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EMBEDDABILITY BETWEEN ORDERINGS AND THE GCH

A b s t r a c t. We provide some statements equivalent in ZFC to GCH, and also to GCH above a given cardinal. These statements express the validity of the notions of replete and well-replete cardinals, which are introduced and proved to be specially relevant to the study of cardinal exponentiation. As a byproduct, a structure theorem for linear orderings is proved to be equivalent to GCH: for every linear ordering L, at least one of L and its converse is universal for the smaller well-orderings.

1. Preliminary Remarks

Two order-theoretic properties of cardinals, the repleteness and the well-repleteness, are defined and the generalized continuum hypothesis (GCH)

is proved to be equivalent to the validity of each of these properties. More precisely, theorem 3.1 (section 3) shows that all cardinals are replete if and only if GCH holds. Then, a strengthening of this concerning the validity of GCH above a given cardinal is proved. Our theorem 3.3 closes section 3, completely characterizing the replete cardinals in terms of the continuum function. After that, theorem 4.1 (section 4) shows that a cardinal is replete if and only if it is well-replete. As an immediate application, we have that GCH is equivalent to the following structure hypothesis for linear orderings.

- (Universality hypothesis) For every linearly ordered set L, at least one of L and its converse is universal for the smaller well-ordered sets (i.e. those of cardinality less than $|L|$).

We say that a linear ordering is universal for a class of linearly ordered sets if it embeds any such set. The universality hypothesis roughly says that in order to arrange a set of points in a line one cannot bypass any of the smaller well-orderings. It is trivially true in the finite realm and the transposition of this regularity to the infinite amounts to GCH.

The equivalence between GCH and the validity of well-repleteness is a natural continuation of the main theorem proved by Erdős and Rado in [1] which, surprisingly, remained unnoticed up until now. Another theorem due to Erdős and Rado is relevant in this paper, the partition theorem, and a suitable form of it is recalled in section 3.

The combination of all our results is summarized in theorem 1.1. The aim of this work is to bring some order-theoretic intuitions to GCH and cardinal exponentiation, generally.

Theorem 1.1. *The following are equivalent in ZFC:*

- *GCH*

- *Every cardinal is replete.*

- *Every cardinal is well-replete.*

- *For every linearly ordered set L, at least one of L and its converse is universal for the smaller well-ordered sets*

2. Replete and Well-Replete Cardinals

We use κ, λ, μ, ν and o as cardinal variables, and α, β and γ as ordinal variables. The converse of an ordinal α is denoted by α^*, and similarly to cardinals and linear orderings in general. We use $\alpha \hookrightarrow L$ to denote the existence of an embedding of α into L, and, similarly, $\alpha^* \hookrightarrow L$ denotes the existence of an embedding of α^* into L. Two notions that are used throughout must be defined.

Definition 2.1. We say that κ is *replete* if for every cardinal $\lambda < \kappa$, every linear ordering of cardinality κ contains a copy of λ or of λ^*. That is, κ is replete if for each $\lambda < \kappa$, there is room for an increasing λ-sequence or for a decreasing λ-sequence in every linear ordering of cardinality κ.

Definition 2.2. We say that κ is *well-replete* if for every ordinal $\alpha < \kappa$, every linear ordering of cardinality κ contains a copy of α or of α^*. That is, κ is well-replete if for each $\alpha < \kappa$, there is room for an increasing α-sequence or for a decreasing α-sequence in every linear ordering of cardinality κ.

Remark 2.3. Recall that an uncountable cardinal κ is weakly compact if and only if every linear ordering of cardinality κ contains a copy of κ or of κ^*. As a consequence, both of the above defined properties are implied by weak compactness. Also, they are both trivially valid up until $\aleph_0$. Indeed, if a linear ordering contains a copy of λ or of λ^*, then it contains, accordingly, a copy of α or of α^* for every $\alpha < \lambda$. Therefore, if κ is finite, ω or weakly compact, then it is well-replete, as it satisfies the stronger property that every linear ordering of cardinality κ contains a copy of κ or of κ^*.

Remark 2.4. The uniform versions of the two previous definitions are easily seen to be equivalent to the given versions. For example, we can say that κ is well-replete if and only if for every linear ordering L with cardinality κ, any ordinal $\alpha < \kappa$ can be embedded in L or the converse α^* of any $\alpha < \kappa$ can be embedded in L. Indeed, if κ is well-replete and L is a linear ordering with cardinality κ, then $\{\alpha < \kappa : \alpha \hookrightarrow L\} \cup \{\alpha < \kappa : \alpha^* \hookrightarrow L\} = \kappa$. Therefore, at least one of $\{\alpha < \kappa : \alpha \hookrightarrow L\}$ and $\{\alpha < \kappa : \alpha^* \hookrightarrow L\}$ is unbounded in κ, which gives the desired conclusion.

Of course, if a cardinal is well-replete, then it is replete. Based on the above remark, the proof of corollary 4.2 establishes the equivalence between the universality hypothesis and the statement that every cardinal

is well-replete. Our theorem 4.1 shows that these are also equivalent to the statement that every cardinal is replete, which is equivalent to GCH by theorem 3.1. Since these auxiliary statements are very useful, it is wise to give names to them:

- (Repleteness hypothesis) Every cardinal is replete.

- (Well-repleteness hypothesis) Every cardinal is well-replete.

Recall that for cardinals κ, λ, μ and $n < \omega$, the notation $\kappa \to (\lambda)^n_\mu$ is used as a shorthand way of saying that for every function f from $[\kappa]^n$ to μ, there is a set $X \subseteq \kappa$ with cardinality λ such that f restricted to $[X]^n$ is constant. Such a set X is said to be homogeneous with respect to f.

Remark 2.5. If $\kappa \to (\lambda)^2_2$, then every linear ordering of cardinality κ contains a copy of λ or of λ^*. In fact, let $f : [\kappa]^2 \to \{0, 1\}$ be such that $f(\{x, y\}) = 1$ if x and y are related by the given linear ordering in the same way as they are related by the natural ordering on κ, and $f(\{x, y\}) = 0$ otherwise. If $X \subseteq \kappa$ is a homogeneous set of cardinality λ, then X endowed with the given linear ordering is a well-ordered set and contains a copy of λ as an initial segment, or X endowed with the inverse of the given linear ordering is a well-ordered set and contains a copy of λ as an initial segment.

3.   The Repleteness Hypothesis and GCH

The statement that every cardinal is replete implies GCH. In fact, it is well-known that for every infinite λ, the cardinal 2^λ lexicographically ordered does not contain an increasing λ^+-sequence or a decreasing λ^+-sequence. (See Lemma 3.17, p. 328, in [3]). If 2^λ is replete, then $2^\lambda \le \lambda^+$. Hence, the repleteness hypothesis implies GCH.

Before proving that, conversely, GCH implies the repleteness hypothesis, recall the Erdős-Rado theorem. The following version is given in [3], p. 327, theorem 3.13:

- If λ and μ are infinite cardinals, $n \ge 2$, $o < \mu$, and $\mu \to (\lambda)^{n-1}_o$, then $(sup_{\nu<\mu} 2^\nu)^+ \to (\lambda)^n_o$.

If we set $\lambda = \mu$, $n = 2$ and $o = 2$, then we get

$$(sup_{\nu < \lambda} 2^\nu)^+ \to (\lambda)^2_2.$$

This partition relation is used in the proof that GCH implies the repleteness hypothesis.

Theorem 3.1. *(ZFC) The repleteness hypothesis is equivalent to GCH.*

Proof. We need only to show that GCH implies the repleteness hypothesis. Let κ be an infinite uncountable cardinal. We prove that κ is replete assuming that GCH holds below κ, and this is enough, for ω and all finite cardinals are already known to be well-replete.

Take $\lambda < \kappa$ infinite and assume that GCH holds below λ. It follows that

$$sup_{\nu < \lambda} 2^\nu = sup_{\nu < \lambda} \nu^+ = \lambda.$$

From Erdős-Rado, $\lambda^+ \to (\lambda)^2_2$ so, by remark 2.5, every linear ordering of cardinality λ^+ contains a copy of λ or of λ^*. Since every linear ordering of cardinality κ contains a linear subordering of cardinality λ^+, every linear ordering of cardinality κ contains a copy of λ or of λ^*. Therefore, κ is replete. $\square$

The previous theorem can be strengthened. Two local lemmas can be easily extracted from the proof. The first one shows that a failure of GCH gives a failure of repleteness. The second gives, conversely, that if κ is not replete, then GCH fails for some $\lambda < \kappa$. Both are used below to strengthen our result.

1. If 2^λ is replete, then $2^\lambda = \lambda^+$.

2. If $(sup_{\nu < \lambda} 2^\nu)^+ \leq \kappa$ for every $\lambda < \kappa$, then κ is replete.

Let κ be an infinite cardinal. We say that GCH holds above κ if $2^\lambda = \lambda^+$, for every $\lambda \geq \kappa$. If every $\mu \geq \kappa^{++}$ is replete, then GCH holds above κ. Indeed, if $\lambda \geq \kappa$, then $2^\lambda \geq \kappa^+$. If $2^\lambda = \kappa^+$, then $\lambda \leq \kappa$, hence $\lambda = \kappa$ and $2^\lambda = \lambda^+$. Otherwise, $2^\lambda \geq \kappa^{++}$, and 2^λ is replete. From (1) above, $2^\lambda = \lambda^+$ also holds in this case.

Conversely, if GCH holds above κ, then every $\mu \geq \kappa^{++}$ is replete. Indeed, the assumption that GCH holds above κ implies that for every $\lambda < \mu$, either $\lambda \leq \kappa$ and

$$(sup_{\nu<\lambda}2^\nu)^+ \leq (2^\lambda)^+ \leq (2^\kappa)^+ = \kappa^{++} \leq \mu,$$

or $\lambda > \kappa$ and

$$(sup_{\nu<\lambda}2^\nu)^+ = (sup_{\kappa\leq\nu<\lambda}2^\nu)^+ = (sup_{\kappa\leq\nu<\lambda}\nu^+)^+ \leq \lambda^+ \leq \mu.$$

Either way, $(sup_{\nu<\lambda}2^\nu)^+ \leq \mu$. From local lemma (2) above, μ is replete.

Therefore, we have proved a stronger version of theorem 3.1, parameterized by an infinite cardinal κ:

Theorem 3.2. *(ZFC) Every $\mu \geq \kappa^{++}$ is replete if and only if GCH holds above κ.*

If $\kappa = \aleph_0$, theorem 3.1 is recovered, for $\aleph_1$, $\aleph_0$ and the finite cardinals are obviously replete. Also, a complete characterization of repleteness in terms of the continuum function can be given as a further application of our lemmas.

Theorem 3.3. *(ZFC) An infinite cardinal κ is replete if and only if given $\mu < \kappa$ and $\nu < \mu$, it holds that $2^\nu < \kappa$.*

Proof. First notice that if there are $\mu < \kappa$ and $\nu < \mu$ such that $2^\nu \geq \kappa$, then no κ-sized subordering of the lexicographic order on 2^ν embeds μ, and κ is not replete in this case. The converse implication is proved in two separate cases, using that κ is either a limit or a successor cardinal.

If κ is a limit cardinal for which the condition holds, then it is a strong limit. From (2) above, it directly follows that if κ is strong limit, then κ is replete.

Now, assume that κ is an infinite successor cardinal, $\kappa = \lambda^+$, and that the condition holds, that is, $2^\nu < \kappa$ for every $\nu < \lambda$. This means that $2^\nu \leq \lambda$ for every $\nu < \lambda$, and $(sup_{\nu<\lambda}2^\nu)^+ \leq \lambda^+ = \kappa$. From (2) above, κ is replete.   □

4. The Equivalence between Repleteness and Well-Repleteness

In view of the previous section, in order to complete the proof that GCH, the repleteness hypothesis and the well-repleteness hypothesis are equivalent, it is enough to prove that repleteness implies well-repleteness. Our

proof relies on the following standard result from the theory of linear orderings, as expounded in [2], section 5.3. Recall that a linear ordering is κ-dense if any (nonempty) open interval has cardinality κ.

- Let κ be an infinite regular cardinal. Every linear ordering with cardinality κ admits an embedding of κ or κ^*, or a κ-dense subordering. (See 3.5., chapter 5, page 140 in [2].)

Theorem 4.1. *(ZFC) A cardinal is replete if and only if it is well-replete. Therefore, the repleteness hypothesis and the well-repleteness hypothesis are equivalent.*

Proof.

It is enough to prove that for every cardinal κ, if κ is replete, then it is well-replete. This is trivial if κ is either finite or a limit cardinal.

Let κ be a replete successor cardinal, and let λ be such that κ is the successor of λ.

Let L be a linear ordering with cardinality κ. From the above mentioned result, we may assume without loss of generality that L is κ-dense. In this case, the cardinality of every (nonempty) open interval of L is κ.

Let $P(\alpha)$ be the property $\alpha < \kappa \to \forall I \subseteq L, \alpha \hookrightarrow I$, where I denotes an open interval of L and $\alpha \hookrightarrow I$ means I embeds α.

Let $Q(\alpha)$ be the property $\alpha < \kappa \to \forall I \subseteq L, \alpha^* \hookrightarrow I$, where I denotes an open interval of L and $\alpha^* \hookrightarrow I$ means I embeds α^*.

If $\forall \alpha (P(\alpha) \wedge Q(\alpha))$, then we are done. Suppose that $\exists \alpha \neg (P(\alpha) \wedge Q(\alpha))$, and let β be the least such ordinal.

Case (i): Suppose that $\neg P(\beta)$.

Let $I \subseteq L$ be an interval such that I does not embed β. Since $P(\gamma)$ holds for every $\gamma < \beta$, we have that β must be regular. Indeed, otherwise, I would embed the cofinality of β, and hence it would embed an increasing sequence whose limit is β, for every ordinal below β can be embedded in every interval of L. From that, I would embed β.

Therefore, $\beta = cf(\beta)$ and β is a cardinal. Being a cardinal below κ, β is at most λ, for $\kappa = \lambda^+$.

Since κ is replete and I is a linear ordering with cardinality κ, we have that $\lambda \hookrightarrow I$ or $\lambda^* \hookrightarrow I$. Since β is contained in λ and I does not embed β, it follows that I does not embed λ. Therefore, $\lambda^* \hookrightarrow I$ and $\beta^* \hookrightarrow I$.

Let $R(\alpha)$ be the property $\alpha < \kappa \to \forall J \subseteq I, \alpha^* \hookrightarrow J$, where J denotes an open interval contained in I. If $\forall \alpha R(\alpha)$, then we are done. Suppose that $\exists \alpha \neg R(\alpha)$, and let γ be the least such ordinal. Let $J \subseteq I$ be such that J is a subinterval of I that does not embed γ^*. Being an interval of L, the cardinality of J is κ.

From the choice of β, every ordinal below β satisfies property Q. So every ordinal below β satisfies property R, for Q implies R for every ordinal. Therefore, $\beta \leq \gamma$. Again, γ must be regular. Otherwise, J would embed the converse of the cofinality of γ, and hence it would embed a decreasing sequence whose limit is γ^*, for the converse of every ordinal below γ can be embedded in every subinterval of I (and of J). From that, J would embed γ^*. Therefore, γ is regular, it is a cardinal, and it is at most λ.

However, κ is replete and J is a linear ordering with cardinality κ. It follows that $\lambda \hookrightarrow J$ or $\lambda^* \hookrightarrow J$. In the first case, we would have that $\beta \hookrightarrow \lambda \hookrightarrow J \hookrightarrow I$, contradicting that I does not embed β. In the second case, we would have $\gamma^* \hookrightarrow \lambda^* \hookrightarrow J$, contradicting that J does not embed γ^*. Therefore, $R(\alpha)$ holds for every α, and L embeds the converse of every ordinal below κ.

Case (ii): Suppose that $\neg Q(\beta)$.

Symmetrically, we conclude that $S(\alpha)$ holds for every α, where $S(\alpha)$ denotes the property $\alpha < \kappa \to \forall J \subseteq I, \alpha \hookrightarrow J$. In this case, L embeds every ordinal below κ.

In any case, the linear ordering L embeds every ordinal below κ or the converse of every ordinal below κ. We conclude that every linear ordering with cardinality κ embeds every ordinal below κ or the converse of every ordinal below κ, and κ is well-replete. $\square$

We have proved that GCH is equivalent to the well-repleteness hypothesis. From this, it is immediate that GCH is equivalent to the the statement that for every linear ordering L, for every well-ordering W, if the cardinality of L is greater than that of W, then L admits an embedding of W or its converse. Moreover, we have the equivalence between GCH and the universality hypothesis, which is another purely order-theoretic principle displaying the uniformity of the usual axioms of ZFC:

Corollary 4.2. *(ZFC) The universality hypothesis is equivalent to GCH.*

Proof. It is enough to prove that the universality hypothesis is equivalent to the well-repleteness hypothesis. Recall, from remark 2.4, that κ is well-replete if and only if for each linear ordering L with cardinality κ, it holds that $\kappa = \{\alpha < \kappa : \alpha \hookrightarrow L\}$ or $\kappa = \{\alpha < \kappa : \alpha^* \hookrightarrow L\}$. Therefore, the well-repleteness hypothesis is equivalent to the statement that for each linear ordering L, it holds that $|L| = \{\alpha < |L| : \alpha \hookrightarrow L\}$ or $|L| = \{\alpha < |L| : \alpha^* \hookrightarrow L\}$. This statement is equivalent to the universality hypothesis. For the first alternative ($|L| = \{\alpha < |L| : \alpha \hookrightarrow L\}$) amounts to L being universal for the smaller well-orderings, and the second alternative ($|L| = \{\alpha < |L| : \alpha^* \hookrightarrow L\}$) amounts to L^* being universal for the smaller well-orderings. $\square$

References

[1] P. Erdős and R. Rado, A Problem on Ordered Sets, *Journal of the London Mathematical Society* **28:4** (1953), 426 – 438.

[2] R. Fraïssé, Theory of Relations, Elsevier, 1986.

[3] A. Levy, Basic Set Theory, Dover, 2002.

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