds. The following example shows that we cannot simply define $\mathsf{c}^{\overline{M}}$ as the restriction to $\mathrm{pow}^+(\overline{U})$ of the choice c^M.

Example 1. *Let* $U = \{a, b, c, d\}$ *and consider the asymmetric relation* $\prec$ *over* U*, where* $a \prec c$ *and* $b \prec d$*. We define the interpretation* M *over* $\{X, Y\}$ *as* $X^M = \{a, b\}$ *and* $Y^M = \{c, d\}$*. Furthermore, let* c^M *be the choice function over the universe* U*, rationalized by* $\prec$*.*

For the restricted interpretation, let $\overline{U} = \{a, d\}$*. We define the interpretation* $\overline{M}$ *over the universe* $\overline{U}$ *as* $X^{\overline{M}} = X^M \cap \overline{U} = \{a\}$ *and* $Y^{\overline{M}} = Y^M \cap \overline{U} = \{d\}$*. Assume that we define the choice* *function* $\mathsf{c}^{\overline{M}}$ *simply as* $\mathsf{c}^{\overline{M}} = \mathsf{c}^M|_{\mathrm{pow}^+(\overline{U})}$, *and consider the choice term* T *equal to* $\mathsf{c}(X \cup Y)$. *Then we have:*

$$\begin{aligned} T^M &= (\mathsf{c}(X \cup Y))^M = \mathsf{c}^M(X^M \cup Y^M) = \mathsf{c}^M(\{a,b,c,d\}) = \{c,d\} \\ T^{\overline{M}} &= (\mathsf{c}(X \cup Y))^{\overline{M}} = \mathsf{c}^{\overline{M}}(X^{\overline{M}} \cup Y^{\overline{M}}) = \mathsf{c}^{\overline{M}}(\{a,d\}) = \{a,d\}. \end{aligned}$$

Comparing the results, we find that $T^{\overline{M}} = \{a,d\}$, *which is not equal to the intersection of* T^M *with* $\overline{U}$, *i.e.,* $\{d\}$. $\lhd$

Note that since T_1 and $\mathsf{c}(T_1)$ are terms in $\mathcal{T}_\varphi$, then T_1^M and $(\mathsf{c}(T_1))^M$ are unions of blocks in the Euler-Venn partition $\mathcal{R}_\varphi^M$. By defining $\Gamma_{T_1}^M$ as the set of the blocks of $\mathcal{R}_\varphi^M$ contained in T_1^M, and similarly for $\Gamma_{\mathsf{c}(T_1)}^M$, namely

$$\Gamma_{T_1}^M \coloneqq \{\rho \in \mathcal{R}_\varphi^M : \rho \subseteq T_1^M\} \quad \text{and} \quad \Gamma_{\mathsf{c}(T_1)}^M \coloneqq \{\rho \in \mathcal{R}_\varphi^M : \rho \subseteq (\mathsf{c}(T_1))^M\}, \tag{3}$$

we have $\bigcup \Gamma_{T_1}^M = T_1^M$ and $\bigcup \Gamma_{\mathsf{c}(T_1)}^M = (\mathsf{c}(T_1))^M = \mathsf{c}^M(T_1^M)$. Therefore, we have $\mathsf{c}^M(\bigcup \Gamma_{T_1}^M) = \bigcup \Gamma_{\mathsf{c}(T_1)}^M$. It would therefore be sufficient that there existed a rationalizable choice $\overline{c}$ over the universe $\overline{U}$ such that $\overline{c}\big((\bigcup \Gamma) \cap \overline{U}\big) = \bigcup \Gamma' \cap \overline{U}$ held, for all nonempty subsets Γ, Γ' of $\mathcal{R}_\varphi^M$ satisfying an identity of the form $\mathsf{c}^M(\bigcup \Gamma) = \bigcup \Gamma'$.

This is guaranteed by the following technical lemma, whose rather intricate and lengthy proof is omitted due to space limits.

Lemma 7. *Let* $c \colon \mathrm{pow}^+(U) \rightrightarrows U$ *be a rationalizable total choice correspondence and* $\Sigma \coloneqq \{B_1, B_2, \dots, B_n\}$ *be an* n*-partition of* U*, for some* $n \geq 1$. *Then for every subset* $U^\star = \{b_1^\star, b_2^\star, \dots, b_n^\star\}$ *of* U *with* n *elements such that* $b_i^\star \in B_i$ *for* $i = 1, 2, \dots, n$, *there exists a rationalizable total choice* $c^\star$ *over* $U^\star$ *such that the following implication holds for all nonempty subsets* Γ, Γ' *of* Σ*:*

$$c(\textstyle\bigcup \Gamma) = \bigcup \Gamma' \implies c^\star\big((\bigcup \Gamma) \cap U^\star\big) = (\bigcup \Gamma') \cap U^\star. \tag{4}$$

Thus, for the rest of the section, we will assume that $\mathsf{c}^{\overline{M}}$ is any choice over $\overline{U}$ such that the implication

$$\mathsf{c}^M(\textstyle\bigcup \Gamma) = \bigcup \Gamma' \implies \mathsf{c}^{\overline{M}}\big((\bigcup \Gamma) \cap \overline{U}\big) = (\bigcup \Gamma') \cap \overline{U} \tag{5}$$

holds for all $\emptyset \neq \Gamma, \Gamma' \subseteq \mathcal{R}_\varphi^M$, whose existence is guaranteed by Lemma 7, so that can prove the identity (2) concerning terms of the form $\mathsf{c}(T_1)$.

The preceding discussion allows us to state the following strengthening of Lemma 6.

Lemma 8. *Assuming that* $\mathsf{c}^{\overline{M}}(T_1^M \cap \overline{U}) = \mathsf{c}^M(T_1^M) \cap \overline{U}$ *for every choice term* $\mathsf{c}(T_1)$ *in* φ, *then the condition* $T^{\overline{M}} = T^M \cap \overline{U}$ *holds for all set terms* T *in* φ.

We claim that $\overline{\boldsymbol{\mathcal{M}}} = (\overline{U}, \overline{M})$ is an Rtl-model for our BSTC-formula φ.

In view of Lemma 4, it suffices to show that, for each atomic subformula $T_1 = T_2$ occurring in φ, we have

$$\boldsymbol{\mathcal{M}} \models_{\mathsf{Rtl}} T_1 = T_2 \iff \overline{\boldsymbol{\mathcal{M}}} \models_{\mathsf{Rtl}} T_1 = T_2.$$

Since

$$\boldsymbol{\mathcal{M}} \models_{\mathsf{Rtl}} T_1 = T_2 \quad \iff \quad T_1^M = T_2^M \qquad\qquad \text{(by definition)}$$

$$\iff \quad T_1^{\overline{M}} = T_2^{\overline{M}} \qquad \text{(by Lemmas 5 and 8)}$$
$$\iff \quad \overline{\boldsymbol{\mathcal{M}}} \models_{\mathsf{Rtl}} T_1 = T_2 \qquad \text{(by definition),}$$

for every atomic equality $T_1 = T_2$ in φ, from Lemma 4 and the hypothesis $\boldsymbol{\mathcal{M}} \models_{\mathsf{Rtl}} \varphi$ it follows that $\overline{\boldsymbol{\mathcal{M}}}$ is an Rtl-model for φ, as claimed. In addition, we have that the universe $\overline{U}$ of $\overline{\boldsymbol{\mathcal{M}}}$ has size $|\mathcal{R}_\varphi^M| < 2^{|\mathcal{T}_\varphi|} \leq 2^{|\varphi|}$, where $|\varphi|$ is the size of φ.

Summing up, we have the following result:

Theorem 1 (Small model property). *A* BSTC*-formula φ is Rtl-satisfiable if and only if it admits an Rtl-model over a universe of size $\mathcal{O}(2^{|\varphi|})$.*

The preceding theorem yields the following trivial decision procedure for the Rtl-satisfiability problem of BSTC.

procedure BSTC-Rtl-test(φ);
1. let $n := |\mathcal{T}_\varphi|$ and let U be any universe of size 2^n;
2. **for** each set assignment $\boldsymbol{\mathcal{M}} = (U, M)$ under rationalizability **do**
3.   **if** $\boldsymbol{\mathcal{M}} \models_{\mathsf{Rtl}} \varphi$ **then**
4.   **return** "φ is Rtl-satisfiable by $\boldsymbol{\mathcal{M}} = (U, M)$";
5. **return** "φ is not Rtl-satisfiable";

end procedure;

Regarding the complexity of the procedure BSTC-Rtl-test, we make the following observations. Given a set assignment $\boldsymbol{\mathcal{M}} = (U, M)$ under rationalizability over a finite universe U of size m, and a collection $\mathsf{V}_0 \cup \mathsf{V}_1$ of size v of individual and set variables:

1. The interpretation M takes $\mathcal{O}(vm)$ space. This is because the interpretation M assigns values to variables from the collection, and since there are v variables and each variable can be assigned a value from a universe of size m, the total space required is proportional to vm.
2. The relation over U that rationalizes c^M can be represented in $\mathcal{O}(m^2)$ space. Here, c^M refers to the rational choice associated with the set assignment $\boldsymbol{\mathcal{M}}$. The relation represents preferences among elements in the universe, and since the universe has size m, storing it requires space proportional to m^2.

Therefore, the total space complexity to store the interpretation M and the relation over U that rationalizes c^M is $\mathcal{O}(vm + m^2)$.

Also, the time needed to check a purported set assignment $\boldsymbol{\mathcal{M}} = (U, M)$ under the same aforementioned conditions is linear in its size $\mathcal{O}(vm + m^2)$ (for instance, when verifying whether the rationalizing relation is indeed devoid of infinite ascending sequences, it suffices to check it for acyclicity, a task that can be accomplished in linear time with respect to its size, which is $\mathcal{O}(m^2)$).

For a given Rtl-satisfiable BSTC-formula φ of size n, we can therefore generate a satisfying Rtl-model $\boldsymbol{\mathcal{M}} = (U, M)$ with a universe of size $m = 2^n$, over a collection of size $\mathcal{O}(n)$ of individual and set variables, in $\mathcal{O}(2^{n^2})$ time and space. In addition, we can check that $\boldsymbol{\mathcal{M}}$ is indeed an Rtl-model for φ in deterministic $\mathcal{O}(2^{n^2})$ time.

Furthermore, the satisfiability problem for BSTC is NP-hard, as the satisfiability problem for propositional logic can readily be reduced to it (in linear time).

To summarize, we can state the following complexity result:

Theorem 2. *The satisfiability problem for* BSTC*-formulae under rationalizability belongs to the complexity classes NP-hard and NEXPTIME.*

5. Conclusions

In this paper, we have explored the implications and characteristics of rational decision-making within the quantifier-free elementary fragment of set theory denoted as BSTC (Boolean Set Theory with a Choice operator). Our primary focus was on the satisfiability problem for BSTC-formulae. By interpreting the choice operator c as a rational choice, we have established that BSTC under rationalizability exhibits a small model property. This significant property has enabled us to demonstrate the decidability of the satisfiability problem for BSTC under rationalizability, classifying it as belonging to the complexity classes NP-hard and NEXPTIME. These findings represent an extension of previous work on the satisfiability problem in the presence of a choice operator.

For future research, it would be worthwhile to explore extensions of BSTC with a predicate $Finite(\cdot)$, which expresses that its argument is a finite set (thus, $\neg Finite(X)$ denotes that the set X is infinite). Additionally, further investigations can be carried out under alternative axiomatizations of the choice operator, such as those associated with (m, n)-rationalizable choices, which have been extensively studied in [2].

Furthermore, we intend to strengthen the small model property, if possible, along lines similar to those that allowed us to prove the NP-completeness of the theory BSTC^{-} as hinted in Remark 1, thereby establishing the NP-completeness of BSTC under rationalizability.

By pursuing these research directions, we can deepen our understanding of the relationship between rational decision-making and set theory, paving the way for new insights and advancements in this interdisciplinary field.

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