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A MORTAR FINITE ELEMENT METHOD USING DUAL SPACES FOR THE LAGRANGE MULTIPLIER

BARBARA I. WOHLMUTH*

Abstract. The mortar finite element method allows the coupling of different discretization schemes and triangulations across subregion boundaries. In the original mortar approach the matching at the interface is realized by enforcing an orthogonality relation between the jump and a modified trace space which serves as a space of Lagrange multipliers. In this paper, this Lagrange multiplier space is replaced by a dual space without losing the optimality of the method. The advantage of this new approach is that the matching condition is much easier to realize. In particular, all the basis functions of the new method are supported in a few elements. The mortar map can be represented by a diagonal matrix; in the standard mortar method a linear system of equations must be solved. The problem is considered in a positive definite nonconforming variational as well as an equivalent saddle-point formulation.

Key words. mortar finite elements, Lagrange multiplier, dual norms, non-matching triangulations, a priori estimates

AMS subject classifications. 65N15, 65N30, 65N55

1. Introduction. Discretization methods based on domain decomposition techniques are powerful tools for the numerical approximation of partial differential equations. The coupling of different discretization schemes or of nonmatching triangulations along interior interfaces can be analyzed within the framework of the mortar methods [6, 7]. In particular, for time dependent problems, diffusion coefficients with jumps, problems with local anisotropies as well as corner singularities, these domain decomposition techniques provide a more flexible approach than standard conforming formulations. One main characteristic of such methods is that the condition of pointwise continuity across the interfaces is replaced by a weaker one. In a standard primal approach, an adequate weak continuity condition can be expressed by appropriate orthogonality relations of the jumps of the traces across the interfaces of the decomposition of the domain [6, 7]. If a saddle point formulation arising from a mixed finite element discretization is used, the jumps of the normal components of the fluxes are relevant [29]. To obtain optimal results, the consistency error should be at least of the same order as the best approximation error. Most importantly, the quality of the a priori error bounds depends strongly on the choice of weak continuity conditions at the interfaces.

Section 2 contains a short overview of the mortar finite element method restricted to the coupling of P_1-Lagrangian finite elements and a geometrically conforming subdivision of the given region. We briefly review the definition of the discrete Lagrange multiplier space and the weak continuity condition imposed on the product space as it is given in the literature. In Section 3, we introduce local dual basis functions, which span the modified Lagrange multiplier space. We also give an explicit formula of projection-like operators and establish stability estimates as well as approximation properties. Section 4 is devoted to the proof of the optimality of the modified nonconforming variational problem. It is shown that we can define a nodal basis function satisfying the constraints at the interface and which at the same time has local support. This is a great advantage of this modified method compared with the

* Math. Institut, Universität Augsburg, Universitätsstr. 14, D-86 159 Augsburg, Germany. Email: wohlmuth@math.uni-augsburg.de, http://wwwhoppe.math.uni-augsburg.de/~wohlmuth

standard mortar methods. Central results such as uniform ellipticity, approximation properties and consistency error are given in separate lemmas. A saddle point formulation, which is equivalent to these nonconforming variational problems is considered in Section 5. Here, the weak continuity condition at the interface enters explicitly in the variational formulation. As in the standard mortar case, we obtain a priori estimates for the discretization error for the Lagrange multiplier. Here, we analyze the error in the dual norm of $H_{00}^{1/2}$, as well as in a mesh dependent L^2-norm. Finally, in Section 6, numerical results indicate that the discretization errors are comparable with the errors obtained when using the original mortar method.

2. Problem setting. We consider the following model problem

$$\begin{aligned} Lu := -\mathrm{div}\,(a\nabla u) + b\,u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma := \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded, polygonal domain in $\mathbb{R}^2$, and that $f \in L^2(\Omega)$. Furthermore, we assume $a \in L^\infty(\Omega)$ to be an uniformly positive function and $0 \le b \in L^\infty(\Omega)$.

We will consider a non-overlapping decomposition of Ω into polyhedral subdomains Ω_k, $1 \le k \le K$,

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k \quad \text{with } \Omega_l \cap \Omega_k = \emptyset, \quad k \ne l.$$

Each subdomain Ω_k is associated with a family of shape regular simplicial triangulations $\mathcal{T}_{h_k}$, $h_k \le h_{k;0}$, where h_k is the maximum of the diameters of the elements in $\mathcal{T}_{h_k}$. The sets of vertices and edges of the subdomains Ω_k and of Ω are denoted by $\mathcal{P}_{h_k}$, $\mathcal{E}_{h_k}$, and $\mathcal{P}_h$, $\mathcal{E}_h$, respectively. We use P_1-conforming finite elements $S_1(\Omega_k, \mathcal{T}_{h_k})$ on individual subdomains and enforce the homogeneous Dirichlet boundary conditions on $\partial\Omega \cap \partial\Omega_k$.

We restrict ourselves to the geometrical conforming situation where the intersection between the boundary of any two different subdomains $\partial\Omega_l \cap \partial\Omega_k$, $k \ne l$, is either empty, a vertex, or a common edge. We only call it an interface in the latter case. The mortar method is characterized by introducing Lagrange multiplier spaces given on the interfaces. A suitable triangulation on the interface is necessary for the definition of a discrete Lagrange multiplier space. Each interface $\partial\Omega_l \cap \partial\Omega_k$ is associated with a one dimensional triangulation, inherited either from $\mathcal{T}_{h_k}$ or from $\mathcal{T}_{h_l}$. In general, these triangulations do not coincide. The interface in question will be denoted by Γ_{kl} and Γ_{lk} if its triangulation is given by that of Ω_k and Ω_l, respectively. We call the inherited one dimensional triangulation on Γ_{kl} and Γ_{lk}, Σ_{kl} and Σ_{lk}, respectively with the elements of Σ_{kl} and Σ_{lk} being edges of $\mathcal{T}_{h_k}$ and $\mathcal{T}_{h_l}$, respectively. We remark that geometrically Γ_{lk} and Γ_{kl} are the same.

Thus, each $\partial\Omega_k$ can be decomposed, without overlap, into

$$\partial\Omega_k = \bigcup_{l \in \bar{\mathcal{M}}(k)} \bar{\Gamma}_{kl},$$

where $\bar{\mathcal{M}}(k)$ denotes the subset of $\{1, 2, \ldots, K\}$ such that $\partial\Omega_l \cap \partial\Omega_k$ is an interface for $l \in \bar{\mathcal{M}}(k)$. The union of all interfaces $\mathcal{S}$ can be decomposed uniquely in

$$\mathcal{S} = \bigcup_{k=1}^{K} \bigcup_{l \in \mathcal{M}(k)} \bar{\Gamma}_{lk}.$$

Here, $\mathcal{M}(k) \subset \bar{\mathcal{M}}(k)$ such that for each set $\{k,l\}$, $1 \le k \le K$, $l \in \bar{\mathcal{M}}(k)$ either $l \in \mathcal{M}(k)$ or $k \in \mathcal{M}(l)$ but not both. The elements of $\{\Gamma_{kl} \mid 1 \le k \le K,\ l \in \mathcal{M}(k)\}$ are called the mortars and those of $\{\Gamma_{lk} \mid 1 \le k \le K,\ l \in \mathcal{M}(k)\}$ the non-mortars. The choice of mortars and non-mortars is arbitrary but fixed. We note that the discrete Lagrange multiplier space will be associated with the non-mortars. To simplify the analysis, we will assume that the coefficients a and b are constant in each subdomain, with $a_k := a_{|\Omega_k}$, $1 \le k \le K$.

It is well known that the unconstrained product space

$$X_h := \prod_{k=1}^{K} S_1(\Omega_k, \mathcal{T}_{h_k})$$

is not suitable as a discretization of (2.1). We also note that in case of non-matching meshes at the interfaces, it is in general not possible to construct a global continuous space with optimal approximation properties. It is shown [6, 7] that weak constraints across the interface are sufficient to guarantee an approximation and consistency error of order h if the weak solution u is smooth enough. The nonconforming formulation of the mortar method is given by:

Find $\tilde{u}_h \in \widetilde{V}_h$ such that

$$a(\tilde{u}_h, v_h) = f(v_h), \quad v_h \in \widetilde{V}_h \tag{2.2}$$

where $a(v,w) := \sum_{k=1}^{K} \int_{\Omega_k} a \nabla v \cdot \nabla w + bv\, w\, dx$, $v, w \in \prod_{k=1}^{K} H^1(\Omega_k)$ and $f(v) := \int_\Omega f v\, dx$, $v \in L^2(\Omega)$. Here, the global space $\widetilde{V}_h$ is defined by

$$\widetilde{V}_h := \{v \in X_h \mid\ b(v,\mu) = 0,\ \mu \in \widetilde{M}_h\},$$

where the bilinear form $b(\cdot,\cdot)$ is given by the duality pairing on $\mathcal{S}$

$$b(v,\mu) := \sum_{k=1}^{K} \sum_{l \in \mathcal{M}(k)} \langle [v], \mu \rangle_{\Gamma_{lk}}, \quad v \in \prod_{k=1}^{K} H^1(\Omega_k),\ \mu \in \prod_{k=1}^{K} \prod_{l \in \mathcal{M}(k)} \left(H^{\frac{1}{2}}(\Gamma_{lk}) \right)'$$

and $[v] := v_{|\Omega_l} - v_{|\Omega_k}$. Here, $(H^{\frac{1}{2}}(\Gamma_{lk}))'$ denotes the dual space of $H^{\frac{1}{2}}(\Gamma_{lk})$.

Of crucial importance is the suitable choice of $\widetilde{M}_h$ in (2.2)

$$\widetilde{M}_h := \prod_{k=1}^{K} \prod_{l \in \mathcal{M}(k)} \widetilde{M}_h(\Gamma_{lk}),$$

where in general the local space $\widetilde{M}_h(\Gamma_{lk})$ is chosen as a modified trace space of finite element functions in $S_1(\Omega_l; \mathcal{T}_{h_l})$. Appropriate spaces [6, 7] can be found satisfying

$$\dim \widetilde{M}_h(\Gamma_{lk}) = \dim\, (H_0^1(\Gamma_{lk}) \cap W_h(\Gamma_{lk})), \quad \widetilde{M}_h(\Gamma_{lk}) \subset W_h(\Gamma_{lk})$$

where $W_h(\Gamma_{lk})$ is the trace space of $S_1(\Omega_l; \mathcal{T}_{h_l})$:

$$W_h(\Gamma_{lk}) := \{v \in C^0(\Gamma_{lk}) \mid v = w_{|\Gamma_{lk}}, w \in S_1(\Omega_l, \mathcal{T}_{h_l})\}.$$

$\widetilde{M}_h(\Gamma_{lk})$ is a subspace of $W_h(\Gamma_{lk})$ of codimension 2 and given by

$$\begin{aligned} \widetilde{M}_h(\Gamma_{lk}) \quad := \quad & \{v \in C^0(\Gamma_{lk}) \mid v = w_{|\Gamma_{lk}}, w \in S_1(\Omega_l, \mathcal{T}_{h_l}), \\ & v_{|e} \in P_0(e), e \in \mathcal{E}_{h_l} \text{contains an endpoint of } \Gamma_{lk}\} \end{aligned}$$

and $N_{lk} := \dim \widetilde{M}_h(\Gamma_{lk}) = N_e - 1$ where N_e denotes the number of elements in Σ_{lk}. Here, we assume that $N_e \geq 2$. The nodal basis functions $\{\phi_i\}_{i=1}^{N_{lk}}$ of $\widetilde{M}_h(\Gamma_{lk})$ are associated with the interior vertices p_j, $1 \leq j \leq N_{lk}$ of Γ_{lk} and given by

$$\phi_i(p_j) = \delta_{ij}.$$

The space $\widetilde{M}_h(\Gamma_{lk})$ and its nodal basis functions $\{\phi_i\}_{i=1}^{N_{lk}}$ are illustrated in Figure 2.1; for a detailed analysis of $\widetilde{M}_h$ see [4, 6, 7, 8].

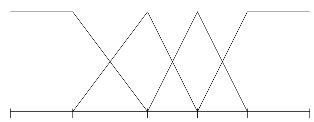


FIG. 2.1. *Lagrange multiplier space*

Let us remark that continuity was imposed at the vertices of the decomposition in the first papers about mortar methods. However, this condition can be removed without loss of stability. Both these settings guarantee uniform ellipticity of the bilinear form $a(\cdot,\cdot)$ on $\widetilde{V}_h \times \widetilde{V}_h$, as well as a best approximation error and a consistency error of $\mathcal{O}(h)$ [6, 7]. Combining the Lemmas of Lax Milgram and Strang, it can be shown that a unique solution of (2.2) exists and that the discretization error is of order h if the solution of (2.1) is smooth; see [6, 7].

In a second, equivalent approach the space $\widetilde{M}_h$ explicitly plays the role of a Lagrange multiplier space. This approach is studied in [4] and used further in [11, 25, 26]. The resulting variational formulation gives rise to a saddle-point problem:

Find $(\tilde{u}_h, \tilde{\lambda}_h) \in X_h \times \widetilde{M}_h$ such that

$$\begin{aligned} a(\tilde{u}_h, v_h) + b(\tilde{\lambda}_h, v_h) &= f(v_h), \qquad v_h \in X_h, \\ b(\mu_h, \tilde{u}_h) &= 0, \qquad \mu_h \in \widetilde{M}_h. \end{aligned} \tag{2.3}$$

In particular, it can be easily seen that the first component of the solution of (2.3) is the unique solution of (2.2). Observing that $\tilde{\lambda}_h$ is an approximation of the normal derivative of u on the interface, it makes sense to consider a priori estimates for $a\nabla u \cdot \mathbf{n}_{lk} - \tilde{\lambda}_h$ in suitable norms. Here $\mathbf{n}_{lk}$ is the outer unit normal of Ω_k restricted to Γ_{lk}. This issue was first addressed in [4] where a priori estimates in the $(H_{00}^{1/2})'$-norm were established. Similar bounds are given in [26] for a weighted L^2-norm. As in the general saddle-point approach [13], the essential point is to establish adequate inf-sup conditions; such bounds have been established with constants independent of h for both these norms; see [4, 26].

In the following, all constants $0 < c \leq C < \infty$ are generic depending on the local ratio between the coefficients b and a, the aspect ratio of the elements and subdomains but not on the mesh size and not on a. We use standard Sobolev notations and

$$\|v\|_1 := \sum_{k=1}^{K} \|v\|_{1;\Omega_k}, \quad |v|_1 := \sum_{k=1}^{K} |v|_{1;\Omega_k}, \quad v \in \prod_{k=1}^{K} H^1(\Omega_k)$$

stand for the broken H^1-norm and semi-norm. The dual space of a Hilbert space X is denoted by X' and the associated dual norm is defined by

$$\|\mu\|_{X'} := \sup_{v \in X} \frac{\langle \mu, v\rangle}{\|\mu\|_X}. \tag{2.4}$$

3. Dual basis functions. The crucial point for the unique solvability of (2.2) and (2.3) is the definition of the discrete space $\widetilde{M}_h$. As we have seen, the discrete space of Lagrange multipliers is closely related to the trace space in the earlier work on mortar methods; these spaces are only modified in the neighborhood of the interface boundaries where the degree of the elements of the test space is lower. We note that it has been shown only recently, see [23], that for P_n-conforming finite elements the finite dimensional space of piecewise polynomials of only degree $\leq n-1$ can be used instead of degree n in the definition of the Lagrange multiplier space without losing the optimality of the discretization error $u-u_h$. However, in none of these studies has duality been used to construct an adequate finite element space for the approximation of the Lagrange multiplier. We recall that the Lagrange multiplier in the continuous setting represents the flux on the interfaces. Even if the weak solution of (2.1) is H^2-regular it does not have to be continuous on the interfaces. This observation has motivated us to introduce a new type of discrete Lagrange multiplier space. We note that local dual basis functions have been used in [22] to define global projection-like operators which satisfy stability and approximation properties; in this paper we use the same dual basis functions to define the discrete Lagrange multiplier space.

Let σ be an edge and $\widetilde{P}_1(\sigma)$ be a polynomial space satisfying $P_0(\sigma) \subset \widetilde{P}_1(\sigma) \subset P_1(\sigma)$, and let $\{\phi_{\sigma;i}\}_{i=1}^N$, $N \in \{1,2\}$, be a basis satisfying $\int_\sigma \phi_{\sigma;i}\, ds \neq 0$. We can then define a dual basis $\{\psi_{\sigma;i}\}_{i=1}^N$, $\psi_{\sigma;i} \in \widetilde{P}_1(\sigma)$ by the following relation

$$\int_\sigma \phi_{\sigma;i}\psi_{\sigma;j}\, ds = \delta_{ij} \int_\sigma \phi_{\sigma;i}\, ds, \quad 1 \leq i,j \leq N. \tag{3.5}$$

The definition (3.5) guarantees that $\{\psi_{\sigma;i}\}_{i=1}^N$ is well defined. Each $\psi_{\sigma;i}$ can be written as a linear combination of the $\phi_{\sigma;i}$, $1 \leq i \leq N$ and the coefficients are obtained by solving a $N \times N$ mass matrix system. Furthermore (3.5) yields

$$\int_\sigma \phi_{\sigma;j} \left(\sum_{i=1}^N \psi_{\sigma;i} - 1 \right) ds = 0, \quad 1 \leq j \leq N$$

and thus $\sum_{i=1}^N \psi_{\sigma;i} = 1$. The $\{\psi_{\sigma;i}\}_{i=1}^N$ also form a linearly independent set. To see this, let us assume that $\sum_{i=1}^N \alpha_i \psi_{\sigma;i} = 0$. Then, it follows

$$\alpha_i = \frac{\int_\sigma \phi_{\sigma;i} \sum_{j=1}^N \alpha_j \psi_{\sigma;j}\, ds}{\int_\sigma \phi_{\sigma;i}} = 0.$$

As a consequence, we obtain $\widetilde{P}_1(\sigma) = \operatorname{span}\{\psi_{\sigma;j},\ 1 \leq j \leq N\}$.

Let us consider the case that $\{\lambda_{\sigma;i}\}_{i=1}^2$ are the nodal basis functions of $P_1(\sigma)$. Then, the dual basis is given by

$$\psi_{\sigma;i} := 2\lambda_{\sigma;i} - \lambda_{\sigma;(i+1)\mathrm{mod}2}, \quad 1 \leq i \leq 2.$$

Based on these observations, we introduce a global space $M_h(\Gamma_{lk})$ on each non-mortar Γ_{lk}, $1 \leq k \leq K$, $l \in \mathcal{M}(k)$ satisfying

$$\dim M_h(\Gamma_{lk}) = \dim \widetilde{M}_h(\Gamma_{lk}).$$

Let $\{\phi_i\}_{i=1}^{N_{lk}}$ be the nodal basis function of $\widetilde{M}_h(\Gamma_{lk})$ as introduced in Section 2. Then, each ϕ_i can be written as the sum of its local contributions

$$\phi_i = \sum_{\substack{\sigma\in\Sigma_{lk}\\ \sigma\subset\text{supp }\phi_i}} \phi_{i|_\sigma} =: \phi_{\sigma;i}$$

where the local contributions are linearly independent. We set $\widetilde{P}_1(\sigma) := \text{span}\{\phi_{\sigma;i}, 1 \leq i \leq N_{lk},\, \sigma \subset \text{supp }\phi_i\}$. In particular by construction, it is guaranteed that $P_0(\sigma) \subset \widetilde{P}_1(\sigma) \subset P_1(\sigma)$ and for $\partial\sigma \cap \partial\Gamma_{lk} = \emptyset$, we have $\widetilde{P}_1(\sigma) = P_1(\sigma)$. Using the local dual basis functions on each σ, the global basis functions of $M_h(\Gamma_{lk})$ are defined as

$$\psi_i := \sum_{\substack{\sigma\in\Sigma_{lk}\\ \sigma\subset\text{supp }\phi_i}} \psi_{\sigma;i}.$$

The support of ψ_i is the same as that of ϕ_i and the $\{\psi_i\}_{i=1}^{N_{lk}}$ form a linear independent system. Figure 3.2 depicts the two different types of dual basis functions.

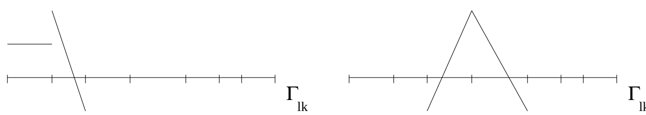


FIG. 3.2. *Dual basis functions in the neighborhood of the boundary of Γ_{lk} (left) and in the interior (right)*

REMARK 3.1. *The following global orthogonality relation holds*

$$\int_{\Gamma_{lk}} \phi_i\psi_j\, ds = \sum_{\substack{\sigma\in\Sigma_{lk}\\ \sigma\subset\text{supp }\Phi_i}} \int_\sigma \phi_{\sigma;i}\psi_{\sigma;j}\, ds = \sum_{\substack{\sigma\in\Sigma_{lk}\\ \sigma\subset\text{supp }\Phi_i}} \delta_{ij}\int_\sigma \phi_{\sigma;i}\, ds = \delta_{ij}\int_{\Gamma_{lk}} \phi_i\, ds. \tag{3.6}$$

We note the similarity with (3.5).

The central point in the analysis of the consistency and approximation error will be the construction of adequate projection-like operators. We refer to [6, 7] for the standard mortar approach. Here, we use different operators onto $M_h(\Gamma_{lk})$, $\widetilde{M}_h(\Gamma_{lk})$ and $W_h(\Gamma_{lk}) \cap H_0^1(\Omega)$.

$P_{lk} : L^1(\Gamma_{lk}) \longrightarrow M_h(\Gamma_{lk})$, is defined by

$$P_{lk}v := \sum_{i=1}^{N_{lk}} \frac{\int_{\Gamma_{lk}} \phi_i v\, ds}{\int_{\Gamma_{lk}} \phi_i\, ds}\psi_i. \tag{3.7}$$

A dual operator $Q_{lk} : L^1(\Gamma_{lk}) \longrightarrow \widetilde{M}_h(\Gamma_{lk})$, is now given by

$$Q_{lk}v := \sum_{i=1}^{N_{lk}} \frac{\int_{\Gamma_{lk}} \psi_i v\, ds}{\int_{\Gamma_{lk}} \phi_i\, ds}\phi_i. \tag{3.8}$$

A detailed discussion of this type of operator can be found in [22]. It is easy to see that P_{lk} and Q_{lk} restricted to $M_h(\Gamma_{lk})$ and $\widetilde{M}_h(\Gamma_{lk})$, respectively, is the identity

$$P_{lk}v = v, \quad v \in M_h(\Gamma_{lk}). \tag{3.9}$$

In addition, using (3.6), (3.7) and (3.8), we find that for $v, w \in L^2(\Gamma_{lk})$,

$$(v - P_{lk}v, w)_{0;\Gamma_{lk}} = (v - P_{lk}v, w - Q_{lk}w)_{0;\Gamma_{lk}} = (v, w - Q_{lk}w)_{0;\Gamma_{lk}},$$

and it therefore makes sense to call Q_{lk} a dual operator to P_{lk}. Furthermore, the operators are L^2-stable. We have

$$\|P_{lk}v\|^2_{0;\sigma} \le \sum_{\substack{1\le i\le N_{lk}\\ \sigma\cap\operatorname{supp}\phi_i\neq\emptyset}} \frac{\int_{\Gamma_{lk}} \phi_i^2\,ds \int_\sigma \psi_i^2\,ds}{(\int_{\Gamma_{lk}} \phi_i\,ds)^2} \|v\|^2_{0;\operatorname{supp}\phi_i} \le C\|v\|^2_{0;D_\sigma}, \tag{3.10}$$

where the domain D_σ is defined by

$$\bar{D}_\sigma := \bigcup \overline{\{\sigma' \in \Sigma_{lk} \mid \partial\sigma' \cap \partial\sigma \neq \emptyset\}}.$$

Here, we have used the fact that D_σ contains at most three elements and that $\|\psi_i\|_{L^\infty(\Gamma_{lk})}$ is bounded by 2 independently of the length of the edges. The same type of estimate holds true for Q_{lk}

$$\|Q_{lk}v\|^2_{0;\sigma} \le \sum_{\substack{1\le i\le N_{lk}\\ \sigma\cap\operatorname{supp}\phi_i\neq\emptyset}} \frac{\int_{\Gamma_{lk}} \psi_i^2\,ds \int_\sigma \phi_i^2\,ds}{(\int_{\Gamma_{lk}} \phi_i\,ds)^2} \|v\|^2_{0;\operatorname{supp}\phi_i} \le C\|v\|^2_{0;D_\sigma}. \tag{3.11}$$

Thus, P_{lk} and Q_{lk} are L^2-projection-like operators which preserve the constants. Using (3.9) and (3.10), it is easy to establish an approximation result.

LEMMA 3.2. *There exist constants such that for $v \in H^s(\Gamma_{lk})$, $0 \le s \le 1$,*

$$\|v - P_{lk}v\|^2_{0;\Gamma_{lk}} \le C \sum_{\sigma\in\Sigma_{lk}} h_\sigma^{2s} |v|^2_{s;\sigma}. \tag{3.12}$$

$$\|v - P_{lk}v\|^2_{(H^{\frac{1}{2}}(\Gamma_{lk}))'} \le c \sum_{\sigma\in\Sigma_{lk}} h_\sigma \|v - P_{lk}v\|^2_{0;\sigma} \le C \sum_{\sigma\in\Sigma_{lk}} h_\sigma^{1+2s} |v|^2_{s;\sigma}. \tag{3.13}$$

Proof. The proof of (3.12) follows by applying the Bramble-Hilbert Lemma and using the stability (3.10) and the identity (3.9); it is important that the constants are contained in the space $M_h(\Gamma_{lk})$. For each v we define a constant c_v in the following way

$$c_v := \frac{1}{|D_\sigma|} \int\limits_{D_\sigma} v\,ds,$$

where $|D_\sigma|$ is the length of D_σ. We remark that the constant c_v depends only on the values of v restricted on D_σ. Now, by means of $P_{lk}c_v = c_v$ we find

$$\begin{aligned} \|v - P_{lk}v\|_{0;\sigma} &\le \|v - c_v\|_{0;\sigma} + \|P_{lk}(v - c_v)\|_{0;\sigma} \\ &\le C\|v - c_v\|_{0;D_\sigma} \le Ch_\sigma^s |v|_{s;D_\sigma}. \end{aligned} \tag{3.14}$$

The global estimate (3.12) is obtained by summing over all local contributions and observing that each σ' is only contained in a fixed number of D_σ.

Although $\dim \widetilde{M}_h(\Gamma_{lk}) < \dim W_h(\Gamma_{lk})$, we get the same type of estimate as (3.14) for Q_{lk} instead of P_{lk} by using (3.11).

For the estimate (3.13) in the dual norm, we use the definition (2.4)

$$\|v - P_{lk}v\|_{(H^{\frac{1}{2}}(\Gamma_{lk}))'} = \sup_{\phi \in H^{\frac{1}{2}}(\Gamma_{lk})} \frac{\int_{\Gamma_{lk}} (v - P_{lk}v)\phi \, ds}{\|\phi\|_{H^{\frac{1}{2}}(\Gamma_{lk})}} = \sup_{\phi \in H^{\frac{1}{2}}(\Gamma_{lk})} \frac{\int_{\Gamma_{lk}} (v - P_{lk}v)(\phi - Q_{lk}\phi) \, ds}{\|\phi\|_{H^{\frac{1}{2}}(\Gamma_{lk})}}.$$

In a next step, we consider the last integral in more detail. Using (3.14) for Q_{lk} instead of P_{lk} and setting $s = 1/2$, we find

$$h_\sigma^{-1} \|\phi - Q_{lk}\phi\|^2_{0;\sigma} \le C |\phi|^2_{H^{\frac{1}{2}}(D_\sigma)}. \tag{3.15}$$

Summing over all $\sigma \in \Sigma_{lk}$ and using that the sum over $|\phi|^2_{H^{1/2}(D_\sigma)}$ is bounded by $|\phi|^2_{H^{1/2}(\Gamma_{lk})}$ yield

$$\|v - P_{lk}v\|^2_{(H^{\frac{1}{2}}(\Gamma_{lk}))'} \le \sum_{\sigma \in \Sigma_{lk}} h_\sigma \|v - P_{lk}v\|^2_{0;\sigma}.$$

Combining this upper bound with (3.14) gives (3.13). □

4. Nonconforming formulation. Replacing the space $\widetilde{M}_h$ in the definition of $\widetilde{V}_h$ by V_h, we get a new nonconforming space

$$V_h := \{v \in X_h \mid \; b(v, \mu) = 0, \; \mu \in M_h\}.$$

The original nonconforming variational problem (2.2) is then replaced by:

Find $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h), \quad v_h \in V_h. \tag{4.16}$$

In what follows, we analyze the structure of an element $v_h \in V_h$. Each $v \in X_h$ restricted to a non-mortar side Γ_{lk} can be written as

$$v_{|\Gamma_{lk}} = \sum_{i=0}^{N_{lk}+1} \alpha_i \phi_i, \tag{4.17}$$

where ϕ_i, $1 \le i \le N_{lk}$ are defined in Section 2 and ϕ_0 and $\phi_{N_{lk}+1}$ are the nodal basis functions of $W_h(\Gamma_{lk})$ associated with the two endpoints of Γ_{lk}. The following lemma characterizes the elements of V_h.

Lemma 4.1. *Let $v \in X_h$ restricted on Γ_{lk} be given as in (4.17). Then, $v \in V_h$ if and only if for each non-mortar Γ_{lk}*

$$\alpha_i = \frac{\int\limits_{\Gamma_{lk}} (v_{|\Omega_k} - \alpha_0\phi_0 - \alpha_{N_{lk}+1}\phi_{N_{lk}+1})\psi_i \, ds}{\int\limits_{\Gamma_{lk}} \phi_i \, ds}, \quad 1 \le i \le N_{lk}. \tag{4.18}$$

The proof follows easily from (4.17) and the global orthogonality relation (3.6).

As in case of $\widetilde{V}_h$ the values of a function $v \in V_h$ at the nodal Lagrange interpolation points in the interior, p_i, $1 \le i \le N_{lk}$, of any non-mortar Γ_{lk} are uniquely determined by its values on the corresponding mortar side Γ_{kl} and the values at the endpoints of

Γ_{lk}. The nodal values in the interior of the non-mortars Γ_{lk} are obtained by combining (4.18) with a basis transformation. In particular, these values can be directly obtained by the simple formula

$$v_{|\Omega_l}(p_i) = \alpha_i = \frac{\int\limits_{\Gamma_{lk}} v_{|\Omega_k} \psi_i \, ds}{\int\limits_{\Gamma_{lk}} \phi_i \, ds}, \quad 2 \le i \le N_{lk} - 1. \tag{4.19}$$

For the two interior nodal points p_1 and $p_{N_{lk}}$ next to the endpoints of Γ_{lk}, we get

$$\begin{aligned} v_{|\Omega_l}(p_1) &= \frac{\int\limits_{\Gamma_{lk}} (v_{|\Omega_k} \psi_1 - v_{|\Omega_l}(p_0)\phi_0) \, ds}{\int\limits_{\Gamma_{lk}} (\phi_1 - \phi_0) \, ds}, \\ v_{|\Omega_l}(p_{N_{lk}}) &= \frac{\int\limits_{\Gamma_{lk}} (v_{|\Omega_k} \psi_{N_{lk}} - v_{|\Omega_l}(p_{N_{lk}+1})\phi_{N_{lk}+1}) \, ds}{\int\limits_{\Gamma_{lk}} (\phi_{N_{lk}} - \phi_{N_{lk}+1}) \, ds}. \end{aligned} \tag{4.20}$$

Here, we have used that $v_{|\Omega_l}(p_0) = \alpha_0 + \alpha_1$, $v_{|\Omega_l}(p_{N_{lk}+1}) = \alpha_{N_{lk}} + \alpha_{N_{lk}+1}$ and that ψ_1 and $\psi_{N_{lk}}$ are identically 1 on the edges next to the endpoints of Γ_{lk}. We note that by definition of the basis functions there exist $1/2 < \beta_1, \beta_2 < 1$ such that

$$\int\limits_{\Gamma_{lk}} (\phi_1 - \phi_0) \, ds = \beta_1 \int\limits_{\Gamma_{lk}} \phi_1 \, ds, \quad \int\limits_{\Gamma_{lk}} (\phi_{N_{lk}} - \phi_{N_{lk}+1}) \, ds = \beta_2 \int\limits_{\Gamma_{lk}} \phi_{N_{lk}} \, ds. \tag{4.21}$$

If we have a closer look at the nodal basis functions of $\widetilde{V}_h$ and V_h we realize that there is a main difference in the structure of the basis functions. Figures 4.3 and 4.4 illustrate this difference for the special situation that we have a uniform but nonmatching triangulation on the mortar and the non-mortar side.

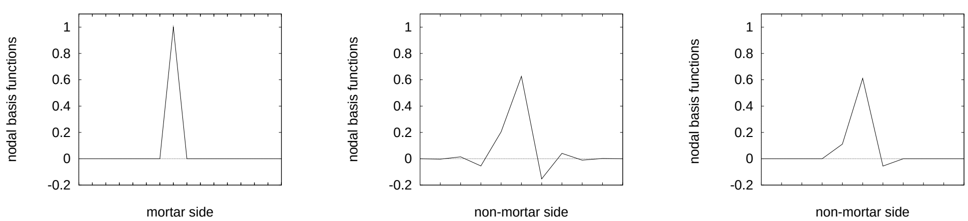


FIG. 4.3. *Nodal basis function on a mortar side (left) and on the non-mortar side in* $\widetilde{V}_h$ *(middle) and in* V_h *(right)*

In Figure 4.3, the mortar side is associated with the finer triangulation whereas in Figure 4.4 it is associated with the coarser one.

As in the standard finite element context, nodal basis functions can be defined for V_h with support contained in a circle of diameter Ch. This is in general not possible for $\widetilde{V}_h$. In the latter case, the support of a nodal basis function associated with a nodal point on the mortar side is a strip of length $|\Gamma_{lk}|$ and width h, see Figure 4.5, and the locality of the basis functions is lost.

We conclude this section, by establishing a priori bounds for the discretization error. As in [6, 7] a mortar projection will be a basic tool in the analysis of the best approximation error. We now use the new Lagrange multiplier space M_h in

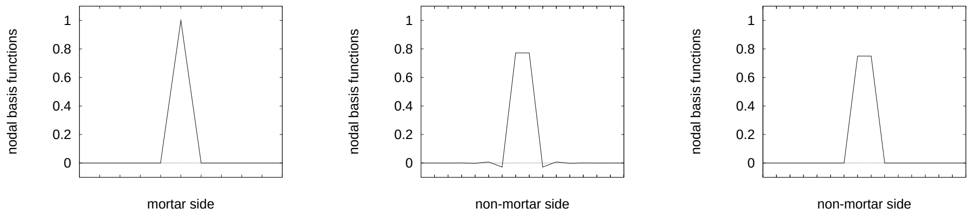


FIG. 4.4. *Nodal basis function on a mortar side (left) and on the non-mortar side in $\widetilde{V}_h$ (middle) and in V_h (right)*

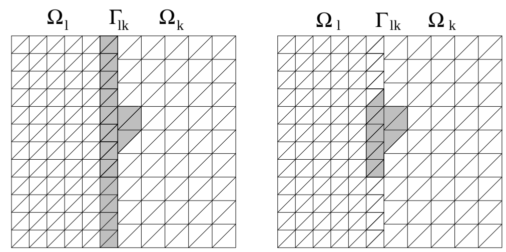


FIG. 4.5. *Support of a nodal basis function in $\widetilde{V}_h$ (left) and in V_h (right)*

the definition of suitable projection-like operators. For each non-mortar side the new mortar projection is given by $\Pi_{lk} : L^2(\Gamma_{lk}) \longrightarrow W_h(\Gamma_{lk}) \cap H_0(\Gamma_{lk})$,

$$\int_{\Gamma_{lk}} (v - \Pi_{lk} v)\mu \, ds = 0, \quad \mu \in M_h(\Gamma_{lk}).$$

By using (4.19) and (4.20), it can be easily seen that the operator Π_{lk} is well defined. To analyze the approximation error, it is sufficient to show that the mortar projection is uniformly stable in suitable norms. The stability in the L^2- and H^1-norms is given in the following lemma.

LEMMA 4.2. *The mortar projection Π_{lk} is L^2- and H^1-stable*

$$\|\Pi_{lk} v\|_{0;\Gamma_{lk}} \leq C \|v\|_{0;\Gamma_{lk}}, \quad v \in L^2(\Gamma_{lk}), \tag{4.22}$$

$$|\Pi_{lk} v|_{1;\Gamma_{lk}} \leq C |v|_{1;\Gamma_{lk}}, \quad v \in H_0^1(\Gamma_{lk}). \tag{4.23}$$

Proof. Using the explicit representation (4.19) and (4.20) where $v_{|\Omega_k}$ has to be replaced by v and $v_{|\Omega_l}(p_0)$ and $v_{|\Omega_l}(p_{N_{lk}+1})$ have to be set to zero, (4.22) is obtained. It can be easily seen that even the local estimate

$$\|\Pi_{lk} v\|_{0;\sigma} \leq C \|v\|_{0;D_\sigma}$$

holds true. By means of an inverse inequality, we find for each $p \in W_h(\Gamma_{lk}) \cap H_0^1(\Gamma_{lk})$ satisfying $p = 0$ if $\partial D_\sigma \cap \partial \Gamma_{lk} \neq \emptyset$ and otherwise $p_{|D_\sigma} = \text{const.}$ and $\int_{D_\sigma} p \, ds = \int_{D_\sigma} v \, ds$

$$|\Pi_{lk} v|_{1;\sigma} = |\Pi_{lk}(v - p)|_{1;\sigma} \leq \frac{C}{h_\sigma} \|\Pi_{lk}(v - p)\|_{0;\sigma} \leq \frac{C}{h_\sigma} \|v - p\|_{0;D_\sigma} \leq C |v|_{1;D_\sigma}.$$

We remark that if $\partial D_\sigma \cap \partial \Gamma_{lk} \neq \emptyset$, then p was set to zero. However, due to the boundary conditions of v we obtain $\|v\|_{0;D_\sigma} \leq C h_\sigma |v|_{1;D_\sigma}$ in this case. ☐

The mortar projection can be extended to the space $H_{00}^{1/2}(\Gamma_{lk})$ in the following way:
$\Pi_{lk} : H_{00}^{1/2}(\Gamma_{lk}) \longrightarrow W_h(\Gamma_{lk}) \cap H_0^1(\Gamma_{lk})$

$$\int\limits_{\Gamma_{lk}} (v - \Pi_{lk} v)\mu \, ds = 0, \quad \mu \in M_h(\Gamma_{lk}).$$

Then, an interpolation argument together with Lemma 4.2 gives the $H_{00}^{1/2}$-stability

$$\|\Pi_{lk} v\|_{H_{00}^{\frac{1}{2}}(\Gamma_{lk})} \le C \|v\|_{H_{00}^{\frac{1}{2}}(\Gamma_{lk})}, \quad v \in H_{00}^{\frac{1}{2}}(\Gamma_{lk}). \tag{4.24}$$

It is of interest to compute the stability constant in (4.22) in the special case of a uniform triangulation of Γ_{lk} with $h := |\sigma|$. Then,

$$\|\Pi_{lk} v\|_{0;\sigma}^2 = \frac{1}{6} h((\Pi_{lk} v(p_0))^2 + (\Pi_{lk} v(p_1))^2 + (\Pi_{lk} v(p_0) + \Pi_{lk} v(p_1))^2)$$

where p_0 and p_1 are the two endpoints of σ. Using the mortar definition and summing over all elements in Σ_{lk}, we get

$$\|\Pi_{lk} v\|_{0;\Gamma_{lk}}^2 \le \frac{17}{6} \|v\|_{0;\Gamma_{lk}}^2 .$$

In general, the constants in a priori estimates depend on the coefficients. Here, we will give a priori estimates which depends explicitly on the coefficient a. For each subdomain Ω_k, $1 \le k \le K$, we define constants α_k, $\tilde{\alpha}_k$ in the following way

$$\begin{aligned} \alpha_k &:= \max\left(\sup_{l \in \mathcal{M}(k)} \min\left(1 + \tfrac{a_l}{a_k}, 1 + (\tfrac{h_l}{h_k})^2\right), \sup_{\substack{1 \le j \le K \\ k \in \mathcal{M}(j)}} \min\left(1 + \tfrac{a_k}{a_j}, 1 + (\tfrac{h_k}{h_j})^2\right)\right), \\ \tilde{\alpha}_k &:= \max\left(\sup_{l \in \mathcal{M}(k)} 1 + \tfrac{a_l}{a_k}, \sup_{\substack{1 \le j \le K \\ k \in \mathcal{M}(j)}} 1 + \tfrac{a_k}{a_j} \right). \end{aligned} \tag{4.25}$$

We note that α_k and $\tilde{\alpha}_k$ are bounded by 2 if the non-mortar side is chosen as that with the smaller value of a.

The uniform ellipticity of the bilinear form $a(\cdot,\cdot)$ on $V_h \times V_h$ is important for the a priori estimates. For the standard mortar space, it is well known, see [6, 7, 8]. Moreover in [8], it is shown that the bilinear form $a(\cdot,\cdot)$ is uniform elliptic on $Y \times Y$, where

$$Y := \{v \in \prod_{k=1}^{K} H^1(\Omega_k) \mid \int\limits_{\Gamma_{lk}} [v] ds = 0, \; 1 \le k \le K, l \in \mathcal{M}(k), \; v_{|\partial\Omega} = 0\}.$$

The starting point of the proof is a suitable Poincaré-Friedrichs type inequality. For general considerations on Poincaré-Friedrichs type inequalities in the mortar situation, we refer to [24]. In [17, Theorem IV.1], it is shown that the ellipticity constant does not depend on the number of subdomains. A similar estimate is given for the three field formulation in [14]. We refer to [17] for a detailed analysis of the constants in the a priori estimates in terms of the number of subdomains and their diameter. Observing that V_h is a subspace of Y, it is obvious that that the bilinear form $a(\cdot,\cdot)$ on $V_h \times V_h$ is uniform elliptic.

4.1. Approximation property. To establish an approximation property for V_h, we follow [6, 7]. One central point in the analysis is an extension theorem. In [9], a discrete extension is used such that the H^1-norm of the extension on Ω_k is bounded by a constant times the $H^{1/2}$-norm on the boundary $\partial\Omega_k$. The support of such an extension is in general Ω_k and it is assumed that the triangulation is quasi-uniform. However, it can be generalized to the locally quasi-uniform case. Combining the approximation property of $\prod_{k=1}^K S_1(\Omega_k; \mathcal{T}_{h_k})$ and using the mortar projection Π_{lk}, we obtain the following lemma.

LEMMA 4.3. *Under the assumption that $u \in \prod_{k=1}^K H^{1+s}(\Omega_k)$, $0 < s \leq 1$, the best approximation error is of order h^s,*

$$\inf_{v_h \in V_h} a(u - v_h, u - v_h) \leq C \sum_{k=1}^K \alpha_k h_k^{2s} a_k \|u\|_{1+s;\Omega_k}^2$$

where the α_k are defined in (4.25).

Proof. The proof follows exactly the same lines as for $\widetilde{V}_h$ and the Laplace operator; we therefore omit the details and refer to [6, 7]. For each subdomain Ω_k, we use the Lagrange interpolation operator I_k. Then, we define $w_h \in X_h$ by $w_{h|_{\Omega_k}} := I_k v$. We note that w_h is not in general, contained in V_h. To obtain an element in V_h, we have to add appropriate corrections. For each interface Γ_{lk}, we consider the jump $[w_h] \in H_0^1(\Gamma_{lk})$ and apply the mortar projection. The result is extended as a discrete harmonic function into the interior of Ω_l. Finally, we define

$$v_h := w_h - \sum_{k=1}^K \sum_{l \in \mathcal{M}(k)} \mathcal{H}_l(\Pi_{lk}[w_h])$$

where $\mathcal{H}_l$ denotes the discrete harmonic extension operator

$$\|\mathcal{H}_l v\|_{1;\Omega_l} \leq C \|v\|_{H^{\frac{1}{2}}(\partial\Omega_l)};$$

see [9, Lemma 5.1]. Here, $\Pi_{lk}[w_h]$ is extended by zero onto $\partial\Omega_l \setminus \Gamma_{lk}$ and $\mathcal{H}_l(\Pi_{lk}[w_h])$ vanishes outside Ω_l. By construction, we have

$$\int_{\Gamma_{lk}} [v_h]\mu \, ds = \int_{\Gamma_{lk}} ([w_h] - \Pi_{lk}[w_h])\mu \, ds = 0, \quad \mu \in M_h(\Gamma_{lk}),$$

and thus $v_h \in V_h$.

A coloring argument yields,

$$\begin{aligned}
a(u - v_h, u - v_h) \quad &\leq \quad C \left(a(u - w_h, u - w_h) + \sum_{k=1}^K \sum_{l \in \mathcal{M}(k)} a_l \|\Pi_{lk}[w_h]\|_{H^{\frac{1}{2}}(\partial\Omega_l)}^2 \right) \\
&\leq C \sum_{k=1}^K \left(a_k h_k^{2s} \|u\|_{1+s;\Omega_k}^2 + \sum_{l \in \mathcal{M}(k)} a_l \|\Pi_{lk}[w_h]\|_{H_{00}^{\frac{1}{2}}(\Gamma_{lk})}^2 \right) \\
&\leq C \sum_{k=1}^K \left(a_k h_k^{2s} \|u\|_{1+s;\Omega_k}^2 + \sum_{l \in \mathcal{M}(k)} a_l \|[u - w_h]\|_{H_{00}^{\frac{1}{2}}(\Gamma_{lk})}^2 \right) \\
&\leq C \sum_{k=1}^K \left(a_k h_k^{2s} \|u\|_{1+s;\Omega_k}^2 + \sum_{l \in \mathcal{M}(k)} a_l \left(h_l^{2s} \|u\|_{H^{\frac{1}{2}+s}(\Gamma_{lk})}^2 + h_k^{2s} \|u\|_{H^{\frac{1}{2}+s}(\Gamma_{lk})}^2 \right) \right) \\
&\leq C \sum_{k=1}^K \alpha_k a_k h_k^{2s} \|u\|_{1+s;\Omega_k}^2 .
\end{aligned}$$

Here, we have used the stability of the harmonic extension; see [9], the stability of the mortar projection (4.24) and the approximation property of the Lagrange interpolant. ◻

4.2. Consistency error. The space V_h is in general not a subspace of $H_0^1(\Omega)$. Therefore, we are in a nonconforming setting and in addition to uniform ellipticity and the approximation property we need to consider the consistency error [10] to obtain a stable and convergent finite element discretization. In Strang's second Lemma, the discretization error is bounded by the best approximation error and the consistency error [10].

LEMMA 4.4. *The consistency error for* $u \in \prod_{k=1}^K H^{1+s}(\Omega_k)$, $1/2 < s \leq 1$ *and* $[a\nabla u \mathbf{n}_{lk}] = 0$ *is of order* h^s

$$\sup_{w_h \in V_h} \frac{\sum\limits_{k=1}^K \sum\limits_{l \in \mathcal{M}(k)} \int_{\Gamma_{lk}} a \frac{\partial u}{\partial \mathbf{n}_{lk}} [w_h] \, ds}{a(w_h, w_h)^{\frac{1}{2}}} \leq C \left(\sum_{k=1}^K \alpha_k h_k^{2s} a_k \|u\|_{1+s,\Omega_k}^2 \right)^{\frac{1}{2}}$$

where α_k *is defined in (4.25).*

Proof. The proof generalizes that given for $\widetilde{V}_h$ in [6, 7]. Here, the Lagrange multiplier space M_h is used and we also consider the effect of discontinuous coefficients. By the definition of V_h, we have

$$\sum_{k=1}^K \sum_{l \in \mathcal{M}(k)} \int_{\Gamma_{lk}} \mu_h [w_h] \, ds = 0, \quad \mu_h \in M_h,$$

and thus

$$\sum_{k=1}^K \sum_{l \in \mathcal{M}(k)} \int_{\Gamma_{lk}} a \frac{\partial u}{\partial \mathbf{n}_{lk}} [w_h] \, ds = \sum_{k=1}^K \sum_{l \in \mathcal{M}(k)} \int_{\Gamma_{lk}} (a \frac{\partial u}{\partial \mathbf{n}_{lk}} - P_{lk}(a \frac{\partial u}{\partial \mathbf{n}_{lk}}))[w_h] \, ds.$$

where P_{lk} is defined in (3.7). Using a duality argument and the continuity of the trace, we get

$$\sum_{k=1}^K \sum_{l \in \mathcal{M}(k)} \int_{\Gamma_{lk}} a \frac{\partial u}{\partial \mathbf{n}_{lk}} [w_h] \, ds \leq \sum_{k=1}^K \sum_{l \in \mathcal{M}(k)} \|\lambda - P_{lk}\lambda\|_{(H^{\frac{1}{2}}(\Gamma_{lk}))'} (|w_h|_{1;\Omega_l} + |w_h|_{1;\Omega_k})$$

where $\lambda := a \frac{\partial u}{\partial \mathbf{n}_{lk}}$. To replace, in the last inequality, the H^1-norm by the H^1-seminorm, we take into account that

$$\Pi_{lk} w_{h|_{\Omega_l}} = \Pi_{lk} w_{h|_{\Omega_k}},$$

where Π_{lk} denotes the L^2-projection onto piecewise constant functions on Γ_{lk}. In the duality argument, the $H^{1/2}$-norm can therefore be replaced by the $H^{1/2}$-semi-norm

$$\begin{aligned} \|[w_h]\|_{H^{\frac{1}{2}}(\Gamma_{lk})} &\leq \|w_{h|_{\Omega_l}} - \Pi_{lk} w_{h|_{\Omega_l}}\|_{H^{\frac{1}{2}}(\Gamma_{lk})} + \|w_{h|_{\Omega_k}} - \Pi_{lk} w_{h|_{\Omega_k}}\|_{H^{\frac{1}{2}}(\Gamma_{lk})} \\ &\leq C \left(|w_{h|_{\Omega_l}} - \Pi_{lk} w_{h|_{\Omega_l}}|_{H^{\frac{1}{2}}(\Gamma_{lk})} + |w_{h|_{\Omega_k}} - \Pi_{lk} w_{h|_{\Omega_k}}|_{H^{\frac{1}{2}}(\Gamma_{lk})} \right). \end{aligned}$$

Finally, Lemma 3.2, which states the approximation property of $P_{lk}\lambda$ in the $(H^{1/2})'$-norm, yields

$$\|\lambda - P_{lk}\lambda\|_{(H^{\frac{1}{2}}(\Gamma_{lk}))'} \leq C h_l^s |\lambda|_{H^{s-\frac{1}{2}}(\Gamma_{lk})} \leq C h_l^s \min(a_l |u|_{1+s;\Omega_l}, a_k |u|_{1+s;\Omega_k}).$$

◻

REMARK 4.5. *In case that the coefficient a is smaller on the non-mortar side then α_k is bounded by 2 independently of the jumps in a. Otherwise the upper bound for α_k depends on the jumps in a. A possibly better bound might depend on the ratio of the mesh size across the interface; see (4.25). However, numerical results have shown that in case of adaptive mesh refinement controlled by an a posteriori error estimator, α_k remains bounded independently of the jump in the coefficients; see [26, 27].*

Using Lemmas 4.3 and 4.4, we obtain a standard a priori estimate for the modified mortar approach (4.16). Under the assumptions that $[a\nabla u \mathbf{n}_{lk}] = 0$ and $u \in \prod_{K=1}^K H^s(\Omega_k)$, $3/2 < s \leq 2$, we find

$$a(u - u_h, u - u_h) \leq C \sum_{k=1}^K \alpha_k a_k h_k^{2(s-1)} \|u\|_{s;\Omega_k}^2. \tag{4.26}$$

REMARK 4.6. *The a priori estimates in the literature [6, 7] are often given in the following form*

$$\|u - u_h\|_1 \leq C \sum_{k=1}^K h_k^{s-1} \|u\|_{s;\Omega_k}.$$

This is weaker than the estimate (4.26), since for $s = 2$ generally we only have

$$\left(\sum_{k=1}^K h_k \|u\|_{2;\Omega_k} \right)^2 \leq K \sum_{k=1}^K h_k^2 \|u\|_{2;\Omega_k}^2.$$

4.3. A priori estimates in the L^2-norm. Finite element discretizations provide, in general, better a priori estimates in the L^2-norm than in the energy norm. In particular, if we assume H^2-regularity, we have the following a priori estimate for $u - \tilde{u}_h$ in the L^2-norm

$$\|u - \tilde{u}_h\|_{0;\Omega} \leq C(a) h^2 \|u\|_{2;\Omega}.$$

The proof can be found in [11] and is based on the Aubin-Nitsche trick. In addition, the nonconformity of the discrete space has to be taken into account. An essential role in the proof of the a priori bound is the following lemma. It shows a relation between the jumps of an element $v \in V_h$ across the interfaces and its nonconformity. The same type of result for $v \in \widetilde{V}_h$ can be found in [26].

LEMMA 4.7. *The weighted L^2-norm of the jumps of an element $v \in V_h$ is bounded by its nonconformity*

$$\sum_{k=1}^K \sum_{l \in \mathcal{M}(k)} \sum_{\sigma \in \Sigma_{lk}} \frac{a_l}{h_\sigma} \|[v]\|_{0;\sigma}^2 \leq \inf_{w \in H_0^1(\Omega)} \sum_{k=1}^K \tilde{\alpha}_k a_k \|v - w\|_{1;\Omega_k}$$

Proof. The proof follows the same ideas as in case for $v_h \in \widetilde{V}_h$, see [26]. We use the orthogonality of the jump and the definition (3.8) to obtain

$$Q_{lk} v_{|\Omega_l} = Q_{lk} v_{|\Omega_k}.$$

Now, it is sufficient to consider an interface Γ_{lk} at a time, and we find

$$\sum_{\sigma \in \Sigma_{lk}} \frac{a_l}{h_\sigma} \|[v_h]\|_{0;\sigma}^2 = \sum_{\sigma \in \Sigma_{lk}} \frac{a_l}{h_\sigma} \|v_{|\Omega_l} - Q_{lk} v_{|\Omega_l} - (v_{|\Omega_k} - Q_{lk} v_{|\Omega_k})\|_{0;\sigma}^2.$$

Using (3.15) and the continuity of the trace operator, we get for each $w \in H_0^1(\Omega)$

$$\begin{aligned}\sum_{\sigma\in\Sigma_{lk}} \tfrac{a_l}{h_\sigma} \|[v_h]\|_{0;\sigma}^2 &\leq C\left(a_l |v_{|\Omega_l} - w|_{H^{\frac12}(\Gamma_{lk})} + a_l |v_{|\Omega_k} - w|_{H^{\frac12}(\Gamma_{lk})}\right) \\ &\leq C a_l \left(\|v-w\|_{1;\Omega_l} + \|v-w\|_{1;\Omega_k}\right).\end{aligned}$$

Summing over the subdomains and using the definition for $\tilde{\alpha}_k$ give the assertion. □

Using the dual problems:
Find $w \in H_0^1(\Omega)$ and $w_h \in V_h$ such that

$$a(w,v) = (u-u_h, v)_{0;\Omega}, \quad v \in H_0^1(\Omega), \quad a(w_h, v) = (u-u_h, v)_{0;\Omega}, \quad v \in V_h$$

gives

$$\|u-u_h\|_{0;\Omega}^2 = a(w-w_h, u-u_h) + \sum_{k=1}^{K} \sum_{l\in\mathcal{M}(k)} \Big(\int_{\Gamma_{lk}} a \frac{\partial w}{\partial \mathbf{n}_{lk}} [u_h]\, ds + \int_{\Gamma_{lk}} a \frac{\partial u}{\partial \mathbf{n}_{lk}} [w_h]\, ds \Big).$$

Then, the H^2-regularity, Lemma 3.2, Lemma 4.7, and observing that the jump of an element in V_h is orthogonal on M_h yield

$$\begin{aligned}\|u-u_h\|_{0;\Omega}^2 \;&\leq\; a(u-u_h, u-u_h)^{1/2} a(w-w_h, w-w_h)^{1/2} \\ &+ \left(\sum_{k=1}^{K} \sum_{l\in\mathcal{M}(k)} \sum_{\sigma\in\Sigma_{lk}} \tfrac{a_l}{h_\sigma} \|[u_h]\|_{0;\sigma}^2 \right)^{1/2} \left(\sum_{k=1}^{K} \sum_{l\in\mathcal{M}(k)} \sum_{\sigma\in\Sigma_{lk}} \tfrac{h_\sigma}{a_l} \|\chi - P_{lk}\chi\|_{0;\sigma}^2 \right)^{1/2} \\ &+ \left(\sum_{k=1}^{K} \sum_{l\in\mathcal{M}(k)} \sum_{\sigma\in\Sigma_{lk}} \tfrac{a_l}{h_\sigma} \|[w_h]\|_{0;\sigma}^2 \right)^{1/2} \left(\sum_{k=1}^{K} \sum_{l\in\mathcal{M}(k)} \sum_{\sigma\in\Sigma_{lk}} \tfrac{h_\sigma}{a_l} \|\lambda - P_{lk}\lambda\|_{0;\sigma}^2 \right)^{1/2} \\ &\leq C(a) \left(\|u-u_h\|_{1;\Omega} \|w-w_h\|_{1;\Omega} + h\|u-u_h\|_{1;\Omega}\|w\|_{2;\Omega} + h\|w-w_h\|_{1;\Omega}\|u\|_{2;\Omega}\right),\end{aligned}$$

where $\chi_{|\Gamma_{lk}} := a\frac{\partial w}{\partial \mathbf{n}_{lk}}$. Using the a priori estimate for the energy norm (4.26), we obtain an a priori estimate for the L^2-norm. The following lemma gives the a priori estimate for the modified mortar approach.

LEMMA 4.8. *Assuming H^2-regularity, the discretization error $u-u_h$ in the L^2-norm is of order h^2.*

5. Saddle point formulation. A saddle point formulation for mortar methods was introduced in [4]. In particular, a priori estimates involving the $(H_{00}^{1/2})'$-norm for the Lagrange multiplier were established in that paper whereas estimates in a weighted L^2-norm were given in [26]. Here, we analyze the error in the Lagrange multiplier for both norms and obtain a priori estimates of the same quality as for the standard mortar approach.

The norm for the Lagrange multiplier is defined by

$$\|\mu\|^2_{(H_{00}^{\frac12}(\mathcal{S}))'} := \sum_{k=1}^{K} \sum_{l\in\mathcal{M}(k)} \frac{1}{a_l} \|\mu\|^2_{(H_{00}^{\frac12}(\Gamma_{lk}))'}, \quad \mu \in \prod_{k=1}^{K} \prod_{l\in\mathcal{M}(k)} (H_{00}^{\frac12}(\Gamma_{lk}))'.$$

The weight a_l^{-1} is related to the fact that we use the energy norm for $u-u_h$ in the a priori estimates.

Working within the saddle point framework, the approximation property on V_h, which is given in Lemma 4.3, is a consequence of the approximation property on X_h, the continuity of the bilinear form $b(\cdot,\cdot)$, and an inf-sup condition [13]. A discrete inf-sup condition is necessary to obtain a priori estimates for the Lagrange multiplier.

The saddle point problem associated with the new nonconforming formulation (4.16) involves the space (X_h, M_h) instead of $(X_h, \widetilde{M}_h)$. We get a new saddle point problem, with exactly the same structure as (2.3):

Find $(u_h, \lambda_h) \in (X_h, M_h)$ such that

$$\begin{aligned} a(u_h, v_h) + b(\lambda_h, v_h) &= f(v_h), \qquad v_h \in X_h, \\ b(\mu_h, u_h) &= 0, \qquad \mu_h \in M_h. \end{aligned} \tag{5.27}$$

The inf-sup condition, established in [4] for the pairing $(X_h, \widetilde{M}_h)$ also holds for (X_h, M_h).

Lemma 5.1. *There exists a constant independent of h such that*

$$\inf_{\substack{\mu_h \in M_h \\ \mu_h \neq 0}} \sup_{\substack{v_h \in X_h \\ a(v_h, v_h) \neq 0}} \frac{b(v_h, \mu_h)}{\|\mu_h\|_{(H_{00}^{\frac{1}{2}}(\mathcal{S}))'} a(v_h, v_h)^{1/2}} \geq c. \tag{5.28}$$

Proof. Using the definition of the dual norm (2.4), we get

$$\begin{aligned} \|\mu_h\|_{(H_{00}^{\frac{1}{2}}(\Gamma_{lk}))'} &= \sup_{\phi \in H_{00}^{\frac{1}{2}}(\Gamma_{lk})} \frac{(\mu_h, \phi)_{0;\Gamma_{lk}}}{\|\phi\|_{H_{00}^{\frac{1}{2}}(\Gamma_{lk})}} = \sup_{\phi \in H_{00}^{\frac{1}{2}}(\Gamma_{lk})} \frac{(\mu_h, \Pi_{lk}\phi)_{0;\Gamma_{lk}}}{\|\phi\|_{H_{00}^{\frac{1}{2}}(\Gamma_{lk})}} \\ &\leq C \sup_{\phi \in H_{00}^{\frac{1}{2}}(\Gamma_{lk})} \frac{(\mu_h, \Pi_{lk}\phi)_{0;\Gamma_{lk}}}{\|\Pi_{lk}\phi\|_{H_{00}^{\frac{1}{2}}(\Gamma_{lk})}} \leq C \max_{\phi \in W_h(\Gamma_{lk}) \cap H_0^1(\Gamma_{lk})} \frac{(\mu_h, \phi)_{0;\Gamma_{lk}}}{\|\phi\|_{H_{00}^{\frac{1}{2}}(\Gamma_{lk})}}. \end{aligned}$$

The maximizing element in $W_h(\Gamma_{lk}) \cap H_0^1(\Gamma_{lk})$ with $H_{00}^{1/2}(\Gamma_{lk})$-norm 1, is called ϕ_{μ_h} and a $v_{lk} \in X_h$ is defined in the following way

$$v_{lk}|_{\Omega \setminus \Omega_l} := 0, \quad v_{lk}|_{\bar{\Omega}_l} := \mathcal{H}_l \phi_{\mu_h},$$

where ϕ_{μ_h} is extended by zero on $\partial\Omega_l \setminus \Gamma_{lk}$. We then find

$$0 \leq (\mu_h, \phi_{\mu_h})_{0;\Gamma_{lk}} = b(\mu_h, v_{lk})$$

and $a(v_{lk}, v_{lk}) \leq C a_l |v_{lk}|^2_{1;\Omega_l}$. Finally, we set

$$v_{\mu_h} := \sum_{k=1}^{K} \sum_{l \in \mathcal{M}(k)} \frac{b(\mu_h, v_{lk})}{a_l} v_{lk},$$

and observe that $a(v_{\mu_h}, v_{\mu_h}) = 0$ if and only if $\mu_h = 0$. A coloring argument gives $a(v_{\mu_h}, v_{\mu_h}) \leq C \|\mu_h\|^2_{(H_{00}^{1/2}(\mathcal{S}))'}$. Summing over all interfaces yields

$$\begin{aligned} \|\mu_h\|^2_{(H_{00}^{\frac{1}{2}}(\mathcal{S}))'} &\leq C \sum_{k=1}^{K} \sum_{l \in \mathcal{M}(k)} \tfrac{1}{a_l} b(\mu_h, v_{lk})^2 \;=\; C\, b(\mu_h, v_{\mu_h}) \\ &\leq C \frac{b(\mu_h, v_{\mu_h})}{a(v_{\mu_h}, v_{\mu_h})^{1/2}} \|\mu_h\|_{(H_{00}^{\frac{1}{2}}(\mathcal{S}))'}. \end{aligned} \tag{5.29}$$

By construction, we have found for each $\mu_h \in M_h$, $\mu_h \neq 0$, a $v_{\mu_h} \in X_h$ with $a(v_{\mu_h}, v_{\mu_h}) \neq 0$ such that

$$\|\mu_h\|_{(H_{00}^{\frac{1}{2}}(\mathcal{S}))'} \leq C \frac{b(\mu_h, v_{\mu_h})}{a(v_{\mu_h}, v_{\mu_h})^{1/2}}.$$

□

The proof of the inf-sup condition (5.28) together with the approximation Lemma 3.2 and the first equation of the saddle point problem gives an a priori estimate similar to (4.26) for the Lagrange multiplier.

LEMMA 5.2. *Under the assumptions* $u \in \prod_{k=1}^K H^s(\Omega)$, $3/2 < s \le 2$ *and* $[a\nabla u \mathbf{n}_{lk}] = 0$, *the following a priori estimate for the Lagrange multiplier holds true*

$$\|\lambda - \lambda_h\|^2_{(H_{00}^{\frac12}(\mathcal{S}))'} \le C \sum_{k=1}^K \alpha_k a_k h_k^{2(s-1)} \|u\|^2_{s;\Omega_k}. \tag{5.30}$$

Proof. Following [4] and using the first equation of the saddle point problem, we get

$$b(\mu_h - \lambda_h, v_h) = a(u_h - u, v_h) + b(\mu_h - \lambda, v_h), \quad v_h \in X_h. \tag{5.31}$$

Taking (5.29) into account, we find that the inf-sup condition even holds if the supremum over X_h is replaced by the supremum over a suitable subspace of X_h. For the proof of (5.30), we start with (5.29) and not with the inf-sup condition (5.28)

$$\|\mu_h - \lambda_h\|^2_{(H_{00}^{\frac12}(\mathcal{S}))'} \le C b(\mu_h - \lambda_h, w), \tag{5.32}$$

where $w := v_{\mu_h - \lambda_h}$ and $v_{\mu_h - \lambda_h}$ is constructed as in the proof of Lemma 5.1. We recall that w is defined as a linear combination of discrete harmonic functions

$$w = \sum_{k=1}^K \sum_{l \in \mathcal{M}(k)} \frac{b(\mu_h - \lambda_h, w_{lk})}{a_l} w_{lk}$$

where $w_{lk} = 0$ on $\mathcal{S} \setminus \Gamma_{lk}$ and $\|[w_{lk}]\|_{H_{00}^{1/2}(\Gamma_{lk})} = 1$. A coloring argument shows that the energy norm of w is bounded by the $H_{00}^{1/2}$–dual norm of $\mu_h - \lambda_h$, moreover we find

$$\begin{aligned} \|[w]\|^2_{H_{00}^{\frac12}(\mathcal{S})} &= \sum_{k=1}^K \sum_{l \in \mathcal{M}(k)} \tfrac{(b(\mu_h - \lambda_h, w_{lk}))^2}{a_l} \|[w_{lk}]\|^2_{H_{00}^{1/2}(\Gamma_{lk})} = \sum_{k=1}^K \sum_{l \in \mathcal{M}(k)} \tfrac{(b(\mu_h - \lambda_h, w_{lk}))^2}{a_l} \\ &\le \sum_{k=1}^K \sum_{l \in \mathcal{M}(k)} \tfrac{1}{a_l} \|\mu_h - \lambda_h\|^2_{(H_{00}^{1/2}(\Gamma_{lk}))'} \|[w_{lk}]\|^2_{H_{00}^{1/2}(\Gamma_{lk})} = \|\mu_h - \lambda_h\|^2_{(H_{00}^{\frac12}(\mathcal{S}))'}. \end{aligned}$$

Now, combining (5.31) and (5.32), we obtain

$$\begin{aligned} \|\mu_h - \lambda_h\|^2_{(H_{00}^{\frac12}(\mathcal{S}))'} &\le C a(u - u_h, u - u_h)^{\frac12} a(w, w)^{\frac12} + \|\mu_h - \lambda\|_{(H_{00}^{\frac12}(\mathcal{S}))'} \|w\|_{H_{00}^{\frac12}(\mathcal{S})} \\ &\le C \ \|\mu_h - \lambda_h\|_{(H_{00}^{\frac12}(\mathcal{S}))'} \left(a(u - u_h, u - u_h)^{\frac12} + \|\mu_h - \lambda\|_{(H_{00}^{\frac12}(\mathcal{S}))'} \right). \end{aligned}$$

Applying the triangle inequality, choosing $\mu_{h|_{\Gamma_{lk}}} := P_{lk}\lambda$ and using

$$\|v\|_{(H_{00}^{\frac12}(\Gamma_{lk}))'} \le \|v\|_{(H^{\frac12}(\Gamma_{lk}))'}, \quad v \in (H^{\frac12}(\Gamma_{lk}))',$$

we find that (3.13) yields, for $s = 1/2$

$$\begin{aligned} \|\lambda - \lambda_h\|^2_{(H_{00}^{\frac12}(\mathcal{S}))'} &\le C \left(a(u - u_h, u - u_h) + \sum_{k=1}^K \sum_{l \in \mathcal{M}(k)} \tfrac{h_l^2}{a_l} |\lambda|^2_{\frac12;\Gamma_{lk}} \right) \\ &\le C \sum_{k=1}^K \alpha_k a_k h_k^2 \|u\|^2_{2;\Omega_k}. \end{aligned}$$

Here, we have used that λ restricted on Γ_{lk} is $a\nabla u\mathbf{n}_{lk}$ and a trace theorem. □

We note that in spite of Lemma 3.2 we cannot obtain a priori estimates of order h for the norm of the dual of $H^{1/2}(\mathcal{S})$. This is due to the fact that the inf-sup condition (5.28) cannot be established for that norm.

REMARK 5.3. *The a priori estimate (5.30) also holds if we replace the $(H_{00}^{1/2})'$-norm by the weighted L^2-norm*

$$\|\mu\|_{h;\mathcal{S}}^2 := \sum_{k=1}^{K} \sum_{l\in\mathcal{M}(k)} \sum_{\sigma\in\Sigma_{lk}} \frac{h_\sigma}{a_l} \|\mu\|_{0;\sigma}^2, \quad \mu \in L^2(\mathcal{S}).$$

Using (3.12) and the techniques of the proof of Lemma 5.2, it is sufficient to have a discrete inf-sup condition similar to (5.28) for the weighted L^2-norm, i.e.

$$\inf_{\substack{\mu_h\in M_h\\ \mu_h\neq 0}} \sup_{\substack{v_h\in X_h\\ a(v_h,v_h)\neq 0}} \frac{b(v_h,\mu_h)}{\|\mu_h\|_{h;\mathcal{S}}\, a(v_h,v_h)^{1/2}} \geq c.$$

The only difference in the proof is the definition of v_{lk}. Instead of using a discrete harmonic extension onto Ω_l, we use a trivial extension by zero, i.e. we set all nodal values on $\partial\Omega_l\backslash\Gamma_{lk}$ and on Ω_l to zero. Then, v_{lk} is non zero only on a strip of length $|\Gamma_{lk}|$ and width h_l and $a(v_{lk}, v_{lk})$ is bounded form below and above by $\sum_{\sigma\in\Sigma_{lk}} \frac{a_l}{h_\sigma}\|v_{lk}\|_{0;\sigma}^2$.

6. Numerical results. We get a priori estimates of the same quality for the error in the weak solution and the Lagrange multiplier as in the standard mortar case [4, 6, 7]. In contrast to $\widetilde{V}_h$, we can define nodal basis functions for V_h which have local supports. Efficient iterative solvers for linear equation systems arising from mortar finite element discretization are very often based on the saddle point formulation or work with the product space X_h instead of the nonconforming mortar space. Different types of efficient iterative solvers are developed in [1, 2, 3, 11, 15, 16, 19, 20, 18, 25]. However, most of these techniques require that each iterate satisfies the constraints exactly. In most studies of multigrid methods, these constraints have to be satisfied even in each smoothing step [11, 12, 18, 25]. If we replace $\widetilde{V}_h$ by V_h the constraints are much easier to satisfy, since instead of solving a mass matrix system, the nodal values on the non-mortar side can be given explicitly.

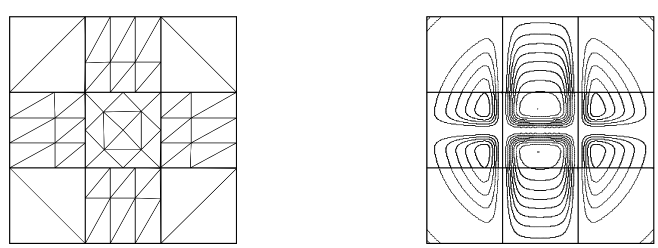


FIG. 6.6. *Decomposition and initial triangulation (left) and solution (right) (Example 1)*

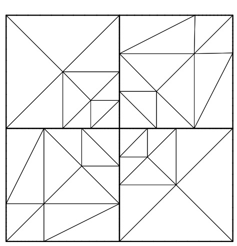
Here, we will present some numerical results illustrating the discretization errors for the standard and the new mortar methods in the case of P_1 Lagrangian finite elements. We recall that in the standard mortar approach the Lagrange multipliers belong to $\widetilde{M}_h$ whereas we use M_h in the new method. We have used a multigrid method which satisfies the constraints in each smoothing step; see [11, 25] for a discussion of the standard mortar case. This multigrid method can be also applied without any modifications to our modified mortar setting. It does not take advantage of the diagonal mass matrix on the non–mortar side of the new formulation. To obtain a speedup in the numerical computations, special iterative solvers for the new mortar setting have to be designed. We will address this issue in a forthcoming paper [28]. We start with an initial triangulation $\mathcal{T}_0$, and obtain the triangulation $\mathcal{T}_l$ on level l by uniform refinement of $\mathcal{T}_{l-1}$.

Both discretization techniques have been applied to the following test example: $-\Delta u = f$ on $(0,1)^2$, where the right hand side f and the Dirichlet boundary conditions are chosen so that the exact solution is $(\exp(-500xx1) - 1) * (\exp(-500xx2) - 1) * (\exp(-500yy) - 1) * (1 - 3rr)^2$. Here $xx1 := (x - 1/3)^2$, $xx2 := (x - 2/3)^2$, $xx := (x - 1/2)^2$, $yy := (y - 1/2)^2$ and $rr := xx + yy$. The solution and the initial triangulation are given in Figure 6.6. The domain is decomposed into nine subdomains defined by $\Omega_{ij} := ((i-1)/3, i/3) \times ((j-1)/3, j/3)$, $1 \le i,j \le 3$ and the triangulations do not match at the interfaces. We observe two different situations at the interface, e.g. the isolines of the solution are almost parallel at $\partial\Omega_{11} \cap \partial\Omega_{12}$ whereas at $\partial\Omega_{11} \cap \partial\Omega_{21}$ the angle between the isolines and the interface is bounded away from zero. In case that the isolines are orthogonal on the interface the exact Lagrange multiplier will be zero.

Table 6.1
Discretization errors (Example 1)

		standard approach		modified approach	
		Lagrange multiplier $\widetilde{M}_h$		Lagrange multiplier M_h	
level	# elem.	L^2-err.	energy err.	L^2-err.	energy err.
0	72	$2.021163e+0$	11.47900	$2.021306e+0$	11.47984
1	288	$1.017372e-1$	3.042101	$1.014502e-1$	3.034778
2	1152	$1.166495e-1$	1.945246	$1.166435e-1$	1.946163
3	4608	$9.482530e-3$	1.114075	$9.476176e-3$	1.113506
4	18432	$2.802710e-3$	0.5928275	$2.797809e-3$	0.5923121
5	73728	$7.130523e-4$	0.2981975	$7.121334e-4$	0.2980159
6	284912	$1.789436e-4$	0.1492382	$1.788082e-4$	0.1491841

In Table 6.1, the discretization errors are given in the energy norm as well as in the L^2-norm for the two different mortar methods. We observe that the energy error is of order h whereas the error in the L^2-norm is of order h^2. There is no significant difference in the accuracy between the two mortar algorithm. The discretization errors in the energy norm as well as in the L^2-norm are almost the same.



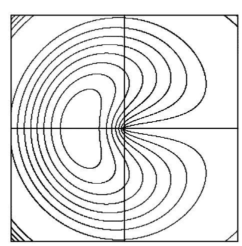


Fig. 6.7. *Decomposition and initial triangulation (left) and solution (right) (Example 2)*

In our second example, we consider the union square with a slit decomposed into four subdomains, see Figure 6.7. Here, the right hand side f and the Dirichlet boundary conditions of $-\Delta u = f$ are chosen so that the exact solution is given by $(1-3r^2)^2 r^{1/2} sin(1/2\phi)$, where $x - 1/2 = r\cos\phi$, and $y - 1/2 = r\sin\phi$. The solution has a singularity in the center of the domain. We do not have H^2-regularity, and we therefore cannot expect an $\mathcal{O}(h)$ behavior for the discretization error in the energy norm.

TABLE 6.2
Discretization errors (Example 2), Energy error in $1e-01$

		standard approach		modified approach	
		Lagrange multiplier $\widetilde{M}_h$		Lagrange multiplier M_h	
level	# elem.	L^2-err.	energy err.	L^2-err.	energy err.
0	44	$4.896283e-02$	6.000955	$4.861265e-02$	6.050778
1	176	$1.651238e-02$	3.553279	$1.619017e-02$	3.584246
2	704	$4.488552e-03$	2.045833	$4.281367e-03$	2.069586
3	2816	$1.254716e-03$	1.232939	$1.125460e-03$	1.252113
4	11264	$3.878438e-04$	0.7824813	$3.046049e-04$	0.7975380
5	45056	$1.401538e-04$	0.5184650	$8.680669e-05$	0.5298379
6	180224	$5.883500e-05$	0.3536026	$2.649174e-05$	0.3619496

The discretization errors are compared in Table 6.2. In this case, we observe a difference in the performance of the different mortar methods. The L^2-error of the modified mortar method is asymptotically better than that of the standard method. The situation is different for the energy error; the standard mortar approach gives slightly better results. A non-trivial difference can only be observed in this example where there is no H^2-regularity. In that case, the modified mortar method gives better results in the L^2-norm.

Our last example illustrates the influence of discontinuous coefficients. We consider the diffusion equation $-\text{div}\, a\nabla u = f$, on $(0,1)^2$, where the coefficient a is discontinuous. The unit square Ω is decomposed into four subdomains $\Omega_{ij} := ((i-1)/2, i/2) \times ((j-1)/2, j/2)$ as in Figure 6.8.

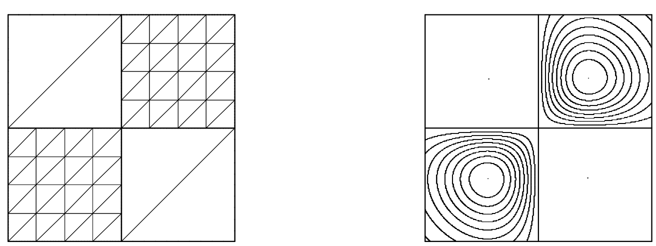


FIG. 6.8. *Decomposition and initial triangulation (left) and solution (right) (Example 3)*

The coefficients on the subdomains are given by $a_{11} = a_{22} = 0.00025$, $a_{12} = a_{21} = 1$. The right hand side f and the Dirichlet boundary conditions are chosen to match a given exact solution, $(x-0.5)(y-0.5)\exp(-10((x-0.5)^2 + (y-0.5)^2))/a$. This solution is continuous with vanishing $[a\nabla u\mathbf{n}]$ on the interfaces. Because of the discontinuity of the coefficients, we use a highly non-matching triangulation at the interface, see Figure 6.8.

The discretization errors in the energy norm as well as in the L^2-norm are given for the two different mortar algorithms in Table 6.3. We observe that the energy error is of order h. As in Example 1, there is only a minimal difference in the performance of the two mortar approaches.

TABLE 6.3
Discretization errors (Example 3), Energy error in $1e-01$

		standard approach		modified approach	
		Lagrange multiplier $\widetilde{M}_h$		Lagrange multiplier M_h	
level	# elem.	L^2-err.	energy err.	L^2-err.	energy err.
0	68	$3.184810e+00$	11.73889	$2.981474e+00$	11.99259
1	272	$9.416096e-01$	6.115732	$9.358117e-01$	6.187439
2	1088	$2.425569e-01$	3.083728	$2.431694e-01$	3.094938
3	4352	$6.093936e-02$	1.545031	$6.103994e-02$	1.546515
4	17408	$1.524479e-02$	0.7729229	$1.525489e-02$	0.7731113
5	69632	$3.811271e-03$	0.3865144	$3.812137e-03$	0.3865380
6	278528	$9.527881e-04$	0.1932641	$9.528569e-04$	0.1932670

The following two figures illustrate the numbers given in Tables 6.1 – 6.3. In Figure 6.9, the errors in the energy norm are visualized whereas in Figure 6.10 the errors in the L^2-norm are shown. In each figure a straight dashed line is drawn below the obtained curves to indicate the asymptotic behavior of the discretization errors.

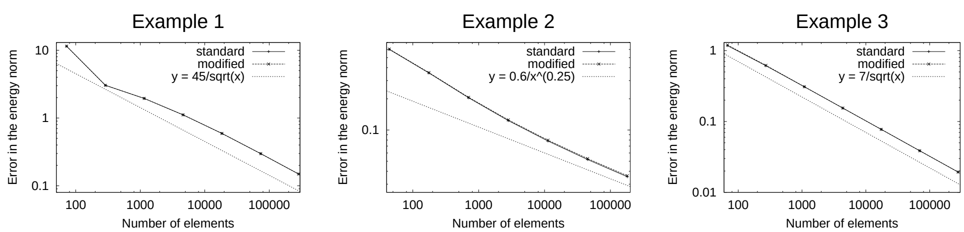


FIG. 6.9. *Discretization errors in the energy norm versus number of elements*

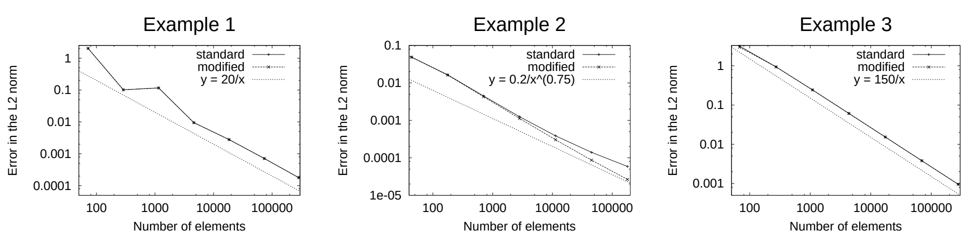


FIG. 6.10. *Discretization errors in the L^2–norm versus number of elements*

In Examples 1 and 2, almost from the beginning on the predicted order h for the energy norm and the order h^2 for the L^2–norm can be observed. In these two examples only one plotted curve for the standard and the new mortar approach can be seen. The numerical results are too close to see a difference in the pictures. In Example 2, where we have no full H^2-regularity, the asymptotic starts late. We observe for both mortar methods an $\mathcal{O}(h^{1/2})$ behavior for the discretization error in the energy norm. During the first refinement steps the error decreases more rapidly. For the L^2-norm the asymptotic rate is given by $\mathcal{O}(h^{3/2})$. Moreover, it seems to be the case that the new mortar method performs asymptotically better than the standard one. However, this cannot be observed for other examples without full regularity.

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