On the security of the initial tropical Stickel protocol and its modification based on Linde-de la Puente matrices

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Abstract

Recently, a more efficient attack on the initial tropical Stickel protocol has been proposed, different from the previously known Kotov-Ushakov attack, yet equally guaranteed to succeed. Given that the Stickel protocol can be implemented in various ways, such as utilizing platforms beyond the tropical semiring or employing alternative commutative matrix "classes" instead of polynomials, we firstly explore the generalizability of this new attack across different implementations of the Stickel protocol. We then conduct a comprehensive security analysis of the Stickel protocol based on Linde-de la Puente (LdlP) matrices. Additionally, we extend the concept of LdlP matrices beyond the tropical semiring, generalizing it to a broader class of semirings and include a discussion of the centralizer of such matrices over the tropical semiring.

Keywords: public key cryptography; cryptographic attack; Stickel protocol
Classification: 94A60, 15A80

1 Introduction

Tropical linear algebra has been recently used as a platform for new supposedly more secure implementations of some cryptographic key exchange protocols including the Stickel protocol [25]. In this context, Grigoriev and Shpilrain [14] introduced the first tropical implementation of the Stickel protocol, which we refer to as the "initial tropical Stickel protocol". The widely accepted attack on this protocol is due to Kotov and Ushakov [18]. This attack successfully breaks the protocol by finding the whole solution set of the underlying tropical linear system imposed by the protocol by enumerating all minimal solutions of such system.

Then, recently, the authors in [22] proposed an alternative attack that breaks the protocol by finding only a single solution of this linear system, rather than enumerating all solutions, which significantly reduces the complexity in relation to the polynomial degree used in the protocol. This attack is possible because the polynomials chosen by Alice and Bob commute with the powers of the public matrices. Notably, this new attack is not guaranteed to succeed on all implementations of the Stickel protocol. Its applicability depends on specific

conditions involving the underlying semiring and the "class" of the commuting matrices being used. Specifically, the attack successfully applies only when the one-sided linear systems over the semiring are easily solvable and the matrices used by Alice and Bob have an obvious finite set of generators with which they commute (e.g., consider matrix powers as generators of matrix polynomials).

Consequently, certain tropical variants of the Stickel protocol may prove resistant to this new attack, one notable candidate being the version based on Linde-de la Puente (LdlP) matrices [19] as proposed by [21]. This variant is also resistant to the Kotov-Ushakov attack which motivates a further investigation of its overall security by exploring the other heuristic means. It turns out that this class of matrices can also be constructed over a wider variety of semirings, possibly offering stronger cryptographic properties when utilized over alternative semirings.

This paper is organized as follows: Section 2 covers preliminaries and basic definitions, particularly those related to matrix algebra and the Stickel protocol over semirings. In Section 3, we present the conditions under which the new attack is applicable and provide its performance comparison with the Kotov-Ushakov attack. In Section 4, we analyze the security of the tropical Stickel protocol based on LdlP matrices against the new attack, the Kotov-Ushakov attack and some other heuristics that were suggested previously. All codes related to the numerical experiments have been made available on GitHub [1].

2 Preliminaries

In this section, we introduce the matrix algebra over semirings followed by the construction of the Stickel protocol over an arbitrary semiring, and how it is typically compromised by the Kotov-Ushakov attack and the new attack put forward in [22]. Note that we use the standard notation $[m] = \{1, \ldots, m\}$ and $[n] = \{1, \ldots, n\}$ for most common index sets. We start by recalling the definition of a semiring.

Definition 2.1 (Semiring)**.** Let S be a non-empty set equipped with two binary operations $\oplus$ and $\otimes$, which satisfy the following properties:

- $(S, \oplus)$ is an Abelian monoid which means that it satisfies associativity, commutativity and existence of an additive identity element ϵ.
- $(S, \otimes)$ is a monoid which means that it satisfies associativity and existence of multiplicative identity element e.
- In $(S, \oplus, \otimes)$ multiplication $\otimes$ distributes over addition $\oplus$.
- The additive identity ϵ satisfies the absorbing property, that is $\epsilon \otimes e = e \otimes \epsilon = \epsilon$.

The semirings of primary interest, particularly for their cryptographic applications in implementing the Stickel protocol, are the tropical (max-plus), fuzzy (max-min), and the max-T semirings. We now present their formal definitions.

[1] https://github.com/suliman1n/On-the-security-of-the-initial-tropical-Stickel-protocol-and-its-modification-based-on-LdlP-matrices

Definition 2.2 (Tropical Semiring)**.** The tropical semiring $\mathbb{R}_{\max}$ is defined by $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, where the tropical addition $\oplus$ and the tropical multiplication $\otimes$ are respectively defined by $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$ for all $a, b \in \mathbb{R}_{\max}$.

Definition 2.3 (Max-min Semiring)**.** The max-min semiring, denoted as $\mathbb{R}_{\max,\min}$, is defined by $\mathbb{R}_{\max,\min} = (\mathbb{R} \cup \{-\infty\} \cup \{\infty\}, \oplus, \otimes)$, with these two operations defined by $a \oplus b = \max\{a, b\}$ and $a \otimes b = \min\{a, b\}$ for all $a, b \in \mathbb{R}_{\max,\min}$.

Definition 2.4 (Max-T Semiring)**.** The max-T semiring is defined as the unit interval $\mathcal{B} = [0, 1]$ equipped with the tropical addition $a \oplus b = \max(a, b)$ and the T-norm multiplication $a \otimes b = T(a, b)$ where $T : \mathcal{B}^2 \to \mathcal{B}$ is a T-norm (see Definition 2.5).

Definition 2.5 (T-norm (e.g., [17]))**.** A T-norm is a binary operation on the unit interval that satisfies the following axioms for all $a, b, d \in [0, 1]$:

1. $T(a, 1) = a$ (boundary condition).
2. $b \leq d$ implies $T(a, b) \leq T(a, d)$ (monotonicity).
3. $T(a, b) = T(b, a)$ (commutativity).
4. $T(a, T(b, d)) = T(T(a, b), d)$ (associativity).

One notable example of a T-norm that has some interesting properties, which will be discussed later, is the Hamacher product, defined as

$$a \otimes b = T(a, b) = \begin{cases} 0, & \text{if } a = b = 0, \\ \frac{ab}{a+b-ab}, & \text{otherwise.} \end{cases} \tag{1}$$

The Stickel key exchange protocol is constructed using matrix algebra over an arbitrary semiring S. We hence present some of the relevant definitions.

Definition 2.6 (Matrix Algebra over Semirings [13])**.** The arithmetic operations over a semiring S are naturally extended to include matrices and vectors. In particular, the operation $A \otimes \alpha = \alpha \otimes A$, where $\alpha \in S, A \in S^{m \times n}$ and $(A)_{ij} = a_{ij}$ for $i \in [m]$ and $j \in [n]$, is defined by

$$(A \otimes \alpha)_{ij} = (\alpha \otimes A)_{ij} = \alpha \otimes a_{ij} \quad \forall i \in [m] \text{ and } \forall j \in [n].$$

The matrix addition $A \oplus B$ of two matrices $A \in S^{m \times n}$ and $B \in S^{m \times n}$, where $(A)_{ij} = a_{ij}$ and $(B)_{ij} = b_{ij}$ for $i \in [m]$ and $j \in [n]$, is defined by

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} \quad \forall i \in [m] \text{ and } \forall j \in [n].$$

The matrix multiplication of two matrices is also similar to the "traditional" algebra. Namely, we define $A \otimes B$ for two matrices, where $A \in S^{m \times p}$ and $B \in S^{p \times n}$, as follows:

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj} = (a_{i1} \otimes b_{1j} \oplus a_{i2} \otimes b_{2j} \oplus \ldots \oplus a_{ip} \otimes b_{pj}) \quad \forall i \in [m] \text{ and } \forall j \in [n].$$

Definition 2.7 (Matrix Powers)**.** For $M \in S^{n\times n}$, the n-th power of M is denoted by $M^{\otimes n}$, and is equal to

$$M^{\otimes n} = \underbrace{M \otimes M \otimes \ldots \otimes M}_{n \text{ times}}.$$

By definition, any square matrix to the power 0 is the identity.

Definition 2.8 (Identity Matrix of a Semiring)**.** The identity matrix $I \in S^{n\times n}$ is of the form $(I)_{ij} = \delta_{ij}$ where

$$\delta_{ij} = \begin{cases} e & \text{if } i = j \\ \epsilon & \text{otherwise} \end{cases}$$

Definition 2.9 (Matrix Polynomials)**.** A matrix polynomial is a function of the form

$$A \mapsto p(A) = \bigoplus_{k=0}^{d} a_k \otimes A^{\otimes k},$$

where $a_k \in S$ for $k = 0, 1, \ldots, d$. Here $A \in S^{n\times n}$ is a square matrix of any dimension n.

Any two matrix polynomials of the same matrix over any semiring commute just like in the classical algebra [13], and this fact was utilized by Grigoriev and Shpilrain to construct an implementation of the Stickel protocol over the tropical semiring after successfully attacking the original implementation [14]. The Stickel protocol can clearly be implemented over any semiring, as this underlying commutativity property remains valid.

Protocol 1 (Stickel Protocol over Semirings)**.**

1. Alice and Bob agree on public matrices A, B, W.
2. Alice chooses two random polynomials $p_1(x)$ and $p_2(x)$ and sends $U = p_1(A) \otimes W \otimes p_2(B)$ to Bob.
3. Bob chooses two random polynomials $q_1(x)$ and $q_2(x)$ and sends $V = q_1(A) \otimes W \otimes q_2(B)$ to Alice.
4. Alice computes her secret key using a public key V obtained from Bob, which is $K_a = p_1(A) \otimes V \otimes p_2(B)$.
5. Bob also computes his secret key using Alice's public key U, which is $K_b = q_1(A) \otimes U \otimes q_2(B)$.

The two parties end up with an identical key due to the commutativity of polynomials of the same matrix. Formally, we have $K_a = p_1(A) \otimes V \otimes p_2(B) = p_1(A) \otimes q_1(A) \otimes W \otimes q_2(B) \otimes p_2(B) = q_1(A) \otimes p_1(A) \otimes W \otimes p_2(B) \otimes q_2(B) = q_1(A) \otimes U \otimes q_2(B) = K_b$.

An intuitive way to attack this protocol is aiming to find the coefficients of two polynomials that can reconstruct the transmitted message (U or V). This is achieved by scanning all solutions of the one-sided linear system corresponding to either message. (Note that

$U = p_1(A) \otimes W \otimes p_2(B)$ is essentially a one-sided linear system of the shape $A \otimes x = b$ with unknowns being the products of polynomial coefficients). The attacker then searches for a solution that satisfies a specific structure arising from the multiplication of two polynomials. This approach was proposed by Kotov and Ushakov to attack the tropical version of the Stickel protocol [18]. The ideas of the attack can be summarized as follows.

The aim is to find two matrices X and Y, where they are expressed as

$$X = \bigoplus_{\alpha=0}^{D} \left(x_\alpha \otimes A^{\otimes\alpha}\right), Y = \bigoplus_{\beta=0}^{D} \left(y_\beta \otimes B^{\otimes\beta}\right),$$

such that D is sufficiently large to exceed the maximal degree of any polynomial that Alice and Bob might use. Then, Alice's message U can be expressed as

$$U = \bigoplus_{\alpha=0}^{D} \left(x_\alpha \otimes A^{\otimes\alpha}\right) \otimes W \otimes \bigoplus_{\beta=0}^{D} \left(y_\beta \otimes B^{\otimes\beta}\right),$$

or equivalently

$$\bigoplus_{\alpha,\beta=0}^{D} x_\alpha \otimes y_\beta \otimes \left(A^{\otimes\alpha} \otimes W \otimes B^{\otimes\beta}\right) = U.$$

We then denote $R^{\alpha\beta} = A^{\otimes\alpha} \otimes W \otimes B^{\otimes\beta}$ and therefore we can write

$$\bigoplus_{\alpha,\beta=0}^{D} x_\alpha \otimes y_\beta \otimes \left(R^{\alpha\beta}\right)_{\gamma\delta} = U_{\gamma\delta} \quad \forall\gamma,\delta \in [n] \times [n]. \tag{2}$$

If we additionally denote $z_{\alpha\beta} = x_\alpha \otimes y_\beta$, we have

$$\bigoplus_{\alpha,\beta=0}^{D} z_{\alpha\beta} \otimes \left(R^{\alpha\beta}\right)_{\gamma\delta} = U_{\gamma\delta} \quad \forall\gamma,\delta \in [n] \times [n]. \tag{3}$$

This is a system of linear equations of the shape $A \otimes x = b$ with coefficients $\left(R^{\alpha\beta}\right)_{\gamma\delta}$ and unknowns $z_{\alpha\beta}$.

The next goal of the attack is to scan all solutions to this system, and get the solution that satisfies $z_{\alpha\beta} = x_\alpha \otimes y_\beta$ for some $x_\alpha, y_\beta \in \mathbb{Z}$ for all $\alpha, \beta \in \{0, 1, \ldots, D\}$. The way how this is done may depend on the theory of $A \otimes x = b$ over the semiring in question. It is known that for the tropical (max-plus) semiring, the max-min semiring and, more generally, for any max-T semiring where T is a continuous T-norm, the system $A \otimes x = b$ has the greatest solution, a finite number of minimal solutions and each solution to $A \otimes x = b$ lies in the box defined by one of the minimal solutions and the greatest solution. For the attacker's purposes, we need to search for a vector $(z_{\alpha\beta})$ in the box defined by one of the minimal solutions and the greatest solution that satisfies $z_{\alpha\beta} = x_\alpha \otimes y_\beta$ for some x_α, y_β. A formal description of the attack is due to Kotov and Ushakov [18] in the tropical case, and a max-min version (which has a straightforward generalization to the max-T case) was suggested in [3].

Different variants of the Stickel protocol (protocol 1) can be implemented using alternative "classes" of commuting matrices. A number of these alternatives are explored in the literature (e.g., [21]). For these protocols using other kinds of commuting matrices, matrix powers can be replaced with other generators, although this may require imposing some mild constraints on the coefficients x_α, y_β, and hence a generalized version of Kotov-Ushakov attack still applies [21]. Formally, X and Y are instead expressed as

$$X = \bigoplus_{\alpha \in \mathcal{A}} (x_\alpha \otimes A_\alpha), Y = \bigoplus_{\beta \in \mathcal{B}} (y_\beta \otimes B_\beta), \tag{4}$$

Here $\{A_\alpha : \alpha \in A\}$ and (respectively) $\{B_\beta : \beta \in B\}$ are finite sets of matrices such that any matrix that can be used by Alice and (respectively) by Bob can be represented as these X and Y. The rest of the attack similarly follows, but may include additional conditions on the coefficients x_α, y_β.

Note that Kotov-Ushakov attack and its generalization [21] are guaranteed to succeed under the (not too restrictive) condition that any matrix used by Alice or Bob can be represented as linear combination of generators in $\mathcal{A}$ and $\mathcal{B}$; for a detailed proof, refer to [21]. However, a significant limitation of these attacks is that they require scanning the entire solution set of the underlying linear system, which involves finding all minimal solutions. As Alice and Bob use polynomials of higher degree (or larger $\mathcal{A}, \mathcal{B}$ in the case of the generalized Kotov-Ushakov attack), the number of minimal solutions in this system grows exponentially, resulting in a corresponding exponential increase in the attack's computational complexity. One way to circumvent this is to seek a particular minimal solution and then hope that the box defined by such solution and the greatest solution contains a solution of the desired structure. Then the resulting attack is of a polynomial time complexity, but the success rate of it may suffer. The heuristic attacks of such type were put forward by Mach [20] and in [1]. In the latter work it was found that a heuristic attack of this kind had 100% success rate when applied to the tropical Stickel protocol based on modified circulants and over 90% success rate when applied to the initial tropical Stickel protocol based on polynomials (the success of a similar attack in the max-min case was, however, much more modest [3]).

Recently, the authors in [22] came up with a better idea to attack the various versions of tropical Stickel protocols, which we next outline. Instead of searching for a special solution of system (3) among all possible solutions—the approach employed in the Kotov-Ushakov attack—it can be observed that any solution $(r_{\alpha\beta})$ to (3) suffices to break the protocol. Indeed, recalling that $V = q_1(A) \otimes W \otimes q_2(B)$ and using the commutation between $A^{\otimes\alpha}$ and $q_1(A)$ on one side and the commutation between $B^{\otimes\beta}$ and $q_2(B)$ on the other side we obtain that for any solution $(r_{\alpha\beta})$ to system (3), the shared secret key K can be recovered by

$$K = \bigoplus_{\alpha,\beta=0}^{D} r_{\alpha\beta} \otimes A^{\otimes\alpha} \otimes V \otimes B^{\otimes\beta}. \tag{5}$$

To prove that, we simply need to verify whether this formula successfully recovers the

key. Given that $(r_{\alpha\beta})$ is any solution to (3), we have

$$\begin{aligned} K &= \bigoplus_{\alpha,\beta=0}^{D} r_{\alpha\beta} \otimes A^{\otimes\alpha} \otimes q_1(A) \otimes W \otimes q_2(B) \otimes B^{\otimes\beta} \\ &= \bigoplus_{\alpha,\beta=0}^{D} r_{\alpha\beta} \otimes q_1(A) \otimes A^{\otimes\alpha} \otimes W \otimes B^{\otimes\beta} \otimes q_2(B) \\ &= q_1(A) \otimes (\bigoplus_{\alpha,\beta=0}^{D} r_{\alpha\beta} \otimes A^{\otimes\alpha} \otimes W \otimes B^{\otimes\beta}) \otimes q_2(B) \\ &= q_1(A) \otimes U \otimes q_2(B) = K_b = K_a. \end{aligned}$$

This attack significantly reduces the burden on the attacker by eliminating the need to explore the entire solution set of system (3). Instead, any solution can be utilized. The new attack is formally described in the following algorithm.

Attack 1 (Attacking Protocol 1 based on (5))**.**

1. Find a solution $r_{\alpha\beta}$ of system (3).
2. Compute the shared secret key K.

$$K = \bigoplus_{\alpha,\beta=0}^{D} r_{\alpha\beta} \otimes A^{\otimes\alpha} \otimes V \otimes B^{\otimes\beta}.$$

In the tropical case, as well as in the max-min case and, more generally, for max-T semirings with lower-semicontinuous T-norms [3], [9], the greatest solution of system (3) can be easily found using an explicit formula and used in Attack 1. Note that the authors in [22] also give algebraic conditions for semirings over which (3) has the greatest solution that is easily computed by an explicit formula. Furthermore, for max-T semirings with upper-semicontinuous T-norms one can find a minimal solution [9] and also use it in Attack 1. Although this may require more time than finding the greatest solution for which there is an explicit formula, it is still better than the Kotov-Ushakov attack where one needs to use a number of minimal solutions and the greatest solution.

Note that Attack 1 begins by solving the linear system (3), where the greatest solution in the tropical case can be computed in $O(D^2n^2)$ time since we need to find the minimum of each entry over D^2 matrices. The subsequent key recovery expression includes classical scalar with matrix multiplication and tropical matrix addition, each with a time complexity of $O(n^2)$. The most computationally demanding step is matrix exponentiation and multiplication, which dominates the overall complexity at $O(D^2n^3)$. In the Kotov-Ushakov attack, the most computationally intensive part is solving a minimal covering problem, which is expected to have exponential time complexity with the number of the subsets (the maximal degree used by Alice and Bob) in the worst case. However, Kotov and Ushakov adopted a heuristic sorting algorithm that significantly reduces the number of enumerated and tested covers.

Figure 1 compares the performance of the Kotov-Ushakov attack and this new attack (Attack 1) on the initial tropical Stickel protocol using the greatest solution to (3) with matrix dimensions of 10 and a range of polynomial degrees, and both matrix entries and polynomial coefficients are random integers from $[-1000, 1000]$, and the average time is computed over 5 trials. All numerical experiments were executed on Windows 11 64-bit, with an Intel(R) Core(TM) i7-9750H CPU @ 2.60GHz and 16.0 GB RAM.

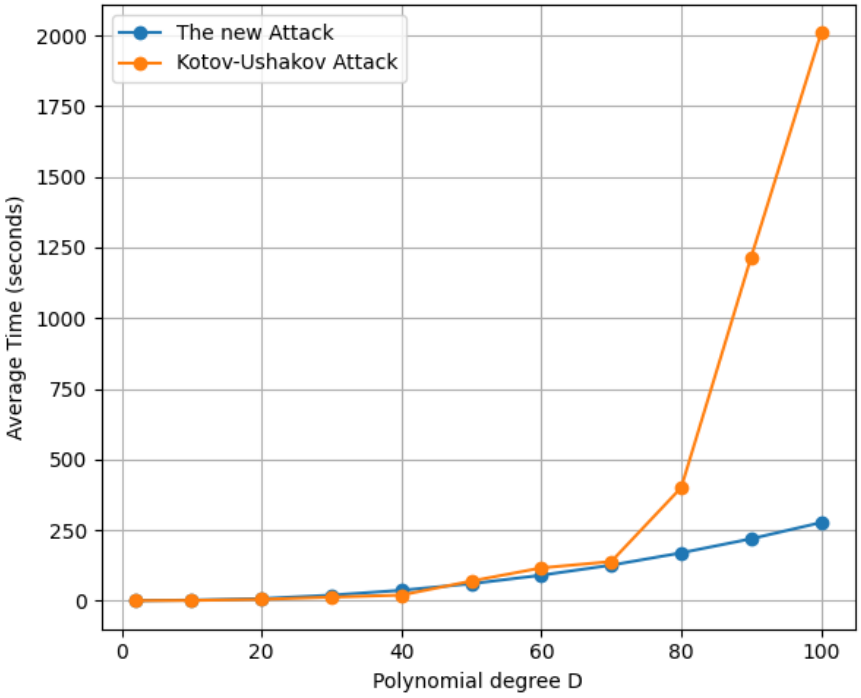

Figure 1: Computational time of Attack 1 vs. Kotov-Ushakov attack

As expected, the computational time of the Kotov-Ushakov attack increases exponentially due to the rapid growth of minimal solutions (enumerated minimal covers) with respect to the used polynomial degree. In contrast, the increase in computational time for the new attack remains relatively small. Note that at lower polynomial degrees, the two attacks show comparable performance, as the computational heavy part in the Kotov-Ushakov attack (enumerating all minimal covers) is not yet dominant.

The computational difficulty of the Kotov-Ushakov attack can be also explained using an observation of [11] that the problem of enumerating all minimal hypergraph transversals can be seen as a special case of the problem of finding all minimal solutions to $A \otimes x = b$ over the tropical (max-plus) semiring when we restrict the entries of A and b to 0 and $-\infty$. While there is a hope (see, e.g., [5]) for an output-polynomial method for finding all minimal hypergraph transversals (i.e., polynomial in the input size and the total number of solutions), the total number of minimal hypergraph transversals is known to grow exponentially with respect to dimension, corresponding to the number of vertices in the hypergraph. We note

here that [22] Theorem 7 has a claim that "determining all the solutions of the system has a computational cost of $o(mn)$". While finding the maximal solution of $A \otimes x = b$ indeed has this computational complexity and serves as a "basis" for finding all minimal solutions, this claim is not supported by a precisely formulated algorithm and seems to be in conflict with the observations mentioned above.

Next, we would like to compare the performance of attack described in [22] with some previously known heuristic implementations of the Kotov-Ushakov attack. Figure 2 also shows the computational time of this new attack and one of the previously proposed heuristics, namely the single cover heuristic from [1], using the same parameter values as in the previous experiment. This may highlight that heuristic attacks can remain valuable, especially when they achieve high success rates, due to their higher efficiency when compared with the guaranteed attacks. However, these two attacks can be also viewed as incomparable as one of them is deterministic (with an appropriate bound) and the other has a significant probability of failure.

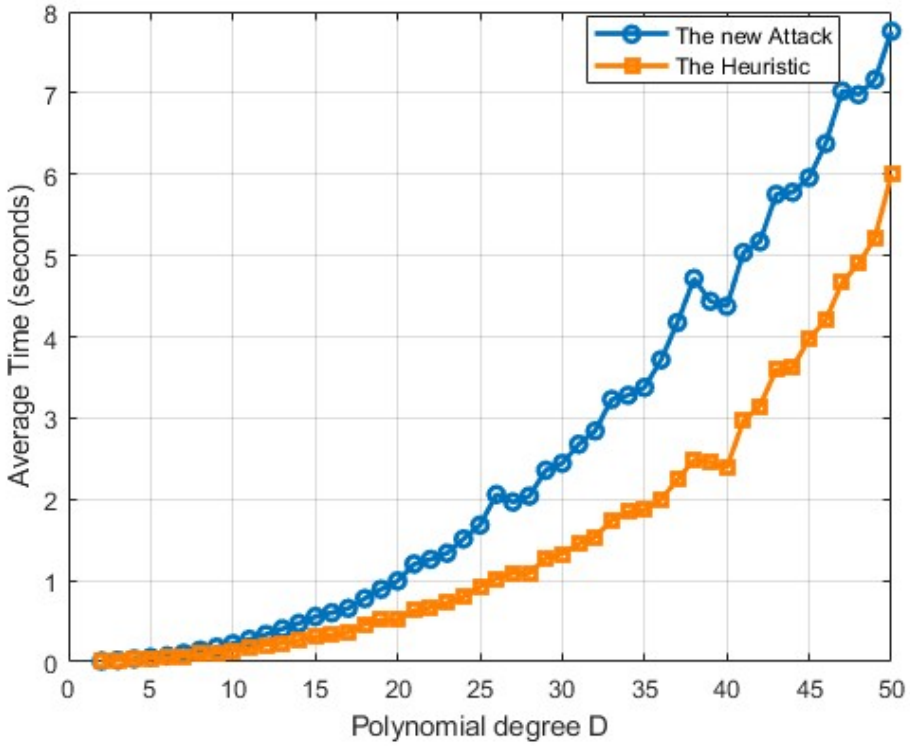

Figure 2: Computational time of Attack 1 vs. the heuristic attack in [1]

The new attack also works for some other implementations of Stickel protocol such as the ones based on the modified circulants and Jones matrices [16], [21]. However, if Alice and Bob use a different implementation of Stickel protocol for which A_α in (4) do not commute with the matrices used by them on the left and/or B_β do not commute with the matrices used by them on the right, then the Kotov-Ushakov attack is still guaranteed to work and the new attack becomes a heuristic (i.e., it is not guaranteed to succeed). The next section

will discuss the tropical Stickel protocol based on Linde-de la Puente matrices (shortly LdlP matrices) for which the attacker has two alternatives: 1) use a small number of generators A_α and B_β that do not commute with LdlP matrices, 2) consider a larger number of different generators that commute with LdlP matrices to ensure the success of this new attack.

3 Security analysis of tropical Stickel protocol based on Linde-de la Puente matrices

The tropical Stickel protocol based on Linde-de la Puente matrices closely resembles the original tropical implementation in [14], but replaces tropical polynomials with matrices of the form $[2r, r]_n^k$ as introduced in [21]. Let us first present the definition of LdlP matrices.

Definition 3.1 ([21], generalizing [19])**.** For arbitrary real number $r \leqslant 0$ and real number $k \geqslant 0$, we denote by $[2r, r]_n^k$ the set of matrices A such that $a_{ii} = k$, for all i and $a_{ij} \in [2r, r]$ for $i \neq j$.

Note that any two matrices of this form commute due to the following theorem.

Theorem 3.2 (LdlP Matrices Commutativity [21])**.** *Let $A \in [2r, r]_n^{k_1}, B \in [2s, s]_n^{k_2}$ for any $r, s \leqslant 0$ and $a_{ii} = k_1 \geqslant 0, b_{ii} = k_2 \geqslant 0$ then*

$$A \otimes B = B \otimes A = k_2 \otimes A \oplus k_1 \otimes B$$

Let us observe that Linde-de la Puente matrices also allow for semiring generalizations. Consider any semiring with idempotent addition ($a \oplus a = a$) in which the order $\leq$ is defined canonically ($a \oplus b = b \Leftrightarrow a \leq b$), the property $a \otimes b \leq a^{\otimes 2} \oplus b^{\otimes 2}$ holds and in which there exists at least one element a with $a^{\otimes 2} \leq a$. In particular, the property $a \otimes b \leq a^{\otimes 2} \oplus b^{\otimes 2}$ (to which we further refer as to the *squares property*) holds in commutative semirings with cancellative condition ($a \otimes b = a \otimes c$ and $a \neq \mathbf{0}$ implies $b = c$) as shown in [10]. The latter condition is sufficient but not necessary: for example, the max-min semiring also satisfies the squares property without being cancellative. Then we can modify the above definition to the following one. Here and below, $\mathbf{0}$ and $\mathbf{1}$ will denote the zero and the unity elements of the semiring.

Definition 3.3. For arbitrary element r such that $r^{\otimes 2} \leq r$ we denote by $[r^{\otimes 2}, r]_n$ the set of matrices A such that $a_{ii} = \mathbf{1}$ for all i and $r^{\otimes 2} < a_{ij} < r$ for $i \neq j$.

Let us show that any two matrices of this form commute, adopting and generalizing an argument of [19].

Theorem 3.4 (LdlP Matrices Commutativity over Semirings)**.** *Consider an idempotent semiring in which the squares property holds, and let $A \in [r^{\otimes 2}, r]_n, B \in [s^{\otimes 2}, s]_n$ for any r, s such that $r^{\otimes 2} \leq r$ and $s^{\otimes 2} \leq s$. Then*

$$A \otimes B = B \otimes A = A \oplus B$$

Proof. We observe that $(A \otimes B)_{ik}$ can be written as

$$\begin{aligned}\bigoplus_j a_{ij} \otimes b_{jk} &= b_{kk} \otimes a_{ik} \oplus a_{ii} \otimes b_{ik} \oplus \bigoplus_{j \neq i,k} a_{ij} \otimes b_{jk} \\ &= \mathbf{1} \otimes a_{ik} \oplus \mathbf{1} \otimes b_{ik} \oplus \bigoplus_{j \neq i,k} a_{ij} \otimes b_{jk} \\ &= a_{ik} \oplus b_{ik} \oplus \bigoplus_{j \neq i,k} a_{ij} \otimes b_{jk}.\end{aligned}$$

Then, note that

$$a_{ij} \otimes b_{jk} \leq r \otimes s \leq r^{\otimes 2} \oplus s^{\otimes 2} \leq a_{ik} \oplus b_{ik},$$

implying that $(A \otimes B)_{ik} = a_{ik} \oplus b_{ik}$, and $(B \otimes A)_{ik} = b_{ik} \oplus a_{ik}$ can be shown similarly. □

As written above, the max-min semiring satisfies the squares property and therefore the above theorem holds for LdlP matrices over it. However, here we have $a^{\otimes 2} = a$ for all a, which trivializes the class of LdlP matrices making it less attractive for cryptographic purposes. We can also consider the max-T semiring with T being the Hamacher product. It can be shown that the Hamacher product is commutative and cancellative and therefore the squares property holds in the max-Hamacher semiring. Furthermore, the intervals $(a^{\otimes 2}, a)$ are non-empty for any $a\colon 0 < a < 1$ (we have $\mathbf{0} = 0$ and $\mathbf{1} = 1$ in any max-T semiring).

The protocol that utilizes the commutativity property of LdLP matrices over tropical semiring is outlined below. Its generalization to commutative idempotent semirings satisfying the squares property ($a \otimes b \leq a^{\otimes 2} \oplus b^{\otimes 2}$) is also obvious, but we will restrict our cryptanalysis to the tropical case in what follows.

Protocol 2 (Tropical Stickel Protocol based on LdlP Matrices [21])**.**

1. Alice and Bob agree on a public matrix $W \in \mathbb{R}_{\max}^{n \times n}$.
2. Alice chooses two random matrices A_1 and A_2, where $A_1 \in [2a_1, a_1]_n^{k_1}$ and $A_2 \in [2a_2, a_2]_n^{k_2}$ such that $a_1, a_2 \leq 0$ and $k_1, k_2 \geq 0$ and sends $U = A_1 \otimes W \otimes A_2$ to Bob.
3. Bob chooses two random matrices B_1 and B_2, where $B_1 \in [2b_1, b_1]_n^{l_1}$ and $B_2 \in [2b_2, b_2]_n^{l_2}$ such that $b_1, b_2 \leq 0$ and $l_1, l_2 \geq 0$ and sends $V = B_1 \otimes W \otimes B_2$ to Bob.
4. Alice computes her secret key using a public key V obtained from Bob, which is $K_a = A_1 \otimes V \otimes A_2$.
5. Bob also computes his secret key using Alice's public key U, which is $K_b = B_1 \otimes U \otimes B_2$.

The two parties end up with an identical key due to the commutativity of Linde-de la Puente matrices. Formally, we have $K_a = A_1 \otimes V \otimes A_2 = A_1 \otimes B_1 \otimes W \otimes B_2 \otimes A_2 = B_1 \otimes A_1 \otimes W \otimes A_2 \otimes B_2 = B_1 \otimes U \otimes B_2 = K_b$.

Before describe the attacks on this protocol let us introduce two kinds of matrices which will serve as generating sets for LdlP matrices.

We firstly introduce the concept of elementary matrices, which (taken together with the identity matrix) can serve as the generators A_α and B_β in the tropical Stickel protocol based on LdlP matrices.

Definition 3.5 (Tropical Elementary Matrices)**.** Let $E^{ij} \in \mathbb{R}_{\max}^{n\times n}$ be a matrix with entries

$$\left(E^{ij}\right)_{kl} = \begin{cases} 0, & \text{if } k = i, l = j \\ -\infty, & \text{otherwise.} \end{cases}$$

for $i, j \in [n]$ and $k, l \in [n]$. Any matrix of this form is called a tropical elementary matrix.

The following elementary LdlP matrices, taken together with the identity matrix I, can also serve as generators of the LdlP matrices.

Definition 3.6 (Elementary LdlP Matrices)**.** Let $F_r^{ij} \in \mathbb{R}_{\max}^{n\times n}$ be a matrix with entries

$$\left(F_r^{ij}\right)_{kl} = \begin{cases} r, & \text{if } k = i, l = j \\ 2r, & \text{otherwise.} \end{cases}$$

for $i, j \in [n]$ and $k, l \in [n]$. Any matrix of this form is called an r-elementary LdlP matrix, and by an elementary LdlP matrix we mean a matrix which is r-elementary LdlP for some r.

Let us formally state and prove what we mean by generating the set of all LdlP matrices by elementary or elementary LdlP matrices.

Proposition 3.7. The following identities hold for any matrix $A \in [2r, r]_n^k$.

$$A = k \otimes I \oplus \bigoplus_{i\neq j} a_{ij} \otimes E^{ij}, \tag{6}$$

$$A = k \otimes I \oplus \bigoplus_{i\neq j} (a_{ij} - r) \otimes F_r^{ij}. \tag{7}$$

Proof. The first identity clearly holds since

$$k \otimes I \oplus \bigoplus_{i\neq j} a_{ij} \otimes E^{ij} = \begin{pmatrix} k \oplus (-\infty) & (-\infty) \oplus a_{12} & \cdots & (-\infty) \oplus a_{1n} \\ (-\infty) \oplus a_{21} & k \oplus (-\infty) & \cdots & (-\infty) \oplus a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (-\infty) \oplus a_{n1} & (-\infty) \oplus a_{n2} & \cdots & k \oplus (-\infty) \end{pmatrix} = A$$

The second identity also holds as

$$k \otimes I \oplus \bigoplus_{i\neq j} (a_{ij} - r) \otimes F_r^{ij} =$$
$$\begin{pmatrix} k \oplus \bigoplus_{i\neq j}(a_{ij} - r) & a_{12} \oplus \bigoplus_{(i,j)\neq(1,2)}(a_{ij} + r) & \cdots & a_{1n} \oplus \bigoplus_{(i,j)\neq(1,n)}(a_{ij} + r) \\ a_{21} \oplus \bigoplus_{(i,j)\neq(2,1)}(a_{ij} + r) & k \oplus \bigoplus_{i\neq j}(a_{ij} - r) & \cdots & a_{2n} \oplus \bigoplus_{(i,j)\neq(2,n)}(a_{ij} + r) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} \oplus \bigoplus_{(i,j)\neq(n,1)}(a_{ij} + r) & a_{n2} \oplus \bigoplus_{(i,j)\neq(n,2)}(a_{ij} + r) & \cdots & k \oplus \bigoplus_{i\neq j}(a_{ij} - r) \end{pmatrix} = A$$

□

The set of elementary matrices has an advantage since there are only $n^2 - n$ of them that are required, and they are independent of r. However, they do not commute with the LdlP matrices in general, as the following example shows:

Example 3.8. Let

$$B_1 = \begin{pmatrix} 6 & -15 & -12 \\ -18 & 6 & -14 \\ -11 & -13 & 6 \end{pmatrix} \quad \text{and} \quad E^{11} = \begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty \end{pmatrix}.$$

Then, note that

$$B_1 \otimes E^{11} = \begin{pmatrix} 6 & -\infty & -\infty \\ -18 & -\infty & -\infty \\ -11 & -\infty & -\infty \end{pmatrix},$$

and

$$E^{11} \otimes B_1 = \begin{pmatrix} 6 & -15 & -12 \\ -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty \end{pmatrix} \neq B_1 \otimes E^{11}.$$

This means that Attack 1, if it uses the elementary matrices, is not guaranteed to succeed since the key recovery formula does not necessarily produce the shared secret key. Namely, after obtaining a solution (r_{ijst}) to the linear system, the secret key is computed by:

$$\begin{aligned} K = \bigoplus_{i,j,s,t=1}^{n} r_{ijst} \otimes E^{ij} \otimes V \otimes E^{st} &= \bigoplus_{i,j,s,t=1}^{n} r_{ijst} \otimes E^{ij} \otimes B_1 \otimes W \otimes B_2 \otimes E^{st} \\ &\neq \bigoplus_{i,j,s,t=1}^{n} r_{ijst} \otimes B_1 \otimes E^{ij} \otimes W \otimes E^{st} \otimes B_2, \end{aligned}$$

since E^{ij} and B_1, as well as E^{st} and B_2, do not generally commute, and we know that

$$\begin{aligned} \bigoplus_{i,j,s,t=1}^{n} r_{ijst} \otimes B_1 \otimes E^{ij} \otimes W \otimes E^{st} \otimes B_2 &= B_1 \otimes \left(\bigoplus_{i,j,s,t=1}^{n} r_{ijst} \otimes E^{ij} \otimes W \otimes E^{st} \right) \otimes B_2 \\ &= B_1 \otimes U \otimes B_2 = K_b = K_a. \end{aligned}$$

The elementary LdlP matrices (being LdlP matrices themselves) commute with any LdlP matrix and hence can be used in Attack 1, but their number, when one takes all non-positive integer values of r, is infinite.

In what follows, we check the work of the Kotov-Ushakov attack using elementary matrices, after which we apply the attack based on the greatest solution and r-elementary LdlP matrices, and discuss the applicability and success of some heuristic attacks which were suggested in the previous literature [1, 2, 21]. These heuristic attacks have previously demonstrated promising results against other tropical implementations of the Stickel protocol. For all numerical experiments, unless stated otherwise, the values of k_1, k_2, l_1, l_2 are chosen randomly from $[0, 100]$, while a_1, a_2, b_1, b_2 are selected from $[-100, 0]$, and the entries

of W are from $[-100, 100]$. Note that the behaviour of all these attacks remains largely unchanged when different ranges are selected, except for the vanishing W attack and the dominant W attack for which the effect of changing some of these ranges will be examined.

- **Kotov-Ushakov attack usisng tropical elementary matrices**

  Let us firstly describe a generalized version of Kotov-Ushakov attack that applies to this protocol followed by an evaluation of its performance. Note that any matrix $A \in [2a, a]_n^k$ chosen in the protocol can be represented as a tropical linear combination of elementary matrices with some restrictions on the coefficients (x, y). Therefore, to break the protocol, we need to find

$$X = \bigoplus_{i,j=1}^{n} \left(x_{ij} \otimes E^{ij}\right), Y = \bigoplus_{s,t=1}^{n} \left(y_{st} \otimes E^{st}\right),$$

  Then, Alice's message U can be expressed as

$$U = \bigoplus_{i,j=1}^{n} \left(x_{ij} \otimes E^{ij}\right) \otimes W \otimes \bigoplus_{s,t=1}^{n} \left(y_{st} \otimes E^{st}\right),$$

  or equivalently

$$\bigoplus_{i,j,s,t=1}^{n} x_{ij} \otimes y_{st} \otimes \left(E^{ij} \otimes W \otimes E^{st}\right) = U.$$

  We then denote $R^{ijst} = E^{ij} \otimes W \otimes E^{st}$ and therefore we can write

$$\bigoplus_{i,j,s,t=1}^{n} x_{ij} \otimes y_{st} \otimes \left(R^{ijst}\right)_{\gamma\delta} = U_{\gamma\delta} \quad \forall \gamma, \delta \in [n] \times [n]. \tag{8}$$

  If we additionally denote $z_{ijst} = x_{ij} \otimes y_{st}$, we have

$$\bigoplus_{i,j,s,t=1}^{n} z_{ijst} \otimes \left(R^{ijst}\right)_{\gamma\delta} = U_{\gamma\delta} \quad \forall \gamma, \delta \in [n] \times [n]. \tag{9}$$

  We then similarly scan the whole solution set of this tropical linear system searching for an appropriate solution through the following attack.

  Attack 2 (Kotov-Ushakov attack on tropical Stickel protocol based on LdlP matrices [21])**.**

  1. Compute

$$c_{ijst} = \min_{\gamma,\delta \in [n]} \left(U_{\gamma\delta} - R_{\gamma\delta}^{ijst}\right), \quad S_{ijst} = \arg\min_{\gamma,\delta \in [n]} \left(U_{\gamma\delta} - R_{\gamma\delta}^{ijst}\right).$$

2. Among all minimal covers of $[n] \times [n]$ by S_{ijst}, that is, all minimal subsets $\mathcal{C} \subseteq [n^2] \times [n^2]$ such that
$$\bigcup_{(ijst)\in\mathcal{C}} S_{ijst} = [n] \times [n],$$
find a cover for which the system
$$\begin{aligned} &x_{ij} + y_{st} = c_{ijst}, \quad \text{if } (i,j,s,t) \in \mathcal{C}, \\ &x_{ij} + y_{st} \leqslant c_{ijst}, \quad \text{if otherwise.} \\ &2a_1 \leqslant x_{ij} \leqslant a_1, \quad 2a_2 \leqslant y_{st} \leqslant a_2, \quad \forall i \neq j, \quad s \neq t, \\ &x_{ii} = k_1, \quad y_{ss} = k_2, \quad \forall i, s, \\ &a_1, a_2 \leqslant 0, \quad k_1, k_2 \geq 0. \end{aligned} \tag{10}$$
is solvable.

Note that the attacker in this case encounters a problem similar to attacking the initial tropical Stickel protocol, namely, finding all minimal covers. However, in this case, the number of minimal covers is significantly higher due to the structure of the sets S_{ijst}. Additionally, the sorting algorithm is ineffective, as all minimal covers are of the same size. Consequently, the required computational time is expected to be higher than in the case of the initial tropical Stickel protocol.

Figure 3 illustrates the computational time required to execute Attack 2 on a single instance of the protocol for different dimensions, showing that the attack is impractical due to the excessively high time consumption, even for relatively low-dimensional cases. This inefficiency arises from the extremely high number of minimal covers, which happens because each S_{ijst} contains only one element (as R^{ijst} has only a single finite element). As a result, the total number of minimal covers becomes $(n^2)^{n^2}$, since each entry is covered by n^2 components. Specifically, for each $(\gamma, \delta) \in [n] \times [n]$, there are n^2 sets S_{ijst} that satisfy $(\gamma, \delta) \in S_{ijst}$. This implies that the time complexity of enumerating all minimal covers is exponential, specifically $O(n^{2n^2})$.

- **The greatest solution attack**
We now explore the applicability of an analogous version of Attack 1, which leverages the greatest solution of the linear system to break the protocol. The attack follows a similar structure, which involves finding the greatest solution to the linear system presented below, followed by the key recovery formula. From the numerical experiments we found that for this attack using tropical elementary matrices makes almost no sense, as the degree of success is close to zero. However, we can achieve more success if we use the r-elementary LdlP matrices. The attack aims to find
$$X = \bigoplus_{i=1}^{n} (x_{ii} \otimes I) \oplus \bigoplus_{\substack{i,j=1 \\ i\neq j}}^{n} \big(x_{ij} \otimes F_a^{ij}\big), \qquad Y = \bigoplus_{s=1}^{n} (y_{ss} \otimes I) \oplus \bigoplus_{\substack{s,t=1 \\ s\neq t}}^{n} \big(y_{st} \otimes F_b^{st}\big).$$
for some $a, b\colon -r_{\max} \leq a, b \leq -1$. Then, Alice's message can be expressed as
$$U = X \otimes W \otimes Y,$$

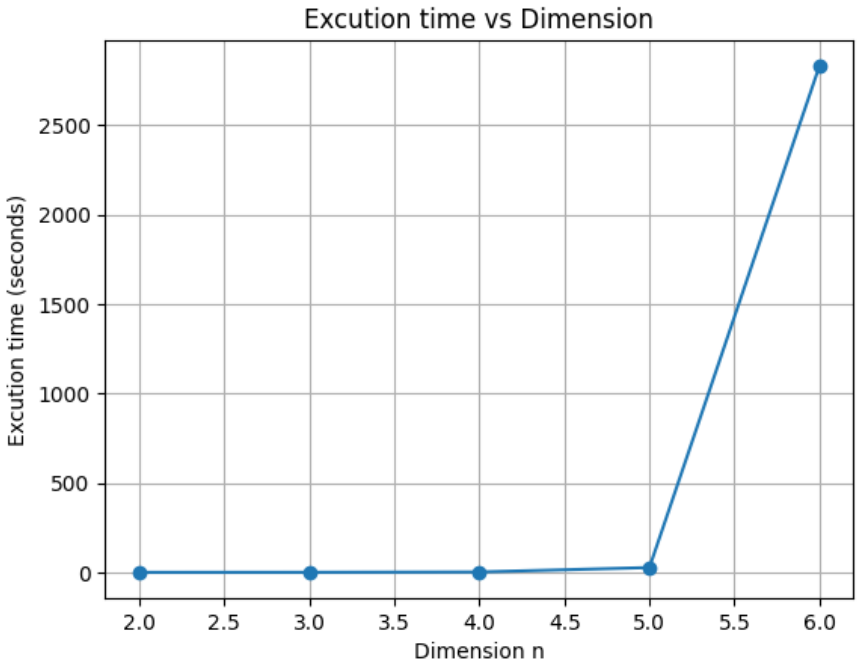

Figure 3: Computional time of Attack 2

or equivalently

$$U = \Big[\bigoplus_{i=1}^{n}(x_{ii}\otimes I) \;\oplus\; \bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}(x_{ij}\otimes F_a^{ij})\Big] \;\otimes\; W \;\otimes\; \Big[\bigoplus_{s=1}^{n}(y_{ss}\otimes I) \;\oplus\; \bigoplus_{\substack{s,t=1\\ s\neq t}}^{n}(y_{st}\otimes F_b^{st})\Big]$$

$$= \bigoplus_{i,j=1}^{n}\big(x_{ij}\otimes G^{ij}\big) \;\otimes\; W \;\otimes\; \bigoplus_{s,t=1}^{n}\big(y_{st}\otimes H^{st}\big)$$

$$= \bigoplus_{i,j,s,t=1}^{n}\big(x_{ij}\otimes y_{st}\big) \;\otimes\; \big(G^{ij}\otimes W\otimes H^{st}\big),$$

where

$$G^{ij} = \begin{cases} I, & i=j,\\ F_a^{ij}, & i\neq j,\end{cases} \qquad H^{st} = \begin{cases} I, & s=t,\\ F_b^{st}, & s\neq t.\end{cases}$$

Also, with $R^{ijstab} = G^{ij}\otimes W\otimes H^{st}$ and $z_{ijst} = x_{ij}\otimes y_{st}$ we have

$$\bigoplus_{i,j,s,t=1}^{n} z_{ijst}\otimes\big(R^{ijstab}\big)_{\gamma\delta} = U_{\gamma\delta} \quad \forall\,\gamma,\delta\in[n]\times[n], \tag{11}$$

where

$$R^{ijstab} = \begin{cases} I \otimes W \otimes I, & i = j,\ s = t, \\ F_a^{ij} \otimes W \otimes I, & i \neq j,\ s = t, \\ I \otimes W \otimes F_b^{st}, & i = j,\ s \neq t, \\ F_a^{ij} \otimes W \otimes F_b^{st}, & i \neq j,\ s \neq t. \end{cases}$$

Attack 3 (The greatest solution attack on tropical Stickel protocol)**.**

1. Compute the greatest solution (c_{ijst}) of system (11).

$$c_{ijst} = \min_{\gamma,\delta \in [n]} \left(U_{\gamma\delta} - R_{\gamma\delta}^{ijstab} \right) \quad \forall i, j, s, t \in [n],\ \forall r \leq -1.$$

2. Compute the shared secret key K.

$$K = \bigoplus_{i,j,s,t=1}^{n} \bigoplus_{a,b=-r_{\max}}^{-1} c_{ijst} \otimes F_a^{ij} \otimes V \otimes F_b^{st}.$$

The attack assumes that Alice and Bob use LdLP matrices with integer r ranging from $-r_{\max}$ to -1 (note that for more clarity it can be also assumed that Alice and Bob generate integer matrices only). It requires time $O(n^7 r_{\max}^2)$. In the first step, computing the greatest solution (c) requires forming $O(n^4 r_{\max}^2)$ terms and performing a matrix multiplication of cost $O(n^3)$ on each term, giving $O(n^4 r_{\max}^2) \cdot O(n^3) = O(n^7 r_{\max}^2)$. The key-recovery formula requires generating $O(n^4 r_{\max}^2)$ intermediate terms and applying an $O(n^3)$ matrix multiplication to each, costing $O(n^7 r_{\max}^2)$. Hence, the overall time complexity of the attack is $O(n^7 r_{\max}^2)$. Figure 4 illustrates the attacker's time consumption for varying values of $r_{\max}$ across different matrix dimensions, and one clear advantage is that for each fixed value of n or r the growth of the time consumption is polynomial. However, our experiments with the key generation also suggest that with fixed n the dependence on r of the time required by Alice and Bob to execute the protocol is very little for $r_{\max}$ in this range. This suggests that by adopting higher parameter values, Alice and Bob can (at least to some extent) resist the attack, with only a slight increase in their computational effort compared to the significantly greater effort required by the attacker.

Figure 5 illustrates the success rate of the attack when the attacker's time is constrained to a reasonable limit, particularly with $r_{\max} = 10$ being the value used. Naturally, the attack achieves a perfect success rate when $r_{\max}$ used in the protocol is lower than 10. However, as the value of $r_{\max}$ in the protocol exceeds 10, the success rate of the attack progressively declines, showing a decreasing trend as the value continues to decrease.

A disadvantage of this attack is that the parameter r used by Alice and Bob can be kept secret unlike the dimension of the matrix and the attacker might overestimate or underestimate this parameter. A possible further idea for the attacker is that they might try to use the generators of the centralizer of all LdlP matrices instead of F_a^{ij} and F_b^{st} in Attack 3 hoping that the number of such generators would be smaller. We give the following standard definition.

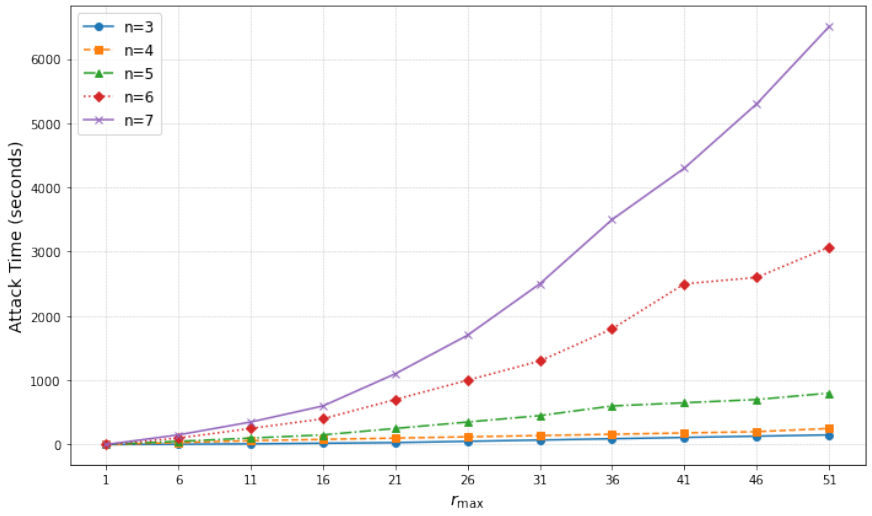

Figure 4: Computational time of Attack 3 and Protocol 2

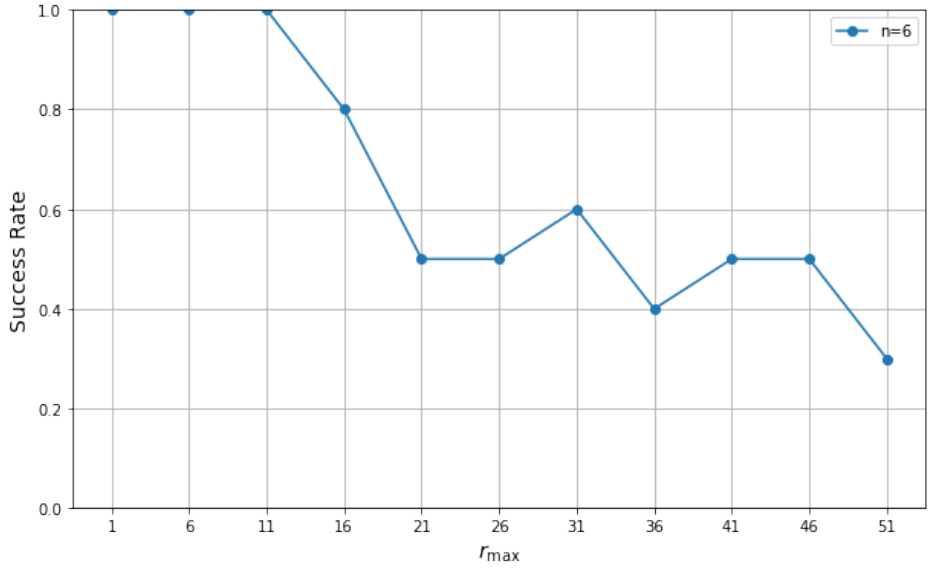

Figure 5: Success rate of Attack 3 using a fixed number of generators (with $r_{\max} = 10$) for $n = 6$: each point is an average of 10 experiments

Definition 3.9. The centralizer of all LdlP matrices of dimension n and negative integer values of r is the set consisting of all matrices X such that $X \otimes A = A \otimes X$ for any such LdlP matrix.

In particular, the following claim can be proved.

Theorem 3.10. *The centralizer of all LdlP matrices of dimension* 2×2 *and negative integer values of* r *is precisely the set of matrices appearing as tropical linear*

combinations of the matrices from the following set:

$$\left\{\begin{bmatrix} 0 & -\infty \\ -\infty & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -\infty & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -\infty \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}\right\} \tag{12}$$

This generating set is minimal with respect to inclusion.

Proof. See Appendix. □

For higher dimensions we conducted a number of experiments trying to find a finite generating set of the integer centralizer (similar to the result of Theorem 3.10) using the tropical double description method of [4]. While such a generating set exists for the centralizer of all matrices in $[2r, r]_n^k$ with fixed n and r, the number of generators rapidly increases with n (we found, e.g., 37 generators already for $n = 3$). Also, based on our experiments, a finite generating set for the centralizer of all LdlP matrices (with fixed n and arbitrary negative integers r) is unlikely to exist.

Conjecture 3.11. The integer centralizer of all LdlP matrices of fixed dimension n and arbitrary negative integer values of r does not admit a finite generating set for any $n \geq 3$.

- **The single cover heuristic attack**
  Kotov and Ushakov observed in their experiment [18] that smaller minimal covers are significantly more likely to "work". A heuristic attack that construct a small sized single minimal cover by iteratively selecting the largest $S_{\alpha\beta}$ until all elements of $[n] \times [n]$ are covered showed to be highly effective against multiple implementations of the Stickel protocol [1]. An adaptation of this attack on protocol 2 is described in Attack 4.

  Attack 4 (The single minimal cover heuristic on tropical Stickel protocol based on LdlP matrices)**.**

  1. Compute

  $$c_{ijst} = \min_{\gamma,\delta\in[n]} \left(U_{\gamma\delta} - R^{ijst}_{\gamma\delta}\right), \quad S_{ijst} = \arg\min_{\gamma,\delta\in[n]} \left(U_{\gamma\delta} - R^{ijst}_{\gamma\delta}\right).$$

  2. For each uncovered $(\gamma, \delta) \in [n] \times [n]$, select the largest S_{ijst} that includes it, and add the indices i, j, s, t to the cover.
  3. Solve the system

  $$\begin{aligned} &x_{ij} + y_{st} = c_{ijst}, \quad \text{if } (i,j,s,t) \text{ is in the cover,} \\ &x_{ij} + y_{st} \leqslant c_{ijst}, \quad \text{if otherwise,} \\ &2a_1 \leqslant x_{ij} \leqslant a_1, \quad 2a_2 \leqslant y_{st} \leqslant a_2, \quad \forall i \neq j, \quad s \neq t, \\ &x_{ii} = k_1, \quad y_{ss} = k_2, \quad \forall i, s, \\ &a_1, a_2 \leqslant 0, \quad k_1, k_2 \geq 0. \end{aligned}$$

This attack also has a polynomial time complexity since it firstly finds the greatest solution of the linear system which requires $O(n^4)$ operations. The second step, extracting the cover, requires iterating over n^2 pairs and selecting the largest set from n^4 possible sets, resulting in a time complexity of $O(n^6)$. Finally, solving the linear system is known to be polynomially solvable.

Figure 6 shows the success rate of this attack over 10 trails for each dimension, which performs poorly probably due to the fact that all minimal covers of system (9) are of equal size (specifically n^2 since each S_{ijst} contains only a single element). As a result, there is no smaller cover that offers a higher probability of solving the linear system (10). Additionally, the large number of minimal covers,as explained in the generalized Kotov-Ushakov attack (Attack 2), probably further reduces the likelihood of finding an appropriate cover.

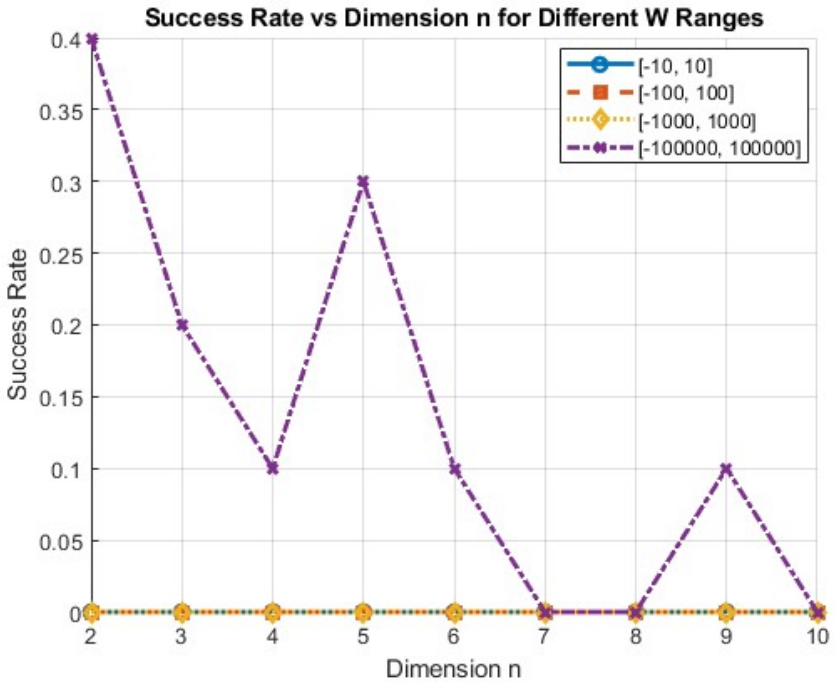

Figure 6: Success rate of Attack 4

- **Tropical Shpilrain attack**
  We now explore the effectiveness of the tropical version of Shpilrain attack [24], building on the approach outlined in [2]. Similar to the other heuristics, this attack aims to avoid the impracticality of the guaranteed attack (Attack 2). The objective of the attack is to find X and Y such that

$$X \otimes W \otimes Y = U$$

where X and Y follow the forms of $[2a_1, a_1]_n^{k_1}$ and $[2a_2, a_2]_n^{k_2}$ respectively. Then, a Mixed-Integer Linear Program (MILP) can be formulated by converting the disjunctive

constraints into linear constraints with Boolean variables [8], and solved using a MILP solver (e.g. [15]). In particular, with x_{ij}, w_{ij}, y_{ij} and u_{ij} being respectively the entries of X, W, Y and U, we have

$$\max_{k,l\in[n]}(x_{ik} \otimes w_{kl} \otimes y_{lj}) = u_{ij} \quad \forall (i,j) \in [n] \times [n],$$

which can be represented as the following set of inequalities

$$x_{ik} \otimes w_{kl} \otimes y_{lj} \leq u_{ij} \quad \forall i,j,k,l \in [n],$$

and with M being a sufficiently large number

$$x_{ik} \otimes w_{kl} \otimes y_{lj} + (1 - z_{klij})M \geq u_{ij} \quad \forall i,j,k,l \in [n],$$
$$\sum_{k} z_{klij} = 1, \quad z_{kij} \in \{0,1\} \quad \forall i,j,k,l \in [n].$$

The details of the attack is described below in Attack 5.

Attack 5 (Shpilrain attack on tropical Stickel protocol based on LdlP matrices)**.**

Solve the following system using a MILP solver

$$\begin{aligned}
&x_{ik} + w_{kl} + y_{lj} \leq u_{ij} \quad \forall i,j,k,l \in [n],\\
&x_{ik} + w_{kl} + y_{lj} + (1 - z_{klij})M \geq u_{ij} \quad \forall i,j,k,l \in [n],\\
&z_{klij} \in \{0,1\},\\
&\sum_{k,l} z_{klij} = 1 \quad \forall i,j \in [n],
\end{aligned}$$

$$\begin{aligned}
&2a_1 \leqslant x_{ij} \leqslant a_1, \quad 2a_2 \leqslant y_{st} \leqslant a_2, \quad \forall i \neq j, \quad s \neq t,\\
&x_{ii} = k_1, \quad y_{ss} = k_2, \quad \forall i,s,\\
&a_1, a_2 \leqslant 0, \quad k_1, k_2 \geq 0.
\end{aligned}$$

This attack has a perfect success rate and shows significantly better time efficiency compared to the Kotov-Ushakov attack (Attack 2), as shown in Figure 7. However, one major limitation of this attack is its high memory usage, which increases with the dimension. The attack demands a substantial amount of memory to encode all the required equations, and in environments like Matlab, it becomes impractical for dimensions larger than 13. Specifically, the attack requires encoding $2n^4+n^2$ equations with $n^4 + 2n^2 + 4$ variables. This also shows that Protocol 2 offers greater resistance to the Shpilrain attack compared to the initial tropical implementation (Protocol 1) since the computational time of the attack increases with the dimension, while in the initial implementation, the attack time remains unchanged, as it does not depend on the polynomial degrees used in the protocol as presented in [2]. The worst-case complexity of solving this MILP grows exponentially with the number binary variables (n^4 binary variables) as it relies on the branch-and-bound method. However, modern solvers improve performance through the use of relaxations, pre-solving techniques, and heuristics.

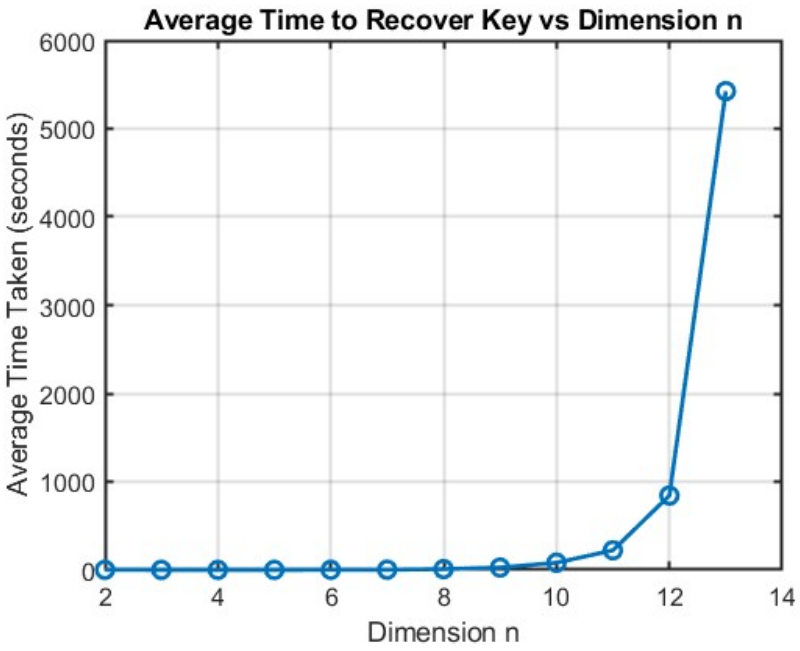

Figure 7: Computational time of Attack 5

- **Vanishing and dominant W heuristic attacks**
  It is occasionally possible to recover the shared secret key using only the public parameters by leveraging the theory of vanishing or dominant W, as outlined in [21], when applicable. The two heuristic approaches for this are illustrated in the following two attacks, where w_{st} denotes the largest entry in W.

  Attack 6 (Vanishing W attack on tropical Stickel protocol based on LdlP matrices)**.**

  1. Compute $l_1 \otimes l_2 = v_{st} \otimes -w_{st}$ and $k_1 \otimes k_2 = u_{st} \otimes -w_{st}$.
  2. Compute the key K as $K = l_1 \otimes l_2 \otimes U \oplus k_1 \otimes k_2 \otimes V$.

  Attack 7 (Dominant W attack on tropical Stickel protocol based on LdlP matrices)**.**

  1. Compute $l_1 \otimes l_2 = v_{st} \otimes -w_{st}$ and $k_1 \otimes k_2 = u_{st} \otimes -w_{st}$.
  2. Compute the key K as $K_{ij} = -w_{st} \otimes (v_{st} \otimes u_{ij} \oplus u_{st} \otimes v_{ij} \oplus u_{it} \otimes v_{sj} \oplus v_{it} \otimes u_{sj})$.

  Both attacks involve matrix shifting (scalar with matrix tropical multiplication) and matrix addition, resulting in a time complexity of $O(n^2)$. The success rate of the two attacks over 100 trials is illustrated in Figure 8. A notable trend is observed: when one attack performs poorly, the other tends to perform well across different ranges of W. As a result, the overall combined success rate is generally high. However, there are specific ranges of W where both attacks underperform, suggesting that Alice and Bob can still effectively resist these two heuristics by carefully selecting certain values of W. For example, Figure 9 highlights a range of W where the performance of both attacks noticeably weakens.

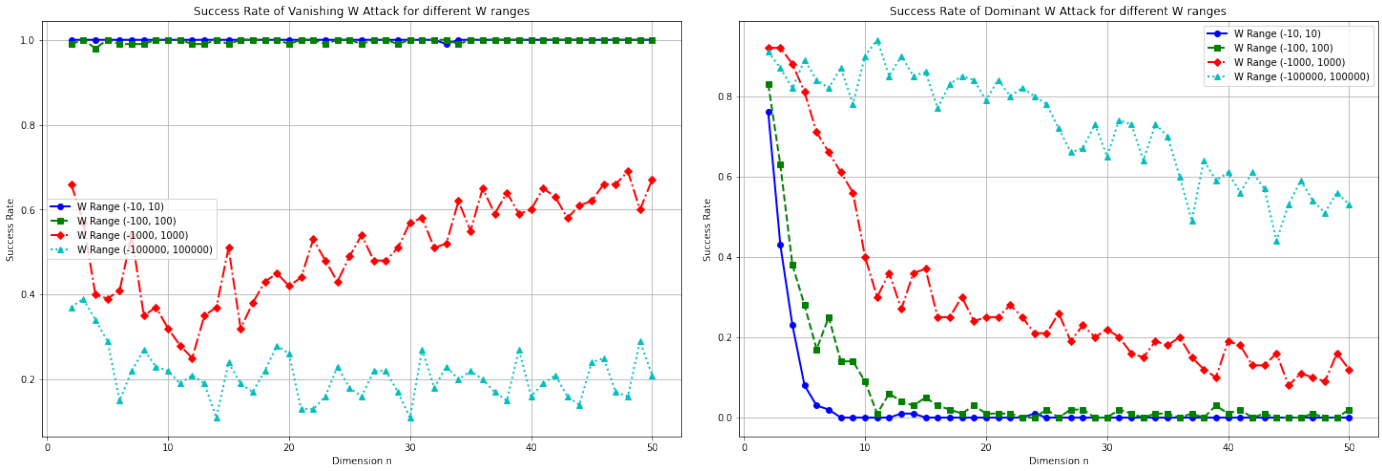

Figure 8: Success rate of Attack 6 and Attack 7

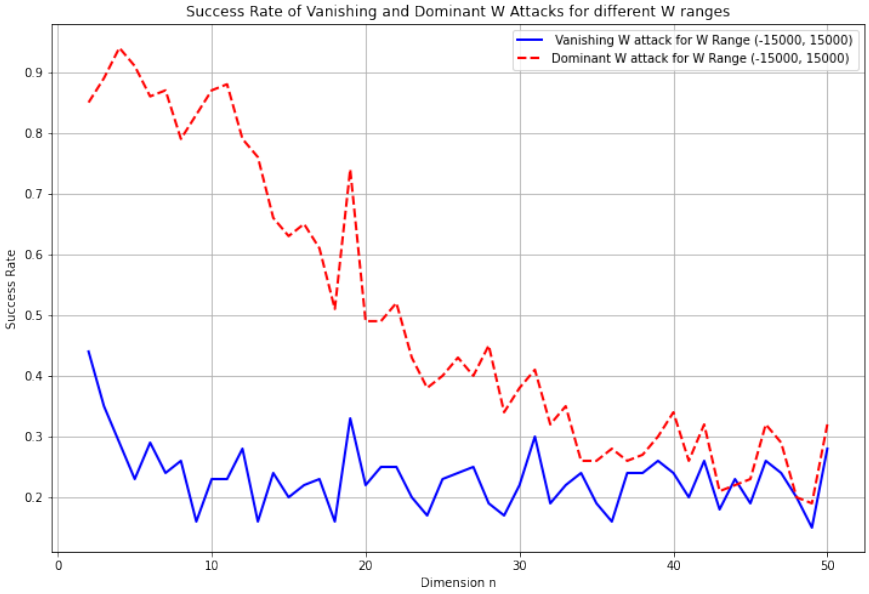

Figure 9: Suboptimal performance of Attack 6 and Attack 7

4 Conclusion

The Kotov-Ushakov attack could now be considered largely obsolete by the introduction of the new attack (Attack 1), which can replace it to attack various implementations of the Stickel protocol. However, the Kotov-Ushakov may still be valuable against some variants that are resistant to the new attack, though it would likely be inefficient due to the significant computations involved in enumerating all solutions of the underlying linear system. For the Kotov-Ushakov attack to remain relevant, new classes of commuting matrices over semirings have to be found. For such cases, the Kotov-Ushakov attack might be the only feasible attack.

While Attack 1 offers a clear advantage over the Kotov-Ushakov attack, it still encounters

some of the same challenges. Firstly, in the case of the Stickel protocol based on polynomials, Alice and Bob can use sparse polynomials with sufficiently large degree D. This is especially easy for them in the case of the implementation based on Jones matrices [16, 21] since they would use rational exponents with high denominator, and the corresponding deformations $A^{(\alpha)}$ and $B^{(\beta)}$ are easy to compute. Secondly, there may still exist semirings over which $A \otimes x = b$ is hard to solve, and in such cases, the new attack is not applicable. Identifying such semirings, however, requires further exploration.

In the case of the tropical semiring, the Stickel protocol based on LdlP matrices is more resistant to the new attack. It also resists the Kotov-Ushakov attack, primarily due to its impracticality as it requires enumerating an exceedingly high number of minimal solutions in this case. Moreover, other heuristic attacks that previously demonstrated promising results against other variants of Stickel protocol showed only limited success here. This indicates that a further cryptanalysis of the tropical Stickel protocol based on LdlP matrices might be of interest. A further idea is to make use of the centralizer of such matrices, as the attacker could aim to find matrices X and Y from that centralizer instead of trying to generate appropriate LdlP matrices. We will also leave this topic for further research.

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A Proof of Theorem 3.10

Before beginning the proof, we first recall some fundamental concepts concerning bases and extremals of tropical cones, based on [6],[12],[7]. A *tropical cone* K, defined as a subset of $(\mathbb{R} \cup \{-\infty\})^n$ closed under componentwise maximization and tropical scalar multiplication, is said to be generated by a set of vectors in $S \subseteq (\mathbb{R} \cup \{-\infty\})^n$, notation $K = \text{span}(S)$ if each $u \in K$ appears as a *tropical linear combination* $u = \bigoplus_{v \in S} \lambda_v \otimes v$, where only a finite number of scalars λ_v are finite. A *basis* for such a cone is a minimal set of generators (with respect to inclusion), meaning that no element can be expressed as a tropical linear combination of others. *Extremals* of a tropical cone K, defined as vectors in K that cannot be expressed as componentwise maxima of other vectors, form a basis. This basis for K is essentially unique, modulo the tropical scalar multiplication of its elements. The essentially unique basis consisting of extremals is known to exist for tropical cones which are closed in the usual product topology of $(\mathbb{R} \cup \{-\infty\})^n$ (the tropical Minkowski theorem [12],[7]), and this applies to the centralizer of LdlP matrices as this is a closed tropical cone. Indeed, the centralizer is defined by the tropical linear equations $A \otimes X = X \otimes A$, hence this is a cone, and the fact that it is closed follows since the tropical arithmetic operations are continuous.

The geometry of tropical cones is closely related to the concept of multiorder convexity, which employs a family of preorder relations $\preceq_i$ for i belonging to some finite set. As observed, e.g., in [12],[7] and further emphasized in [23], for the tropical space $(\mathbb{R} \cup \{-\infty\})^n$ one can introduce relations $\preceq_i$ for $i = 1, \ldots, n$. For each i this relation compares vectors u, v with finite ith coordinates by normalizing their components relative to the i-th coordinate: $u \preceq_i v$ holds if $u_j - u_i \leq v_j - v_i$. The multiorder principle asserts that a vector y belongs to $K = \operatorname{span}(S)$ if and only if, for every finite coordinate y_i, there exists a generator $v \in S$ such that $v \preceq_i y$. This principle generalizes set-covering conditions and underpins key results like the Tropical Carathéodory theorem, which states that any vector $y \in K$ can be expressed as a max combination of at most $|\operatorname{supp}(y)|$ generators from S. Extremals, in this context, are characterized by their minimality under these preorders: a vector is an extremal in a tropical cone $K = \operatorname{span}(S)$ precisely when it is minimal in S (equivalently, minimal in K) with respect to $\preceq_i$ for some i in its support.

With these preliminaries in place, we now turn to the proof of the claim. Let

$$M = \begin{bmatrix} k & a \\ b & k \end{bmatrix} \in [2r, r]_2^k,$$

then, we note that by Proposition 3.7 any LdlP matrix M can be generated as a tropical linear combination of

$$A_{2r}^r = \begin{bmatrix} 0 & r \\ 2r & 0 \end{bmatrix}, \quad A_r^{2r} = \begin{bmatrix} 0 & 2r \\ r & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 0 & -\infty \\ -\infty & 0 \end{bmatrix}.$$

For a set of matrices $S \subseteq (\mathbb{R} \cup \{-\infty\})^{2\times 2}$, denote the centralizer of S by $C(S)$. Then, since we have the following inclusions

$$\{A_{2r}^r,\, A_r^{2r}\} \subseteq [2r, r]_2^k \subseteq \operatorname{span}\{I,\, A_{2r}^r,\, A_r^{2r}\},$$

we have the reverse inclusions for centralizers

$$C\left(\{I, A_{2r}^r, A_r^{2r}\}\right) \subseteq C\left([2r, r]_2^k\right) \subseteq C\left(\{A_{2r}^r, A_r^{2r}\}\right).$$

Since the left-hand side and the right-hand sides coincide, we obtain that the centralizer of $[2r, r]_2^k$ can be characterized as the solution set of the system:

$$A_{2r}^r \otimes X = X \otimes A_{2r}^r$$
$$A_r^{2r} \otimes X = X \otimes A_r^{2r}$$

which can be written as the following set of equations

$$\max(x_{11},\, x_{21} + r) = \max(x_{11},\, x_{12} + 2r) \tag{E1}$$
$$\max(x_{12},\, x_{22} + r) = \max(x_{11} + r,\, x_{12}) \tag{E2}$$
$$\max(x_{11} + 2r,\, x_{21}) = \max(x_{21},\, x_{22} + 2r) \tag{E3}$$
$$\max(x_{12} + 2r,\, x_{22}) = \max(x_{21} + r,\, x_{22}) \tag{E4}$$
$$\max(x_{11},\, x_{21} + 2r) = \max(x_{11},\, x_{12} + r) \tag{E5}$$
$$\max(x_{12},\, x_{22} + 2r) = \max(x_{11} + 2r,\, x_{12}) \tag{E6}$$
$$\max(x_{11} + r,\, x_{21}) = \max(x_{21},\, x_{22} + r) \tag{E7}$$
$$\max(x_{12} + r,\, x_{22}) = \max(x_{21} + 2r,\, x_{22}) \tag{E8}$$

Hence the centralizer of the set of all LdlP matrices for all integer values $r \leq -1$ is the solution set of the (infinite) system of equations (E1)-(E8) written for all such values of r.

Let us now examine the candidate minimal generators proposed in the claim (further referred to as *candidate set*):

$$G_1 = \begin{bmatrix} 0 & -\infty \\ -\infty & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 1 \\ -\infty & 0 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 & -\infty \\ 1 & 0 \end{bmatrix},$$
$$G_4 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad G_5 = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$$

and determine whether they indeed represent the extremals of the solution set of this system. Following the approach of [7], this involves firstly verifying that each candidate satisfies the system (i.e., belongs to the centralizer), and then checking their minimality with respect to the partial order $\preceq_{ij}$, where $A \preceq_{ij} B$ denotes $A - a_{ij} \leq B - b_{ij}$, and then by verifying that for every matrix X in the centralizer, there exists a generator G among these such that $G \preceq_{ij} X$. Let us first observe that for each matrix G_l in the candidate set, there exists at least one coordinate (i, j) such that, for any other matrix G_k in this set, we have $G_k \not\preceq_{ij} G_l$. The corresponding distinguishing coordinates are as follows: $(1, 1)$ and $(2, 2)$ for G_1; $(1, 2)$ for G_2; $(2, 1)$ for G_3; $(2, 2)$ for G_4; and $(1, 1)$ for G_5.

Now, to verify that the matrices in the candidate set are indeed the extremals of the centralizer, we first observe that each of these matrices satisfies the above system (E1)-(E8) for any r. This can be verified by direct substitution, and thus they are part of the centralizer. Then, to verify their minimality, we assume $r = -1$ (as we will see, any smaller values of r do not have to be considered) and check the following cases

- Case $x_{11} = x_{22}$:
  - We clearly have $G_1 \preceq_{11} X$ and $G_1 \preceq_{22} X$, so it only remains to find good generators for $\preceq_{12}$ and $\preceq_{21}$: see below.
  - We have $\begin{bmatrix} -1 & -\infty \\ 0 & -1 \end{bmatrix} \preceq_{21} X$, and this can be shown by contradiction. Indeed, assume this relation does not hold, meaning that we assume $x_{11} < -1$, and $x_{21} = 0$ (note that relation $\preceq_{21}$ is insensitive to scalar shifts of X and this allows us to assume $x_{21} = 0$ and further $x_{11} < -1$ to seek a contradiction). Then, from equation (E1) we obtain $x_{12} = 1$ and substituting into (E4) leads to a contradiction. Alternatively, we can assume $x_{22} < -1$ and $x_{21} = 0$ and use equation (E4) to obtain $x_{12} = 1$. Substituting this value into (E8) then also leads to a contradiction.
  - We have $\begin{bmatrix} -1 & 0 \\ -\infty & -1 \end{bmatrix} \preceq_{12} X$, which also can be proved by contradiction. Indeed, assume that this relation does not hold, which means that we can assume $x_{11} < -1$ and $x_{12} = 0$. We then obtain $x_{21} = 1$ from (E5), and substituting it into (E1) leads to a contradiction. Alternatively, we can assume $x_{22} < -1$ and $x_{12} = 0$, and then from equation (E8) we obtain $x_{21} = 1$. Substituting this value into (E4) then also leads to a contradiction.

- Case $x_{11} > x_{22}$:
  - We have $\begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix} \preceq_{11} X$, and we can prove this by contradiction. Namely, taken $x_{11} = 0$ assume $x_{12} < -1$ or $x_{21} < -1$ or $x_{22} < -2$. With $x_{12} < -1$, we can check equation (E2) to get $x_{22} = 0$, contradicting $x_{11} > x_{22}$. Similarly, with $x_{21} < -1$, we can check equation (E7) to get $x_{22} = 0$, contradicting $x_{11} > x_{22}$. Lastly, with $x_{22} < -2$, we can check (E4) and (E8), we then must have $x_{12} < -1$, and $x_{21} < -1$ which reduces to the previous scenarios and again leads to a contradiction.
  - We have $\begin{bmatrix} -1 & -\infty \\ 0 & -1 \end{bmatrix} \preceq_{21} X,$. Indeed, to prove this we assume $x_{11} < -1$, and $x_{21} = 0$ by the partial order. We then use equations (E1) and (E5) like in the case $x_{11} = x_{22}$ to get a contradiction based on the value $x_{12} = 1$ inferred from (E1). When $x_{22} < -1$, we use equations (E4) and (E8) to get a contradiction based on the value of $x_{12} = 1$ inferred from (E4).
  - We have $\begin{bmatrix} -1 & 0 \\ -\infty & -1 \end{bmatrix} \preceq_{12} X,$. Indeed, to prove this we first assume $x_{11} < -1$ and $x_{12} = 0$. We then use equations (E1) and (E5) to get a contradiction based on the value $x_{21} = 1$ inferred from (E5). When $x_{22} < -1$, we use equations (E4) and (E8) to get a contradiction based on the value of $x_{21} = 1$ inferred from (E8).

- Case $x_{11} < x_{22}$:
  - We have $\begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix} \preceq_{22} X$, and we can prove this by contradiction. Namely, taken $x_{22} = 0$ assume $x_{12} < -1$ or $x_{21} < -1$ or $x_{11} < -2$. With $x_{12} < -1$, we can check equation (E2) to get $x_{11} = 0$, contradicting $x_{11} < x_{22}$. Similarly, with $x_{21} < -1$, we can check equation (E7) to get $x_{11} = 0$, contradicting $x_{11} < x_{22}$. Lastly, with $x_{11} < -2$, we can check (E4) and (E8), we then must have $x_{12} < -1$, and $x_{21} < -1$ which reduces to the previous scenarios and again leads to a contradiction.
  - We also have $\begin{bmatrix} -1 & -\infty \\ 0 & -1 \end{bmatrix} \preceq_{21} X$. The proof follows the same lines as in the case $x_{11} > x_{22}$.
  - We also have $\begin{bmatrix} -1 & 0 \\ -\infty & -1 \end{bmatrix} \preceq_{12} X,$. The proof follows the same lines as in the case $x_{11} > x_{22}$.